Existence and Regularity Results of a Ferroelectric Phase-Field Model

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Abstract

In this thesis, we investigate the existence and regularity results of a ferroelectric phase-field model, which is a state-of-the-art model arising in recent years from the engineering area for the ferroelectric study. The ferroelectric phase-field model describes the relationship between the mechanical displacement $\boldsymbol{u}$, the electric field ϕ and the spontaneous polarization $\boldsymbol{P}$. Mathematically, the model is given by a coupled differential system, by the means that the mechanical displacement $\boldsymbol{u}$ and the electric field ϕ are given as the unique solution of a second order elliptic system, whose coefficients will also depend on the spontaneous polarization $\boldsymbol{P}$; The dynamics of $\boldsymbol{P}$ will be given by the nonlinear evolutionary law

$$-D_{\boldsymbol{P}}\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) \in \partial\Psi(\boldsymbol{P}'), \tag{LAW}$$

where $\Psi(\boldsymbol{P})$ is a convex and weakly lower semicontinuous dissipation functional, and $\mathcal{H}$ is an entropy functional which contains a nonlinearity that is given as a quadratic function of the gradient of $(\boldsymbol{u}, \phi)$. We discuss two different cases by imposing different dissipation functionals for the model.

We first consider the case that the dissipation functional $\Psi(\boldsymbol{P})$ is given as the sum

$$\Psi(\boldsymbol{P}) = \Psi_1(\boldsymbol{P}) + \beta\|\boldsymbol{P}\|_{L^2}^2,$$

where β is some given fixed positive constant and Ψ_1 is a convex, weakly lower semicontinuous and positively 1-homogenous functional. In this case, we will discretize the time interval $[0, T]$ into subintervals with step size $\tau > 0$ and construct certain time discrete interpolant solutions corresponding to the discretization, which will satisfy a discrete version of (LAW). We then use several variational identities and inequalities to show that by pushing τ to zero, one obtains a weak limit of the time discrete interpolant solutions, which turns out to be the actual solution (the so called viscous solution) of (LAW). Finally, by using certain Γ-convergence theory, we are able to push β to zero to obtain a weak limit of the (parameterized) viscous solutions, the so called vanishing viscosity solution, that will also satisfy a parameterized evolutionary law derived from (LAW).

On the other hand, we will also consider the case

$$\Psi(\boldsymbol{P}) = \frac{\beta}{2}\|\boldsymbol{P}\|_{L^2}^2$$

with some fixed $\beta > 0$. In this case, (LAW) is reduced to a coupled semilinear parabolic system. We will then use the theory of maximal parabolic operators and fixed point theorem to construct a unique local solution of (LAW). At the end, we show that by setting $\beta = 0$ in the first case, one also obtains a global solution of (LAW) in the parabolic case.

Zusammenfassung

In dieser Arbeit untersuchen wir Existenz- und Regularitätsresultate für ein ferroelektrisches Phasenfeldmodell auf dem aktuellen Stand der Wissenschaft, welches in den letzten Jahren aus den ferroelektrischen Studien in den Ingenieurwissenschaften entstanden ist. Das ferroelektrische Phasenfeldmodell beschreibt die Beziehung zwischen der mechanischen Verschiebung $\boldsymbol{u}$, dem elektrischen Feld ϕ und der spontanen Polarisation $\boldsymbol{P}$. Mathematisch ist das Modell durch ein gekoppeltes Differentialgleichungssystem gegeben: Die mechanische Verschiebung $\boldsymbol{u}$ und das elektrische Feld ϕ sind als die eindeutige Lösung eines elliptischen Systems zweiter Ordnung angegeben, dessen Koeffizienten auch von der spontanen Polarisation abhängig sind; Die Dynamik von $\boldsymbol{P}$ ist durch das nichtlineare Evolutionsgesetz

$$-D_{\boldsymbol{P}}\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) \in \partial\Psi(\boldsymbol{P}') \tag{LAW}$$

gegeben. Hier ist $\Psi(\boldsymbol{P})$ ein konvexes, schwach unterhalbstetiges Dissipationsfunktional; $\mathcal{H}$ ist ein Entropiefunktional, das eine Nichlinearität in Form einer quadratischen Funktion des Gradienten von $(\boldsymbol{u}, \phi)$ enthält. Wir diskutieren zwei verschiedene Fälle, bei denen wir unterschiedliche Dissipationsfunktionale zu Grunde legen.

Wir betrachten zunächst den Fall, in dem das Dissipationsfunktional $\Psi(\boldsymbol{P})$ als die Summe

$$\Psi(\boldsymbol{P}) = \Psi_1(\boldsymbol{P}) + \beta\|\boldsymbol{P}\|_{L^2}^2$$

gegeben ist, wobei β eine festgelegte positive Konstante und Ψ_1 ein konvexes, schwach unterhalbstetiges und positiv 1-homogenes Funktional sind. In diesem Fall zerlegen wir das Zeitintervall $[0, T]$ in Teilintervalle mit Schrittweite $\tau > 0$ und konstruieren zeitdiskrete Interpolationslösungen, die eine diskrete Version des Gesetzes (LAW) erfüllen. Mit Hilfe einiger Variationsgleichungen und -Ungleichungen lässt sich mit verschwindender Zeitschrittweite die Konvergenz der Interpolationslösungen zu einem schwachen Grenzwert zeigen, welcher dann eine Lösung von (LAW) ist. Da man β als künstlichen Viskositätsparameter interpretieren kann, werden diese Lösungen auch Viskositätslösungen genannt. Schließlich erhalten wir unter Verwendung von Γ-Konvergenztheorie (nach Umparametrisierung) einen schwachen Grenzwert der Viskositätslösungen für verschwindendes β. Dieser Grenzwert wird viskositätsverschwindende Lösung genannt und erfüllt ein parametrisiertes Evolutionsgesetz, das von (LAW) abgeleitet wird.

Auf der anderen Seite betrachten wir auch den Fall

$$\Psi(\boldsymbol{P}) = \frac{\beta}{2}\|\boldsymbol{P}\|_{L^2}^2$$

mit einer festgelegten Konstante $\beta > 0$. In diesem Fall lässt sich (LAW) auf ein gekoppeltes semilineares parabolisches System reduzieren. Wir werden dann die Theorie der maximalen parabolischen Operatoren und einen Fixpunktsatz verwenden, um eine eindeutige lokale Lösung von (LAW) zu konstruieren. Schließlich zeigen wir auch, dass man durch Einsetzen von $\beta = 0$ im ersten Fall eine globale Lösung von (LAW) für den parabolischen Fall erhält.

Inhaltsverzeichnis

Chapter 1

Introduction

This thesis is devoted to the study of a phase-field model for ferroelectric materials arising from the engineering area. The study of ferroelectrics began with J. Valasek's paper [65], where the ferroelectric material Rochelle salt was studied and it was the first time that a hysteresis curve, which describes the relation between the external electric field and the polarization, was proposed. From 1942 to 1946, ferroelectrics were discovered in the material Barium Titanate ($BaTiO_3$) independently by scientists in USA, Russia and Japan, which becomes one of the most important ferroelectric materials nowadays. Thereafter, more and more ferroelectric materials have been found, and corresponding ferroelectric study has been rapidly developed. For an introduction of the history of ferroelectric materials, we refer to [11, 27].

We give a brief introduction of the physical principles which characterize the ferroelectric materials. One can imagine that electric charge is distributed on certain dielectric material. The electric charge can then be classified into two categories: applying an external electric field (denoted by $\boldsymbol{E}$ in the following) to the material, some of the electric charge will move along the direction of the electric field and in this case, electric current occurs. This part of electric charge is called the *free charge*. An example is the conductor material, where the free charge has the dominating effect in the material; However, there is also a part of electric charge which can only limitedly move under the application of an external electric field. Such part of electric charge is called the *bound charge*. An example is the insulator material, where the bound charge dominates the free charge and electric current can barely happen by an existing external electric field. Physically, the bound charge is described by a vector-valued physical variable $\boldsymbol{P}$, whose direction is the direction pointing from the negative bound charge to the positive bound charge, and whose absolute value stands for the electric dipole moment density. The letter $\boldsymbol{P}$ stands for the word "polarization", which is due to the fact that the direction of the bound charge is tending to the same direction when increasing the strength of the external electric field and the procedure can be seen as kind of polarization progress.

While in most dielectric materials, the polarization is a linear function of the electric field, ferroelectric materials are those dielectric materials which show a nonlinear evolutionary behavior. In particular, if one starts with zero initial polarization, increases the electric field to saturation state and then decreases it to zero, the polarization will not reduce to zero but equal a remaining nonzero part. In order to cancel out the remaining polarization, one has to increase the strength of the external electric field, but with an opposite direction as shown in Fig. 1.1. Therefore, the polarization will also depend on the historical state of the electric field and such memory effect forms the so called ferroelectric hysteresis loop.

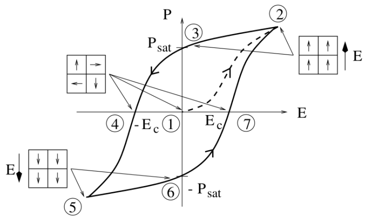

Figure 1.1: Polarization versus electric field [35].

On the other hand, piezoelectric effect is also obtained in ferroelectric materials. That is, the mechanical potential and the electric potential can be mutually transformed to each other. We use the Fig. 1.2 to give an explanation of the piezoelectricity. As we see, electric charge is distributed on the piezoelectric body (the plates) as shown in Fig. 1.2. Therefore, displacement or deformation of the piezoelectric body (for instance increasing the distance between the surface plates, which are interpreted by the bold arrows drawn in Fig. 1.2) will cause the occurrence of electric dipole moments, and hence the applied electric field can be strengthened or weakened. Conversely, if an external electric field is applied, the piezoelectric body can be compressed or stretched by the electric force induced by the electric charge (for instance the forces indicated by the blue arrows given in Fig. 1.2).

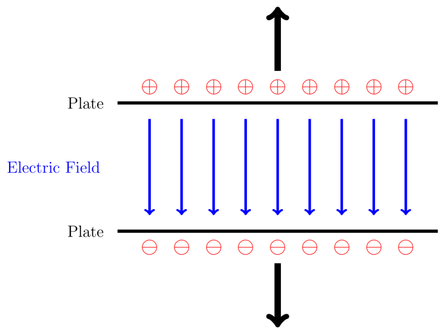

Figure 1.2: Piezoelectric plate separation by applying an external electric field.

Due to the mentioned properties, ferroelectric materials are widely used in capacitors, sensors and other products with memory effects. For a more detailed introduction of applications of ferroelectric materials, we refer to [35, 41, 66] and the references therein.

The following methods are mainly used for ferroelectric study:

1. Models with constitutive laws based on the second law of thermodynamics, which is considered from a macroscopic viewpoint. See [57, 61].

2. Models which consider single ferroelectric grains, which is a model from a microscopic viewpoint. See [4, 31].

3. Since ferroelectric materials show a nonlinear evolutionary behavior, which corresponds to certain hysteresis looping curve, there are also models concerning the so called Preisach operator, which is a basic tool for studying materials with hysteresis behavior. See [5, 54].

Our model is based on the phase-field model given by [55], which is derived from the so called Ginzburg-Landau-Devonshire-theory (see [43] and the references therein). In this model, the domain switching effect is also taken into account (for a precise introduction of the domain wall switching effect, we refer to Fig. 2.1 given in Section 2.4 below), which plays an important role in the phase-field theory. Roughly speaking, in order to describe such effect mathematically, certain gradient energy term is added to the model, which is to indicate the interfacial energy between domain walls that separate polarization vectors with different directions. Such modification makes the model here significantly independent to the above mentioned ones and gives a more accurate description of the physical principles about the ferroelectric materials.

1.1 A brief introduction of the mathematical setting and methodologies

In Section 2.4, we will give a precise derivation of the main model studied in this thesis, after we have introduced some necessary notation and definitions at the beginning of Chapter 2. Nevertheless, we give a first description of the mathematical setting of the main model for introductory purpose. The model involves three variables: the mechanical displacement $\boldsymbol{u}$, the electric field ϕ and the polarization $\boldsymbol{P}$. The former two variables are given by an elliptic piezo-system, with coefficients which are functions in variable $\boldsymbol{P}$ (see (2.14) below). Thus roughly speaking, once $\boldsymbol{P}$ is given, the variable $(\boldsymbol{u}, \phi)$ can be uniquely determined by certain elliptic existence theory. Hence it suffices to find a solution $\boldsymbol{P}$ which fulfills the evolutionary law

$$\partial\Psi(\boldsymbol{P}') \ni -D_{\boldsymbol{P}}\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) \tag{1.1}$$

(given as (2.14e) below), where $-D_{\boldsymbol{P}}\mathcal{H}$ is the system entropy corrsponding to $\boldsymbol{P}$ and Ψ is the dissipation functional (for details see Section 2.4). In recent papers [47, 55], the dissipation functional Ψ is assumed to be of polynomial growth. For the very special case that Ψ is taken to be the absolute value functional (or L^1-norm in variational expression), the evolutionary law (1.1) is rate-independent, and existence result for the rate-independent ferroelectric model, or more precisely, the existence of a so called *energetic solution*, is already given in the seminal paper [50]. However, we are not able to directly apply the result given in [50] to our model, due to the following reasons:

- We point out that the functional $\mathcal{H}$ given in (1.1) is in general not convex in $\boldsymbol{P}$ by our model. Such nonconvexity is troublesome when we apply the energetic solution ansatz given in [50]. More precisely, it is well known that the nonconvexity of the energy functional in variable $\boldsymbol{P}$ will cause non physically reasonable occurrence of jumps of the energetic solutions before critical times. We refer to [48] for a survey of this phenomenon.

- Actual physical experiments [62] show that in general, the dissipation functional can only be assumed to be rate-independent if the external loadings have a relatively low frequency. For some materials, the rate-independence of the dissipation functional can not hold even if the external loadings have low frequency [60].

The above mentioned reasons suggest a study with a dissipation functional Ψ of mixed type, which admits the expression

$$\Psi = \Psi_1 + \beta\Psi_2, \tag{1.2}$$

where Ψ_1 is the rate-independent part and Ψ_2 is the rate-dependent part of the dissipation functional Ψ (in this thesis, Ψ_2 is assumed to be of polynomial growth of order 2, that is, a quadratic functional); β here is a positive constant, and scaling β gives us the opportunity to control the rate-dependency of the dissipation functional on $\boldsymbol{P}$, which makes it possible to apply our model to either (approximately) rate-independent or rate-dependent systems. On the other hand, as stated in [48], the rate-dependent part Ψ_2 can be seen as kind of viscosity, which delays the occurrence of jumps (that occur in the limiting case $\beta \to 0$, see Section 3.8) and gives a more physically accurate description of the jumping points of the solution.

We mainly follow the lines of [39] to obtain existence results for the model with dissipation functional of mixed type, namely, for every given fixed positive β, we utilize the viscosity method given in [39] to obtain a global viscous solution $\boldsymbol{P}_\beta$ of the evolutionary law (1.1). More precisely, we construct time discrete interpolant solutions which satisfy a discrete evolutionary law related to (1.1), and then show that such time discrete interpolant solutions converge weakly to the actual solution of the law (1.1) by using certain Rothe's approximation process.

We point out that the main difficulty to apply the vanishing viscosity method to our model is the insufficient regularity of the solution $(\boldsymbol{u}, \phi)$ given by the piezo-system. More precisely, by using Lax-Milgram we can only expect that the solution $(\boldsymbol{u}, \phi)$ is of class H^1 in general. However, coupled nonlinearities appearing in the functional $D_{\boldsymbol{P}}\mathcal{H}$ will have similar expression as the term $|\boldsymbol{\varepsilon}(\boldsymbol{u})|^2|\boldsymbol{P}|$, where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ is the small strain tensor generated by $\boldsymbol{u}$. Thus we need to estimate a product of Lebesgue functions of the form $L^2 \times L^2 \times L^p$, and it is clear that no Hölder's inequality is applicable to estimate such product unless $p = \infty$, which is not the case in our model.

To overcome such difficulties, we apply the regularity results given in [28] (Proposition 3.18 below) for elliptic systems to obtain higher integrability of the solution $(\boldsymbol{u}, \phi)$ of the piezo-system. More precisely, assuming Gröger-type geometric conditions on the underlying domain $\boldsymbol{\Omega}$ (see Section 2.1) and uniform boundedness of the coefficients of the elliptic piezo-system, we are able to infer that the solution $(\boldsymbol{u}, \phi)$ is of class $W^{1,p}$ for some $p > 2$, assuming that the external loadings are of class $W^{-1,p}$. We also point out that mixed boundary conditions (Dirichlet-Neumann or purely Dirichlet) are allowed in this case, due to the geometric profile of the underlying domain that characterized by the Gröger-type geometric conditions, which is another surprising result obtained from our analysis. However, due to the lacking of certain Sobolev's embedings in three dimensional space, we need to replace the gradient energy term $\|\nabla \boldsymbol{P}\|_{L^2}^2$ by a fractional one for three dimensional case, which is in order to guarantee that each $\boldsymbol{P}$ in the underlying space can be embedded to L^p space for all $p \in [1, \infty)$, see Section 3.2 for details.

At the end, we will also push β to zero, with help of the arclength parametrization and Γ-convergence theory given in [49], to investigate the limiting behavior of the viscous model. It turns out that the arclength parameterized viscous solution $(\tilde{t}_\beta, \tilde{\boldsymbol{P}}_\beta)$ (see Section 3.8) converges (within $W^{1,\infty}$-weak-$*$ topology) to some vanishing viscosity limit function $(\tilde{t}, \tilde{\boldsymbol{P}})$ as $\beta \to 0$, which has certain physical interpretation in terms of rate-independent content, see Section 3.9 for details. We postpone the details of the precise strategy for proving the existence results of the main model to Section 2.4.1, since some necessary definitions and mathematical notation, which are given in Chapter 2, have still to be imposed.

On the other hand, we are also interested in the model with purely quadratic dissipation functional, which is the one studied in [55]. To be more precise, the dissipation functional Ψ is assumed to be equal to

$$\Psi(\boldsymbol{P}) = \frac{\beta}{2}|\boldsymbol{P}|^2$$

in local form, or

$$\Psi(\boldsymbol{P}) = \frac{\beta}{2}\|\boldsymbol{P}\|_{L^2}^2$$

in variational expression, where β is in this case a fixed positive constant. Then, since Ψ is differentiable in $\boldsymbol{P}$, the evolutionary law (1.1) reduces to a semilinear parabolic equation

$$\beta \boldsymbol{P}' = \kappa\Delta \boldsymbol{P} + S(t, \boldsymbol{u}, \phi, \boldsymbol{P}), \tag{1.3}$$

where κ is some positive constant and S is some nonlinear functional depending on $(t, \boldsymbol{u}, \phi, \boldsymbol{P})$, see Section 4.1 for details. Again, since $(\boldsymbol{u}, \phi)$ can be uniquely determined by a given t and $\boldsymbol{P}$, we can reduce our problem to finding a solution $\boldsymbol{P}$ of the equation (1.3). Our strategy is to utilize the fixed point theorem given in [10] (stated as Theorem 4.3 in the following) to obtain local solutions of (1.3). The goal is then to show that the contraction Assumption (S) in Theorem 4.3 is satisfied (where the Assumption (A) in Theorem 4.3 is evident by our case).

For two dimensional case, the verification of Assumption (S) relies on a direct Hölder type estimation of the difference of $S(t, \boldsymbol{P}_1)$ and $S(t, \boldsymbol{P}_2)$ in Assumption (S) for test functions $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, which is inspired by the analysis for the model with dissipation functional of mixed type. Thus Gröger-type geometric conditions and uniform boundedness of the coefficients will be imposed. In this case, mixed boundary conditions are also allowed.

For three dimensional case, the elliptic regularity result Proposition 3.18, which plays the main role in the analysis for two dimensional case, is no more applicable, since the integrability of $(\boldsymbol{u}, \phi)$ obtained from Proposition 3.18 is only expected to be greater than 2, and the verification of Assumption (S) will require an integrability of $(\boldsymbol{u}, \phi)$ larger than 3, namely greater than the number of space dimension. Instead of assuming mixed boundary type conditions, if we restrict ourselves to the case that the piezo-system admits overall Dirichlet boundary conditions, then we are able to obtain that the integrability of $(\boldsymbol{u}, \phi)$ is greater than 3, assuming that the boundary of the domain is sufficiently smooth. More precisely, a greater than 3 integrability of $(\boldsymbol{u}, \phi)$ can be obtained by assuming that the boundary of the domain is of class C^1 [14] or the domain is a cuboid [1].

However, to show that Assumption (S) of Theorem 4.3 is satisfied for three dimensional case, a direct difference comparison of $S(t, \boldsymbol{P}_1)$ and $S(t, \boldsymbol{P}_2)$ as done for the two dimensional case is not straightforwardly applicable, since the inverse norm of the $W_0^{1,p}$-$W^{-1,p}$ isomorphism given by the piezo-system is in general no more uniform for all test functions $\boldsymbol{P}$ by the application of the regularity results from [14] and [1]. We will utilize certain continuity arguments inspired by [46] to solve this problem. In particular, no uniform boundedness condition of the coefficients of the elliptic piezo-system is required, which is another relaxation of conditions compared to the two dimensional case.

At the end, we will extend our local existence result to polyhedral domains, with help of the elliptic regularity results given in [44] and assuming, in brief words, that the Dirichlet boundary data is zero near the geometric singularities (edges and vertices) of the polyhedral domain. See Assumption 4.51 for details.

On the other hand, by taking Ψ_1 equal to zero in the mixed type case, we are able to obtain a global solution of the model with quadratic dissipation functional, by using the similar Rothe's approximation method as the one for the model with mixed type dissipation. In this case, uniform boundedness of the coefficients and replacement of the gradient energy (in three dimensional space) will have to be imposed.

1.2 Outline of the thesis

In Chapter 2, we first introduce the mathematical notation appearing in this thesis, then we give a precise derivation of the main model and a detailed explanation of the methodologies for obtaining the main results of the thesis. In Chapter 3 we will deal with the model with dissipation functional of mixed type. In Chapter 4 we will deal with the model with quadratic dissipation functional. At the end, we will give a summary of the main results in Chapter 5.

Chapter 2

Notation, definitions and basic physics

2.1 Domain and boundary characterization

Domain with Lipschitz boundary

Let $\boldsymbol{\Omega} \subset \mathbb{R}^d$ with $d \geq 2$ be a bounded domain. Then $\boldsymbol{\Omega}$ is said to be a domain with Lipschitz boundary, if for every point $\boldsymbol{x} \in \partial\boldsymbol{\Omega}$ there exists a neighborhood $\boldsymbol{U} \subset \mathbb{R}^d$ of $\boldsymbol{x}$, a new coordinate system $\{\boldsymbol{y}_1, ..., \boldsymbol{y}_d\}$ and a real positive number $\alpha > 0$ such that

- $\boldsymbol{U}$ is an open cube which can be interpreted by the new coordinates

$$\boldsymbol{U} = \{(\boldsymbol{y}_1, ..., \boldsymbol{y}_d) : \boldsymbol{y}_i \in (-\alpha, \alpha), i = 1, ..., d\}.$$

- There exists a Lipschitz function $a : \boldsymbol{U}' \to \mathbb{R}$, where

$$\boldsymbol{U}' = \{\boldsymbol{y}' = (\boldsymbol{y}_1, ..., \boldsymbol{y}_{d-1}) : \boldsymbol{y}_i \in (-\alpha, \alpha), i = 1, ..., d-1\},$$

  such that

$$\begin{aligned} |a(\boldsymbol{y}')| &\leq \frac{\alpha}{2} \text{ for all } \boldsymbol{y}' \in \boldsymbol{U}', \\ \boldsymbol{\Omega} \cap \boldsymbol{U} &= \{\boldsymbol{y} \in \boldsymbol{U} : \boldsymbol{y}_d < a(\boldsymbol{y}')\}, \\ \partial\boldsymbol{\Omega} \cap \boldsymbol{U} &= \{\boldsymbol{y} \in \boldsymbol{U} : \boldsymbol{y}_d = a(\boldsymbol{y}')\}. \end{aligned}$$

G1-regular and G2-regular sets

We introduce the concept of the so called G1-regular and G2-regular sets, which are originally given by Gröger [26] and are used to characterize the geometric profile of the underlying domain $\boldsymbol{\Omega}$ and its boundary $\partial\boldsymbol{\Omega}$: A set $\boldsymbol{W} \subset \mathbb{R}^d$ is called G1-regular, if the interior of $\boldsymbol{W}$ is a bounded domain and for every $\boldsymbol{x} \in \partial\boldsymbol{W}$, there exist $\boldsymbol{U}_1, \boldsymbol{U}_2 \subset \mathbb{R}^d$ and a bi-Lipschitz transformation $\Phi : \boldsymbol{U}_1 \to \boldsymbol{U}_2$, such that $\boldsymbol{x} \in \boldsymbol{U}_1$ and $\Phi(\boldsymbol{U}_1 \cap \boldsymbol{\Omega})$ is one of the following sets:

- $\boldsymbol{M}_1 := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| < 1, \boldsymbol{x}_d < 0\}$,
- $\boldsymbol{M}_2 := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| < 1, \boldsymbol{x}_d \leq 0\}$,
- $\boldsymbol{M}_3 := \{\boldsymbol{x} \in \boldsymbol{M}_2 : \boldsymbol{x}_d < 0 \text{ or } \boldsymbol{x}_1 > 0\}$.

If in addition the absolute value of the Jacobian of each local chart Φ is equal to one almost everywhere, then the set is called G2-regular.

Remark 2.1. In this thesis we will mainly deal with regular sets $\boldsymbol{W}$ with the expression $\boldsymbol{W} = \boldsymbol{\Omega} \cup \boldsymbol{\Gamma}$, where $\boldsymbol{\Omega}$ is the underlying bounded domain and $\boldsymbol{\Gamma}$ is a closed subset of $\partial\boldsymbol{\Omega}$. △

l-sets

The following concept of an l-set is originally given by [33, Def. 1.1], which characterizes the sufficient conditions for certain Sobolev extension theorem from partial boundary part $\boldsymbol{\Gamma} \subset \partial\boldsymbol{\Omega}$ to the whole space $\mathbb{R}^d$ (see Lemma 3.8 below): Let l be a real number with $0 < l \leq d$. Let $\boldsymbol{M} \subset \mathbb{R}^d$ be closed and ρ be the restriction of the l-dimensional Hausdorff measure $\mathcal{H}_l$ to $\boldsymbol{M}$. Then $\boldsymbol{M}$ is called a l-set, if there exist two positive constants c_1, c_2 that satisfy

$$\forall \boldsymbol{x} \in \boldsymbol{M}, r \in (0,1) : c_1 r^l \leq \rho\big(B(\boldsymbol{x}, r) \cap \boldsymbol{M}\big) \leq c_2 r^l,$$

where $B(\boldsymbol{x}, r)$ is the ball with center $\boldsymbol{x}$ and radius r in $\mathbb{R}^d$.

2.2 Function spaces and subdifferential

Sobolev and Sobolev-Slobodeckij spaces

Let $\boldsymbol{\Omega} \subset \mathbb{R}^d$ be an open set. The Sobolev space $W^{s,p}(\boldsymbol{\Omega})$ with non negative integer s and real number $p \in [1, \infty]$ is defined by

$$W^{s,p}(\boldsymbol{\Omega}) := \{f \in L^p(\boldsymbol{\Omega}) : D^{\boldsymbol{k}} f \in L^p(\boldsymbol{\Omega}) \text{ for } |\boldsymbol{k}| \leq s\}$$

with the norm

$$\|f\|_{W^{s,p}(\boldsymbol{\Omega})} := \sum_{|\boldsymbol{k}| \leq s} \|D^{\boldsymbol{k}} f\|_{L^p(\boldsymbol{\Omega})},$$

where $D^{\boldsymbol{k}}$ is the usual multi-differential symbol for a non negative integer multi-index $\boldsymbol{k}$. For noninteger $s \in (0, \infty) \setminus \mathbb{N}$ and $p \in [1, \infty)$, the Sobolev-Slobodeckij space $W^{s,p}(\boldsymbol{\Omega})$ is defined by

$$W^{s,p}(\boldsymbol{\Omega}) := \{f \in W^{\lfloor s \rfloor, p}(\boldsymbol{\Omega}) : \|f\|_{W^{s,p}(\boldsymbol{\Omega})} < \infty\},$$

where the norm $\|\cdot\|_{W^{s,p}(\boldsymbol{\Omega})}$ is defined by

$$\|f\|_{W^{s,p}(\boldsymbol{\Omega})} := \|f\|_{W^{\lfloor s \rfloor, p}(\boldsymbol{\Omega})} + \sum_{|\boldsymbol{k}| = \lfloor s \rfloor} \int_{\boldsymbol{\Omega} \times \boldsymbol{\Omega}} \frac{|D^{\boldsymbol{k}} f(\boldsymbol{x}) - D^{\boldsymbol{k}} f(\boldsymbol{y})|^p}{|\boldsymbol{x} - \boldsymbol{y}|^{d + p(s - \lfloor s \rfloor)}} d\boldsymbol{x} d\boldsymbol{y}$$

and $\lfloor \cdot \rfloor$ is the Gaußinteger function. For a closed set $\boldsymbol{\Gamma} \subset \partial\boldsymbol{\Omega}$, the space $W^{1,p}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ with $p \in (1, \infty)$ is defined as the closure of the space

$$\{\phi|_{\boldsymbol{\Omega}} : \phi \in C_0^\infty(\mathbb{R}^d), \operatorname{supp}(\phi) \cap \boldsymbol{\Gamma} = \emptyset\}$$

in the space $W^{1,p}(\boldsymbol{\Omega})$ w.r.t. the $W^{1,p}$-norm. The dual space of $W^{1,p}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ is denoted by $W^{-1,p'}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$, where p' is the Hölder conjugate of p with $\frac{1}{p} + \frac{1}{p'} = 1$. We also write

$$H^s(\boldsymbol{\Omega}) = W^{s,2}(\boldsymbol{\Omega}) \text{ and } H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}) = W^{1,2}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}).$$

The dual space of $H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ is denoted by $H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$. For $\boldsymbol{\Gamma} = \partial\boldsymbol{\Omega}$, $W^{1,p}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ and $H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ are denoted by $W^{1,p}_0(\boldsymbol{\Omega})$ and $H^1_0(\boldsymbol{\Omega})$ and their dual spaces are denoted by $W^{-1,p'}(\boldsymbol{\Omega})$ and $H^{-1}(\boldsymbol{\Omega})$ respectively. For most cases we will also neglect the symbol $\boldsymbol{\Omega}$ in the norm index for notational convenience, for instance $\|\cdot\|_{L^2(\boldsymbol{\Omega})} = \|\cdot\|_{L^2}$. For $s \in [1,2)$ and $\boldsymbol{P}, \bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, the bilinear form $\langle \boldsymbol{P}, \bar{\boldsymbol{P}}\rangle_s$ is defined by

$$\langle \boldsymbol{P}, \bar{\boldsymbol{P}}\rangle_s = \begin{cases} \int_{\boldsymbol{\Omega}} \nabla \boldsymbol{P} : \nabla \bar{\boldsymbol{P}} d\boldsymbol{x}, & \text{if } s = 1; \\ \int_{\boldsymbol{\Omega}\times\boldsymbol{\Omega}} \frac{\big(\nabla \boldsymbol{P}(x) - \nabla \boldsymbol{P}(y)\big):\big(\nabla \bar{\boldsymbol{P}}(x) - \nabla \bar{\boldsymbol{P}}(y)\big)}{|x-y|^{d+2(s-\lfloor s\rfloor)}} d\boldsymbol{x}d\boldsymbol{y}, & \text{if } s \neq 1. \end{cases}$$

Here, for two matrices $\boldsymbol{M}, \boldsymbol{N} \in \mathbb{R}^{m\times n}$, the symbol $\boldsymbol{M} : \boldsymbol{N}$ is defined by

$$\boldsymbol{M} : \boldsymbol{N} := \sum_{i=1}^{m}\sum_{j=1}^{n} \boldsymbol{M}_{ij}\boldsymbol{N}_{ij}.$$

Sobolev trace space

Let $\boldsymbol{\Omega}$ be a bounded domain with Lipschitz boundary. Then due to compactness of $\partial\boldsymbol{\Omega}$, one can find a finite collection of local coordinate systems $(\boldsymbol{U}_r, a_r)_{r=1}^m$, given by the definition of domain with Lipschitz boundary, such that the local coordinate system covers $\partial\boldsymbol{\Omega}$. The Sobolev trace space $W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega})$ for $p \in (1,\infty)$ is defined as the space of measurable functions $f : \partial\boldsymbol{\Omega} \to \mathbb{R}$ such that

$$\|f\|_{W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega})} := \sum_{r=1}^{m} \|f\big(\cdot, a_r(\cdot)\big)\|_{W^{1-\frac{1}{p},p}\left(a_r^{-1}\left(a_r(\boldsymbol{U}_r)\cap\partial\boldsymbol{\Omega}\right)\right)} < \infty.$$

Particularly, $\|\cdot\|_{W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega})}$ defines a norm on the space $W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega})$ and $W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega})$ endowed with the norm $\|\cdot\|_{W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega})}$ is a Banach space. Moreover, the trace function tr defined by

$$\begin{aligned} \mathrm{tr} : W^{1,p}(\boldsymbol{\Omega}) &\to W^{1-\frac{1}{p},p}(\partial\boldsymbol{\Omega}), \\ f|_{\boldsymbol{\Omega}} &\mapsto f|_{\partial\boldsymbol{\Omega}} \end{aligned}$$

is well-defined (which is understood as the unique linear extension of tr defined on $C^\infty(\overline{\boldsymbol{\Omega}})$ to $W^{1,p}(\boldsymbol{\Omega})$) and surjective. For a proof of the mentioned properties related to the Sobolev trace space and the trace operator, we refer to [51, Chap. 2].

Besov spaces

For a $(d-1)$-set $\boldsymbol{\Gamma} \subset \mathbb{R}^d$ and $0 < \alpha < \infty, 1 \leq p, q \leq \infty$, the symbol $B^{p,q}_\alpha(\boldsymbol{\Gamma})$ denotes the Besov space with components α, p, q defined on the set $\boldsymbol{\Gamma}$. The Besov spaces $B^{p,q}_\alpha(\boldsymbol{\Gamma})$ are seen as the trace spaces of Besov spaces $B^{p,q}_\alpha(\mathbb{R}^{\boldsymbol{d}})$ [34, Chap. V, 2.2], which can also be defined as interpolation spaces of Lebesgue spaces and Sobolev spaces [2, Chap. 7] or can be defined as the set of functions whose difference quotients are integrable of certain orders [34, page 7]. In this thesis we use the definition of Besov space defined on $(d-1)$-set from [34, Chap. V, 2.2]. Due to the complicated construction of Besov spaces defined on submanifolds and its limited application in the thesis, we refer to [34, Chap. V, 2.2] for the detailed definition of the Besov spaces.

2.3 Some more preliminary notation

Vector- and Gâteaux-derivatives

For a function $\boldsymbol{f} : \boldsymbol{\Lambda} \subset \mathbb{R}^m \to \mathbb{R}^n$, where $\boldsymbol{\Lambda}$ is an open subset of $\mathbb{R}^m$, the symbols $D_{\boldsymbol{x}}\boldsymbol{f}(\cdot)$, $\partial_{\boldsymbol{x}}\boldsymbol{f}(\cdot)$ or $\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\cdot)$ denote the usual derivative of $\boldsymbol{f}$ in the Euclidian space; for Banach spaces X, Y and function $f : X \to Y$, the symbol $D_{\boldsymbol{x}}f(\cdot)[\bar{\boldsymbol{x}}]$ denotes the Gâteaux-differential of f in direction $\bar{\boldsymbol{x}} \in X$.

If $\boldsymbol{\Lambda}$ is an open subset of $\mathbb{R}$, then the derivative of $\boldsymbol{f}$ will also be denoted by $\boldsymbol{f}'$, which stands for the time derivative of $\boldsymbol{f}$.

Vector divergence and divergence operator

For a vector function $\boldsymbol{v} : \boldsymbol{\Omega} \subset \mathbb{R}^d \to \mathbb{R}^d$, its divergence div$\boldsymbol{v}$ is defined by

$$\mathrm{div}\boldsymbol{v} := \sum_{i=1}^{d} \partial_i \boldsymbol{v}_i.$$

For $\boldsymbol{f} \in \left(L^p(\boldsymbol{\Omega})\right)^d$, $p \in (1, \infty)$, the operator $-\mathrm{Div} : \left(L^p(\boldsymbol{\Omega})\right)^d \to \left(W^{1,p'}(\boldsymbol{\Omega})\right)^*$ is defined by

$$-\mathrm{Div}\boldsymbol{f}[g] := \int_{\boldsymbol{\Omega}} \boldsymbol{f} \cdot \nabla g d\boldsymbol{x}$$

for all $g \in W^{1,p'}(\boldsymbol{\Omega})$.

Linear mappings between finite dimensional spaces

For finite dimensional spaces V and W over the real field, we denote by $\mathrm{Lin}(V, W)$ the space of all linear mappings from V to W. If $\dim(V) = \dim(W) = d$, then $\mathrm{Lin}_{\mathrm{sym}}(V, W)$ denotes the space of all linear mappings from V to W whose matrix representation w.r.t. the standard ordered basis in $\mathbb{R}^d$ is symmetric.

Linear and continuous functions between Banach spaces

For Banach spaces X and Y, we denote by $L(X, Y)$ the space of all functions $f : X \to Y$ which are linear and continuous. If $X = Y$, then we define $L(X) := L(X, X)$. We also denote by $LH(X, Y)$ the subset of $L(X, Y)$ whose elements are additionally bijective.

Remark 2.2. Note that the set $LH(X, Y)$ is not a subspace, since for $f \in LH(X, Y)$, $\boldsymbol{0} = f + (-f)$ is not in $LH(X, Y)$. △

Remark 2.3. Using open mapping theorem we know that $f^{-1} \in LH(Y, X)$ for $f \in LH(X, Y)$, thus $LH(X, Y)$ is the collection of all linear homeomorphisms (for which the word "LH" stands) from X to Y. △

Subdifferential of a convex functional

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function on a Banach space X which allows infinite values. Then for $x^0 \in X$ the subdifferential $\partial f(x^0)$ is defined by

$$\partial f(x^0) := \{f^* \in X^* : f(x) \geq f(x^0) + \langle f^*, x - x^0 \rangle_X \ \forall x \in X\},$$

where X^* is the dual space of X and $\langle \cdot, \cdot \rangle_X$ denotes the dual product of X and X^*. An element in the subdifferential is called a subgradient. The convex function f is called positively 1-homogeneous if $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$ and $x \in X$.

Remark 2.4. Taking $x = \mathbf{0}$ and $\lambda = 2$ in the condition $f(\lambda x) = \lambda f(x)$ we immediately see that $f(\mathbf{0}) = 0$. △

2.4 Model derivation

In this section and in the rest of the thesis, we will make extensive use of the following physical quantities:

Time interval	$(0,T) \subset [0,\infty)$
Ferroelectric body	$\boldsymbol{\Omega} \subset \mathbb{R}^d,\ d \in \{2,3\}$
Mechanical displacement	$\boldsymbol{u} : (0,T) \times \boldsymbol{\Omega} \to \mathbb{R}^d$
Infinitesimal small strain tensor	$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right)$
Electric potential	$\phi : (0,T) \times \boldsymbol{\Omega} \to \mathbb{R}$
Electric field	$\boldsymbol{E} = \boldsymbol{E}(\phi) := -\nabla\phi$
Polarization	$\boldsymbol{P} : (0,T) \times \boldsymbol{\Omega} \to \mathbb{R}^d$
Cauchy stress tensor	$\boldsymbol{\sigma} : (0,T) \times \boldsymbol{\Omega} \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$
Dielectric displacement	$\boldsymbol{D} : (0,T) \times \boldsymbol{\Omega} \to \mathbb{R}^d$.
Elastic stiffness tensor	$\mathbb{C} : \mathbb{R}^d \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$ with $\mathbb{C}_{ij}^{kl} = \mathbb{C}_{ji}^{lk}$
Symmetric plastic strain tensor	$\boldsymbol{\varepsilon}^0 : \mathbb{R}^d \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$
Coupling effect tensor	$\boldsymbol{e} : \mathbb{R}^d \to \mathrm{Lin}(\mathbb{R}^{d\times d}, \mathbb{R}^d)$
Symmetric dielectric matrix	$\boldsymbol{\epsilon} : \mathbb{R}^d \to \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$

Remark 2.5. In this thesis, the tensors $\mathbb{C}$, $\boldsymbol{\varepsilon}^0$, $\boldsymbol{e}$, $\boldsymbol{\epsilon}$ are assumed to be dependent on the polarization $\boldsymbol{P}$. Since $\boldsymbol{P}$ takes value in $\mathbb{R}^d$, we will thus assume that these tensors are defined on the domain $\mathbb{R}^d$. Alternatively, these functions can also be assumed to be functions defined on the underlying domain $\boldsymbol{\Omega}$, which is the case given in the recent paper [38]. △

Based on [55], we introduce the main phase-field model from the following physics. The model describes the relationship between the mechanical displacement $\boldsymbol{u}$, the electric field $\boldsymbol{E}$ and the polarization $\boldsymbol{P}$. During the modeling we generally assume that all functionals are smooth and integrable, and limit and integration can always be interchanged.

Now we start to derive the precise model. Consider the problem on the time interval $(0,T)$ with some $T \in (0,\infty)$. Assuming quasistatic state, we obtain from the Cauchy's momentum equation and the Gauss' law given in the Maxwell's equations that

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{2.1a}$$

$$\operatorname{div}\boldsymbol{D} = f_2 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{2.1b}$$

where $\boldsymbol{f}_1$ is the mechanical volume force, f_2 is the free space charge. Our next step is to derive a precise expression formula for $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ in terms of $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$ and to derive an evolutionary law for the polarization $\boldsymbol{P}$. This will be done with help of the second law of thermodynamics. We first define the following physical quantities, which are necessary notation and definitions for a reasonable mathematical formulation of the evolutionary law of $\boldsymbol{P}$:

1. The symbol $\mathcal{G}$ denotes the free energy, which is thought as the energy part stored in the system and can be transformed to heat or dissipation. We adopt the explicit formula for $\mathcal{G}$ given by [55]:

$$\mathcal{G} = \mathcal{G}_1(\boldsymbol{u}, \phi, \boldsymbol{P}) + \mathcal{G}_2(\boldsymbol{P}),$$

which is a sum of two energy functionals. Here, the functional $\mathcal{G}_1$ corresponds to the free energy part related to the mechanical displacement and free electric charge and is defined by

$$\begin{aligned} &\mathcal{G}_1(\boldsymbol{u}, \phi, \boldsymbol{P}) \\ &:= \int_{\boldsymbol{\Omega}} G_1\big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{E}(\phi), \boldsymbol{P}\big) d\boldsymbol{x} \\ &:= \int_{\boldsymbol{\Omega}} \frac{1}{2}\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) : \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \Big(\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) \\ &\quad - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) + \frac{1}{2}\boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{E}(\phi) + \boldsymbol{P}\Big) \cdot \boldsymbol{E}(\phi) d\boldsymbol{x}, \end{aligned} \tag{2.2}$$

where $\mathbb{C}, \boldsymbol{\varepsilon}^0, \boldsymbol{e}, \boldsymbol{\epsilon}$ are assumed to be functions of the polarization $\boldsymbol{P} \in \mathbb{R}^d$; $\mathcal{G}_2$ corresponds to the free energy part w.r.t. the polarization $\boldsymbol{P}$. To model the energy part $\mathcal{G}_2$, we use the fact that phase transition occurs in ferroelectric materials. More precisely, when applying an external electric field to the ferroelectric body, the direction of the bound charge is aligned to the same direction of the electric field. Therefore we obtain the ferroelectric phase transition phenomenon as shown in Fig. 2.1.

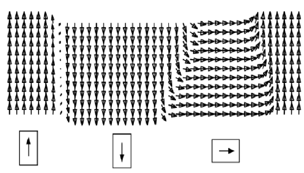

Figure 2.1: Phase transition phenomenon of the polarization [55].

A general tool for modeling phase transition phenomenon is the Ginzburg-Landau-Devonshire-theory (for ferroelectrics see for instance [43]), which suggests that the free energy is approximated by a polynomial ω (of some physically reasonable accurate order, which is chosen to be of order six here due to [55]) and the corresponding coefficients of ω can be calculated with help of the maximum dissipation principle. We refer to [56] for the precise calculation method for the polynomial coefficients. On the other hand, to include the interfacial effects between different phases, a gradient energy term is suggested to be added into the energy functional. Thus we assume that

$$\begin{aligned} \mathcal{G}_2(\boldsymbol{P}) &= \mathcal{G}_{2,GLD}(\boldsymbol{P}) + \mathcal{G}_{2,grad}(\boldsymbol{P}) \\ &=: \int_{\boldsymbol{\Omega}} \omega(\boldsymbol{P}) d\boldsymbol{x} + \int_{\boldsymbol{\Omega}} \frac{\kappa}{2} |\nabla \boldsymbol{P}|^2 d\boldsymbol{x}, \end{aligned} \tag{2.3}$$

where ω is a sixth order polynomial and κ is a positive constant.

2. The symbol $\mathcal{U}$ denotes the energy part which is done by the external loadings. Thus we have

$$\mathcal{U} = \int_{\partial\boldsymbol{\Omega}} \boldsymbol{f}_1^S \cdot \boldsymbol{u} + f_2^S \phi + \boldsymbol{\pi} \cdot \boldsymbol{P} d\boldsymbol{S} + \int_{\boldsymbol{\Omega}} \boldsymbol{f}_1 \cdot \boldsymbol{u} - f_2 \phi + \boldsymbol{f}_3 \cdot \boldsymbol{P} d\boldsymbol{x}, \tag{2.4}$$

where $\boldsymbol{f}_1^S, f_2^S$ are the mechanical force and free charge on the surface, $\boldsymbol{\pi}, \boldsymbol{f}_3$ are the surface and volume force related to $\boldsymbol{P}$.

3. The symbol $\mathcal{D}$ denotes the system dissipation.

We generally assume that the work done by the external loadings is transformed to two parts: one part is the dissipation, and another part is the energy part which is stored in the system and can be dissipated. Due to the definition, the second part is exactly the free energy part. Thus we obtain that

$$\mathcal{D} = \mathcal{U} - \mathcal{G}.$$

Now the second law of thermodynamics states that the dissipation is always increasing, which can be mathematically seen that its time derivative is always nonnegative. Therefore

$$\mathcal{D}' = \mathcal{U}' - \mathcal{G}' \geq 0.$$

Assuming quasistatic loadings, the derivatives of the external loadings can be neglected. Thus due to chain rule we infer that the variable $(\boldsymbol{u}, \phi, \boldsymbol{P})$ should satisfy the inequality

$$\begin{aligned} &\Big(\int_{\partial\boldsymbol{\Omega}} \boldsymbol{f}_1^S \cdot \boldsymbol{u}' + f_2^S \phi' + \boldsymbol{\pi} \cdot \boldsymbol{P}' d\boldsymbol{S} \\ &\quad + \int_{\boldsymbol{\Omega}} \boldsymbol{f}_1 \cdot \boldsymbol{u}' - f_2 \phi' + \boldsymbol{f}_3 \cdot \boldsymbol{P}' d\boldsymbol{x} \Big) - \frac{d}{dt} \mathcal{G}(\boldsymbol{u}, \phi, \boldsymbol{P}) \geq 0. \end{aligned} \tag{2.5}$$

Our goal is to find $(\boldsymbol{u}, \phi, \boldsymbol{P})$ such that (2.1) and (2.5) are satisfied.

In the following we will simplify the equations (2.1) and (2.5) to the differential system (2.14) given below by using appropriate balancing and constitutive ansatz, whose solution will automatically be a solution of (2.1) and (2.5). It turns out that the simplified model (2.14) exhibits a much clearer mathematical structure and hence it will be our main model in the rest of the thesis. Denote by $\boldsymbol{n}$ the unit outer normal vector. Assuming the force balance on boundary:

$$\boldsymbol{\sigma n} = \boldsymbol{f}_1^S, \quad \boldsymbol{D} \cdot \boldsymbol{n} = f_2^S \quad \text{on } (0, T) \times \partial\boldsymbol{\Omega},$$

we obtain using divergence theorem that

$$\begin{aligned} \int_{\boldsymbol{\Omega}} \boldsymbol{f}_1 \cdot \boldsymbol{u}' d\boldsymbol{x} &= \int_{\boldsymbol{\Omega}} - \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{u}' d\boldsymbol{x} \\ &= \int_{\boldsymbol{\Omega}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}') d\boldsymbol{x} - \int_{\partial\boldsymbol{\Omega}} \boldsymbol{\sigma n} \cdot \boldsymbol{u}' d\boldsymbol{S} \\ &= \int_{\boldsymbol{\Omega}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}') d\boldsymbol{x} - \int_{\partial\boldsymbol{\Omega}} \boldsymbol{f}_1^S \cdot \boldsymbol{u}' d\boldsymbol{S} \end{aligned}$$

or equivantly

$$\int_{\boldsymbol{\Omega}} \boldsymbol{f}_1 \cdot \boldsymbol{u}' d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}} \boldsymbol{f}_1^S \cdot \boldsymbol{u}' d\boldsymbol{S} = \int_{\boldsymbol{\Omega}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}') d\boldsymbol{x}. \tag{2.6}$$

Analogously we obtain that

$$-\int_{\boldsymbol{\Omega}} f_2 \phi' d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}} f_2^S \phi' d\boldsymbol{S} = \int_{\boldsymbol{\Omega}} \boldsymbol{D} \cdot \nabla \phi' d\boldsymbol{x} = -\int_{\boldsymbol{\Omega}} \boldsymbol{D} \cdot \boldsymbol{E}(\phi') d\boldsymbol{x}. \tag{2.7}$$

Now using chain rule we deduce that

$$\begin{aligned}&\frac{d}{dt}\mathcal{G}(\boldsymbol{u},\phi,\boldsymbol{P})\\&=\int_{\boldsymbol{\Omega}}\frac{\partial G_1}{\partial\boldsymbol{\varepsilon}}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{E}(\phi),\boldsymbol{P}\big):\boldsymbol{\varepsilon}(\boldsymbol{u}')+\frac{\partial G_1}{\partial\boldsymbol{E}}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{E}(\phi),\boldsymbol{P}\big)\cdot\boldsymbol{E}(\phi')d\boldsymbol{x}\\&\quad+D_{\boldsymbol{P}}\mathcal{G}(\boldsymbol{u},\phi,\boldsymbol{P})[\boldsymbol{P}'].\end{aligned}\tag{2.8}$$

Inserting (2.6), (2.7) and (2.8) into (2.5), we obtain that

$$\begin{aligned}&\int_{\boldsymbol{\Omega}}\Big(\boldsymbol{\sigma}-\frac{\partial G_1}{\partial\boldsymbol{\varepsilon}}\Big):\boldsymbol{\varepsilon}(\boldsymbol{u}')-\Big(\boldsymbol{D}+\frac{\partial G_1}{\partial\boldsymbol{E}}\Big)\cdot\boldsymbol{E}(\phi')d\boldsymbol{x}\\&\quad-\Big(D_{\boldsymbol{P}}\mathcal{G}(\boldsymbol{u},\phi,\boldsymbol{P})[\boldsymbol{P}']-\int_{\boldsymbol{\Omega}}\boldsymbol{f}_3(t)\cdot\boldsymbol{P}'d\boldsymbol{x}-\int_{\partial\boldsymbol{\Omega}}\boldsymbol{\pi}(t)\cdot\boldsymbol{P}'d\boldsymbol{S}\Big)\geq 0.\end{aligned}\tag{2.9}$$

A canonical ansatz (which is also a standard argument in rational continuum thermodynamics, see [55, (9)]) that makes (2.9) satisfied is to assume that

$$\boldsymbol{\sigma}=\frac{\partial G_1}{\partial\boldsymbol{\varepsilon}}=\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)+\boldsymbol{e}(\boldsymbol{P})^T\nabla\phi,\tag{2.10}$$

$$\boldsymbol{D}=-\frac{\partial G_1}{\partial\boldsymbol{E}}=\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)-\boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi+\boldsymbol{P}\tag{2.11}$$

and

$$-D_{\boldsymbol{P}}\mathcal{H}(t,\boldsymbol{u},\phi,\boldsymbol{P})[\boldsymbol{P}']\geq 0,\tag{2.12}$$

where

$$\mathcal{H}(t,\boldsymbol{u},\phi,\boldsymbol{P})=\mathcal{G}(\boldsymbol{u},\phi,\boldsymbol{P})-\int_{\boldsymbol{\Omega}}\boldsymbol{f}_3(t)\cdot\boldsymbol{P}d\boldsymbol{x}-\int_{\partial\boldsymbol{\Omega}}\boldsymbol{\pi}(t)\cdot\boldsymbol{P}d\boldsymbol{S}.$$

The functional $-D_{\boldsymbol{P}}\mathcal{H}$ evaluated at $\boldsymbol{P}'$ can be understood as the entropy part involving the polarization, and (2.12) reads that the entropy part concerning $\boldsymbol{P}$ is always increasing. In order to guarantee (2.12), it suffices to assume that there exists a convex and non negative dissipation functional Ψ such that $\Psi(\boldsymbol{0})=0$ and

$$-D_{\boldsymbol{P}}\mathcal{H}(t,\boldsymbol{u},\phi,\boldsymbol{P})\in\partial\Psi(\boldsymbol{P}').\tag{2.13}$$

Thus the variable $(\boldsymbol{u},\phi,\boldsymbol{P})$ will satisfy (2.1) and (2.5) if $(\boldsymbol{u},\phi,\boldsymbol{P})$ satisfies the following differential system:

$$\begin{aligned}&\boldsymbol{\sigma}=\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)+\boldsymbol{e}(\boldsymbol{P})^T\nabla\phi && \text{in }(0,T)\times\boldsymbol{\Omega}, && \text{(2.14a)}\\&\boldsymbol{D}=\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)-\boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi+\boldsymbol{P} && \text{in }(0,T)\times\boldsymbol{\Omega}, && \text{(2.14b)}\\&\operatorname{div}\boldsymbol{\sigma}=\boldsymbol{f}_1 && \text{in }(0,T)\times\boldsymbol{\Omega}, && \text{(2.14c)}\\&\operatorname{div}\boldsymbol{D}=f_2 && \text{in }(0,T)\times\boldsymbol{\Omega}, && \text{(2.14d)}\\&\partial\Psi(\boldsymbol{P}')\ni-D_{\boldsymbol{P}}\mathcal{H}(t,\boldsymbol{u},\phi,\boldsymbol{P}) && \text{in }(0,T). && \text{(2.14e)}\end{aligned}$$

In order to obtain physically and mathematically reasonable existence results, different boundary and initial conditions and dissipation functionals Ψ will be imposed for (2.14), which are formulated and discussed in the forthcoming content.

2.4.1 An overview of the main results for dissipation functionals of two different types

We discuss ferroelectric models with different dissipation functionals Ψ, where Ψ is given by the expression (1.2). Here, we focus on two types of dissipation functionals which are mostly studied in recent mathematical research: the dissipation functional of mixed type and the dissipation functional of quadratic growth.

Viscous and vanishing viscosity solutions

Most of the ferroelectric models are rate-independent. The reason is that most of these models utilize the so called Preisach operator to describe the ferroelectric hysteresis loop, and the Preisach operator is rate-independent, see for instance [8]. However, smart actuators and sensors are usually rate-dependent. The magnetostrictive actuators investigated in [62] can be considered rate-independent if the external loading has smaller than 5Hz frequency. For larger frequencies, the rate-independency is inappropriate any longer and one must consider the model as a rate-dependent one. In [60] it is pointed out that piezoceramics are rate-dependent even at low frequencies. These facts show that a reasonable candidate of Ψ should include both rate-independent and- dependent effects. Thus in this thesis, we analyse general dissipation functionals of the expression

$$\Psi(\boldsymbol{P}) = \Psi_1(\boldsymbol{P}) + \beta\Psi_2(\boldsymbol{P}), \tag{2.15}$$

where Ψ_1, Ψ_2 are non negative real valued convex functions defined on $\mathbb{R}^d$ with $\Psi_1(\mathbf{0}) = \Psi_2(\mathbf{0}) = 0$ and β is some positive constant. In particular, Ψ_1 is positively 1-homogeneous, i.e.

$$\forall\lambda > 0\ \forall\boldsymbol{P} \in \mathbb{R}^d : \Psi(\lambda\boldsymbol{P}) = \lambda\Psi(\boldsymbol{P})$$

and Ψ_1 corresponds to the rate-independent dissipation part. On the other hand, Ψ_2 corresponds to the rate-dependent part (namely it is not positively 1-homogeneous). In recent papers [47, 55], this part is assumed to be of polynomial growth. Among all rate-dependent dissipation functionals of polynomial growth, the quadratic one is of particular interest, which is the most command one appearing in engineering simulation. In this thesis we will thus assume that

$$\Psi_2(\boldsymbol{P}) = |\boldsymbol{P}|^2.$$

We see that one may scale the size of β to control the rate dependence. If $\beta = 0$, the model becomes fully rate-independent and the so called energetic solution for rate-independent models has been given in [50]. For $\beta > 0$, we see that solutions given by this ansatz are closely connected to the so called viscous solutions (see for instance [39] for an application of the viscous method for the damage model). We utilize the same strategy as in [39] to show the existence of viscous solutions. In what follows, we briefly outline the idea for showing the existence of viscous solutions:

- From the constitutive law (2.14) it is natural to formulate a weak expression for the piezo-system (assuming mixed boundary conditions on $(\boldsymbol{u}, \phi)$ for (2.14a) to (2.14d)) by multiplying (2.14c) and (2.14d) with test function $(\bar{\boldsymbol{u}}, \bar{\phi})$, which is given by (3.2a) below. In this case, the underlying spaces for $(\boldsymbol{u}, \phi, \boldsymbol{P})$ are the corresponding Sobolev spaces with certain orders. More precisely, we will be looking for solutions

$$(\boldsymbol{u}, \phi, \boldsymbol{P}) : (0, T) \to \left(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\right)^d \times H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega}) \times \left(H^1(\boldsymbol{\Omega})\right)^d$$

for the problem (3.2). Also, the dissipation potentials Ψ_1 and Ψ_2 are generalized to variational expressions, i.e., they are understood as functionals defined on the underlying space $\left(H^1(\boldsymbol{\Omega})\right)^d$ for $\boldsymbol{P}$ (more precisely, these are integrals of the ones given in (2.15) over $\boldsymbol{\Omega}$). On the other hand, in order to guarantee certain three dimensional Sobolev's compact embeddings from Sobolev spaces into Lebesgue spaces with sufficient integrability orders, the gradient energy has to be replaced by a fractional energy term by the case $d = 3$ (for details, see Section 3.2). In this case, the underlying space for $\boldsymbol{P}$ is replaced by $\left(H^s(\boldsymbol{\Omega})\right)^3$ for some $s \in [\frac{3}{2}, 2)$.

Remark 2.6. A general question is if such replacement of gradient energy is physically acceptable. So far, no experimental and numerical results are given for the ferroelectric model with a fractional gradient energy due to technical difficulties involving fractional calculus. But the study with such a fractional gradient energy might be of interest in the following sense: the gradient energy term $\|\nabla \boldsymbol{P}\|_{L^2}^2$ indicates that the interaction effect between different bound charge is irrelevant of their distance. However, an accurate physical description of the interaction between bound charge is that the interaction effect is inversely proportional to the distance of the bound charge. In order to indicate such inversely proportional effect, the gradient energy term should be replaced by a fractional term, which gives a better description of the interaction effect between different bound charge. △

- Next, we see that the coefficient tensor $\mathbb{B}_1$ of the elliptic problem (2.14a) to (2.14d) defined by (3.3) is not symmetric. This leads to the problem that when taking the test function equal to the solution of the piezo-system to obtain a variational formulation, the coupling terms of the variational problem will cancel out. In this case, the minimizer of the variational problem is an Euler-Lagrange critical point corresponding to a piezo-system without coupling effect. To solve this problem, we will use the Legendre transform to formulate an equivalent variational problem (3.20) involving the energy $\mathcal{E}$ given by (3.14), which takes the variable $(\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ with divergence free $\boldsymbol{D}$ of class L^2, such that, roughly speaking, the energy $\mathcal{E}$ has a symmetric coefficient tensor $\mathbb{B}_2$ (defined by (3.19)), which will guarantee the remanence of the coupling terms. Now the new formulated energy $\mathcal{E}$ will have a unique global minimizer $(\boldsymbol{u}, \boldsymbol{D})$ (since $\mathcal{E}$ is strict convex in $(\boldsymbol{u}, \boldsymbol{D})$) for each $\boldsymbol{P}$, and the minimization property of $(\boldsymbol{u}, \boldsymbol{D})$ will be essential for the regularity results given in Section 3.5.

- Since the global minimizer $(\boldsymbol{u}, \boldsymbol{D})$ is uniquely determined for each $\boldsymbol{P}$, one can formulate an equivalent variational problem (3.34) for (3.20), which involves a reduced energy functional $\mathcal{I}(t, \boldsymbol{P})$ having only the variables t and $\boldsymbol{P}$. In this case, we also point out that Proposition 3.18 guarantees that the solution $(\boldsymbol{u}, \phi)$ and the minimizer $(\boldsymbol{u}, \boldsymbol{D})$ are of class $W^{1,q} \times W^{1,q}$ and $W^{1,q} \times L^q$ respectively for some $q \in (2, \infty)$ if the domain is sufficiently regular (say, G1-regular) and the external loadings are sufficiently smooth (roughly speaking, they are of class $W^{-1,q}$), and this will ensure that the functionals $\mathcal{H}$ and $D_{\boldsymbol{P}}\mathcal{H}$ are well-defined by using certain Sobolev's embeddings and Hölder's inequalities.

- Now we concentrate on the reduced problem (3.34). Using Rothe's method one can construct a sequence of time discrete interpolant solutions $\hat{\boldsymbol{P}}_\tau, \bar{\boldsymbol{P}}_\tau, \underline{\boldsymbol{P}}_\tau$ with time finess $\tau > 0$, which satisfy a discrete evolutionary law (3.60) derived from (3.34). We will show that $\hat{\boldsymbol{P}}_\tau, \bar{\boldsymbol{P}}_\tau, \underline{\boldsymbol{P}}_\tau$ converge (up to subsequence) to some $\boldsymbol{P}_\beta$ (within certain weak topology and pointwise (in time) weak convergence due to the Helly's selection theorem, see the proof of Theorem 3.30 below for details) as $\tau \to 0$ such that

$$\boldsymbol{P}_\beta \in H^1(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d).$$

From the previously mentioned weak convergence of $\hat{\boldsymbol{P}}_\tau, \bar{\boldsymbol{P}}_\tau, \underline{\boldsymbol{P}}_\tau$ to $\boldsymbol{P}_\beta$, the convergence properties of functionals given by Corollary 3.24 below and the initial value condition that $D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)$ is of class L^2, we infer that $\boldsymbol{P}_\beta$ is a solution of (3.34). This completes the proof.

We are also interested in the behavior of $\boldsymbol{P}_\beta$ as β tends to zero. To achieve this, we define for a viscous solution $\boldsymbol{P}_\beta$ the quantity

$$s_\beta(t) := t + \int_0^T \|\boldsymbol{P}'_\beta(\sigma)\|_{H^s} d\sigma$$

and let $S_\beta := s_\beta(T)$. Define the arc length parametrization $\tilde{t}_\beta : [0,S_\beta] \to [0,T]$ and $\tilde{\boldsymbol{P}}_\beta : [0,S_\beta] \to \big(H^s(\boldsymbol{\Omega})\big)^d$ by

$$\begin{aligned} \tilde{t}_\beta(\sigma) &:= s_\beta^{-1}(\sigma), \\ \tilde{\boldsymbol{P}}_\beta(\sigma) &:= \boldsymbol{P}_\beta\big(\tilde{t}_\beta(\sigma)\big). \end{aligned}$$

One can show that $\{S_\beta\}_{\beta>0}$ is a bounded sequence (Lemma 3.33), thus up to a converging subsequence of $\{S_\beta\}_{\beta>0}$ we have $S_\beta \to S$ as $\beta \to 0$ for some $S \geq T$. Using the rescaling argument given in [16] we may w.l.o.g. consider the parameterized trajectories on the fixed time interval $[0,S]$. Particularly, $\tilde{t}_\beta$ and $\tilde{\boldsymbol{P}}_\beta$ satisfy the variational formulation (3.111). Using certain Γ-convergence theory given in [48], it is possible to push β to 0 and the arclength parametric solutions $(\tilde{t}_\beta, \tilde{\boldsymbol{P}}_\beta)$ will converge to some *vanishing viscosity solution* $(\tilde{t}, \tilde{\boldsymbol{P}})$ (up to subsequence and within weak-$*$-topology) such that

$$(\tilde{t}, \tilde{\boldsymbol{P}}) \in W^{1,\infty}(0,S;[0,T] \times \big(H^s(\boldsymbol{\Omega})\big)^d)$$

and $(\tilde{t}, \tilde{\boldsymbol{P}})$ satisfies a parametric limiting version (3.114) of (3.111). A vanishing viscosity solution admits certain physical interpretation (see Section 3.9). Since $\mathcal{H}$ is in general not convex in $\boldsymbol{P}$, the vanishing viscosity solution differs from the energetic solutions given in [50] at jumping points. See [48] for a survey of both solution concepts.

Dissipation functional with quadratic growth

On the other hand, we are also interested in the model given in [55], that is, Ψ_1 in (2.15) is constantly equal to zero. Thus we have (with a prefactor $\frac{\beta}{2}$ due to scaling convenience)

$$\Psi(\boldsymbol{P}) = \frac{\beta}{2}|\boldsymbol{P}|^2.$$

We will show local and global existence results. For local results, note that Ψ is now differentiable in $\boldsymbol{P}$, thus (2.14d) reduces to a semilinear parabolic equation

$$\beta \boldsymbol{P}' = \kappa\Delta\boldsymbol{P} + S(t,\boldsymbol{u},\phi,\boldsymbol{P}) \quad \text{in } (0,T)\times\boldsymbol{\Omega}.$$

Here, the functional $S(t,\boldsymbol{u},\phi,\boldsymbol{P})$ is defined by

$$S(t,\boldsymbol{u},\phi,\boldsymbol{P}) = -D_{\boldsymbol{P}}G_1\big(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{E}(\phi),\boldsymbol{P}\big) - D_{\boldsymbol{P}}\omega(\boldsymbol{P}) + \boldsymbol{f}_3(t),$$

where G_1, ω, $\boldsymbol{f}_3$ are the functionals given by (2.2), (2.3), (2.4) respectively. We will hence look for a solution $(\boldsymbol{u}, \phi, \boldsymbol{P})$ which satisfies the differential system

$$\boldsymbol{\sigma} = \mathbb{C}(\boldsymbol{P})\big(\varepsilon(\boldsymbol{u}) - \varepsilon^0(\boldsymbol{P})\big) + \boldsymbol{e}(\boldsymbol{P})^T\nabla\phi \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{2.16a}$$

$$\boldsymbol{D} = \boldsymbol{e}(\boldsymbol{P})\big(\varepsilon(\boldsymbol{u}) - \varepsilon^0(\boldsymbol{P})\big) - \boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi + \boldsymbol{P} \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{2.16b}$$

$$\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{2.16c}$$

$$\operatorname{div}\boldsymbol{D} = f_2 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{2.16d}$$

$$\beta\boldsymbol{P}' = \kappa\Delta\boldsymbol{P} + S(t,\boldsymbol{u},\phi,\boldsymbol{P}) \quad \text{in } (0,T)\times\boldsymbol{\Omega}. \tag{2.16e}$$

Note again that $(\boldsymbol{u}, \phi)$ can be uniquely determined by a given $\boldsymbol{P}$, with help of certain elliptic existence result. Thus we can reduce our problem to seeking a solution $\boldsymbol{P}$ of (2.16e). In this case, $S(t,\boldsymbol{u},\phi,\boldsymbol{P})$ can also be reduced to the functional $S(t,\boldsymbol{P})$ which depends only on t and $\boldsymbol{P}$. We utilize the idea given in [46], namely, we apply the fix point theorem from [10] (Theorem 4.3 below), which is based on the Banach fixed point theorem and the so called *maximal parabolic property* (see Definition 4.1), to obtain the existence of local solutions. We will show that for certain $p > d$ and $r > \frac{2p}{p-d}$ (depending on given assumptions), (2.16e) has a unique local weak solution

$$\boldsymbol{P} \in W^{1,r}(0,\hat{T}; \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \cap L^r(0,\hat{T}; \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d)$$

for some $\hat{T} \in (0,T]$, as long as the initial value $\boldsymbol{P}_0$ satisfies the condition

$$\boldsymbol{P}_0 \in (\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d, \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d)_{\frac{1}{r},r}.$$

In particular, $\boldsymbol{P}$ is also Hölder continuous w.r.t. time and space (see Proposition 4.13). Inserting $\boldsymbol{P}$ to the elliptic system (2.16a) to (2.16d) involving $\boldsymbol{u}$ and ϕ, we easily obtain the existence, uniqueness and regularity of $(\boldsymbol{u}, \phi)$ (different from case to case due to different imposed boundary conditions and given assumptions, thus we omit the details here due to the tedious classification and refer to Chapter 4 for details). In what follows, we give the precise methodologies for obtaining local existence results under different situations:

- We first consider the case that $d = 2$ and mixed boundary conditions are imposed for $(\boldsymbol{u}, \phi, \boldsymbol{P})$. Using Proposition 3.18, the elliptic system (2.16a) to (2.16d) has a unique weak solution $(\boldsymbol{u}, \phi)$ for each given t and $\boldsymbol{P}$, and the r.h.s. operator $S(t,\boldsymbol{u},\phi,\boldsymbol{P})$ of (2.16e) reduces to some functional $S(t,\boldsymbol{P})$ given by (4.24) which depends only on t and $\boldsymbol{P}$. Thus Theorem 4.3 is applicable to the equation

$$\beta\boldsymbol{P}' = \kappa\Delta\boldsymbol{P} + S(t,\boldsymbol{P}). \tag{2.17}$$

  In this case, it suffices to show that the Assumptions (A) and (S) of Theorem 4.3 are satisfied under the given assumptions stated in Section 4.2. The verification of the Assumption (A) is trivial by our case, thus only the fulfillment of the Assumption (S) in Theorem 4.3 has to be checked. However, it turns out that the Assumption (S) is in fact a statement of the Lipschitz continuity of $S(t,\boldsymbol{P})$ in the variable $\boldsymbol{P}$, thus it can be similarly proved as in the proofs of the regularity results given in Section 3.5. We also point out that in order to apply Proposition 3.18, the uniform boundedness of the coefficient tensor is necessary. However, the uniform boundedness of the derivatives of the coefficient tensor is not needed anymore as it was in the case of mixed type dissipation, since the underlying space for $\boldsymbol{P}$ is continuously embedded to some Hölder space with certain order (thus uniformly bounded on $\overline{\boldsymbol{\Omega}}$), while the underlying space for $\boldsymbol{P}$ of the mixed type dissipation case is only embedded to Lebesgue spaces with finite order.

- Now we consider the case $d = 3$. In this case, Proposition 3.18 is no more applicable: by imposing mixed boundary conditions on $(\boldsymbol{u}, \phi)$ and assuming uniform boundedness of the coeffcients, the solution $(\boldsymbol{u}, \phi)$ of the elliptic system (2.16a) to (2.16d) is expected to be of class $W^{1,q}$ for some $q > 2$ under the application of Proposition 3.18. However, the condition $q > d = 3$ is essential for our analysis, since from this condition we infer that the term $D_{\boldsymbol{P}}\tilde{H}$, which is the most complicated and nonregular summand in $S(t, \boldsymbol{P})$, is of class $W^{-1,q}$ and this will be necessary for the verification of the Assumption (S) of Theorem 4.3.

  If we restrict ourselves to the case that Dirichlet boundary conditions are imposed for the elliptic system (2.16a) to (2.16d) and the underlying domain $\boldsymbol{\Omega}$ has C^1-boundary, then we are able to infer the condition $q > d = 3$ by applying the regularity result from [14, Lem. 2]. In other words, for each time point t and each $\boldsymbol{P}$ which is uniformly continuous on $\overline{\boldsymbol{\Omega}}$, the weak formulation given by (2.16a) to (2.16d) defines an isomorphism from the space $W_0^{1,q}$ to $W^{-1,q}$ for all $q \in (1, \infty)$. In particular, unlike the two dimensional case with mixed boundary conditions, no uniform boundedness conditions of the coefficient tensors are needed for the application of [14, Lem. 2]. However, the inverse norm of the isomorphism from $W_0^{1,q}$ to $W^{-1,q}$ is in general no more uniform in $\boldsymbol{P}$ (which was the case by Proposition 3.18), thus the proof for two dimensional case can not be directly applied for the three dimensional case here. We will use the continuity arguments given by [46] to fix this problem.

- A natural question is if we can obtain similar results for less regular domains in three dimensional space, i.e., the boundary is less regular than C^1. We see that the $W_0^{1,q}$ to $W^{-1,q}$ isomorphism property of the elliptic system (2.16a) to (2.16d) for some $q > d$ is essential for applying the existence results from [10]. If we restrict to a cuboid, then the $W_0^{1,q}$ to $W^{-1,q}$ isomorphism property can be directly obtained by using the L^p-regularity results with $p > 3$ given in [1]; for general polyhedrons, we point out that the model given in [55] is closely related to Lamé operator and Laplace operator. Then the corresponding L^p-regularity results can be obtained from [44], which are based on some subtle spectral analysis near the geometric singularities of a polyhedron (namely the edges and vertices). However, unlike the case for a cuboid, since the regularity results of [44] rely on a self-adjoint elliptic operator theory, some additional Dirichlet boundary conditions on $\boldsymbol{P}$ have to be imposed. We refer to Section 4.4 for details.

We will also show global results: we apply similar Rothe's method as the one applied to the model with dissipation functional of mixed type to obtain global results. More precisely, it suffices to set $\Psi_1 = 0$ in the model with dissipation functional of mixed type and then to apply the previously constructed Rothe's approximation given in Chapter 3. Thus uniform boundedness of the coefficients and their derivatives and replacement of the gradient energy in 3D-case will still have to be imposed.

Chapter 3

Existence results for dissipation functional of mixed type

Throughout this chapter we assume that $d \in \{2,3\}$.

3.1 Main problem formulation

First recall from (2.14) that

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) + \boldsymbol{e}(\boldsymbol{P})^T \nabla\phi && \text{in } (0,T)\times\boldsymbol{\Omega}, \\ \boldsymbol{D} &= \boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi + \boldsymbol{P} && \text{in } (0,T)\times\boldsymbol{\Omega} \end{aligned}$$

and (with appropriate sign changes of the equations for calculation convenience)

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma} &= \boldsymbol{f}_1 && \text{in } (0,T)\times\boldsymbol{\Omega}, \\ -\operatorname{div}\boldsymbol{D} &= f_2 && \text{in } (0,T)\times\boldsymbol{\Omega}, \\ \boldsymbol{0} &\in \partial\Psi_\beta(\boldsymbol{P}') + D_{\boldsymbol{P}}\mathcal{H}(t,\boldsymbol{u},\phi,\boldsymbol{P}) && \text{in } (0,T). \end{aligned}$$

We also impose the following mixed boundary conditions

$$\begin{aligned} \boldsymbol{u}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}} &= \boldsymbol{u}_D && \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \\ \boldsymbol{\sigma}\boldsymbol{n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} &= \boldsymbol{t} && \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}, \\ \phi|_{\partial\boldsymbol{\Omega}_\phi} &= \phi_D && \text{on } (0,T)\times\partial\boldsymbol{\Omega}_\phi, \\ \boldsymbol{D}\cdot\boldsymbol{n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}} &= \rho && \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{D}} \end{aligned}$$

and initial value condition

$$\boldsymbol{P}(0) = \boldsymbol{P}_0,$$

where $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}, \partial\boldsymbol{\Omega}_\phi, \partial\boldsymbol{\Omega}_{\boldsymbol{D}}$ are subsets of $\partial\boldsymbol{\Omega}$. Assume also the existence of the functions $(\boldsymbol{u}_D, \phi_D) : (0,T)\times\boldsymbol{\Omega} \to \mathbb{R}^d \times \mathbb{R}$ with

$$\boldsymbol{u}_D|_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}} = \boldsymbol{u}_D \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \tag{3.1a}$$

$$\phi_D|_{\partial\boldsymbol{\Omega}_\phi} = \phi_D \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_\phi. \tag{3.1b}$$

Then multiplying (2.14a) to (2.14d) with test function $(\bar{\boldsymbol{u}}, \bar{\phi})$ and then integrating over $\boldsymbol{\Omega}$, we obtain the following weak formulation of the main problem:

Main Problem

Find $(\boldsymbol{u},\phi,\boldsymbol{P}):(0,T)\to\big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d\times H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})\times\big(H^1(\boldsymbol{\Omega})\big)^d$ such that

$$\int_{\boldsymbol{\Omega}}\mathbb{B}_1\big(\boldsymbol{P}(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\\ \nabla\phi(t)\end{pmatrix}:\begin{pmatrix}\bar{\boldsymbol{\varepsilon}}\\ \nabla\bar{\phi}\end{pmatrix}d\boldsymbol{x}=l_{t,\boldsymbol{P}(t)}(\bar{\boldsymbol{u}},\bar{\phi}), \tag{3.2a}$$

$$\boldsymbol{0}\in D_{\boldsymbol{P}}\mathcal{H}\big(t,\boldsymbol{u}(t),\phi(t),\boldsymbol{P}(t)\big)+\partial\Psi_\beta\big(\boldsymbol{P}'(t)\big), \tag{3.2b}$$

$$\boldsymbol{P}(0)=\boldsymbol{P}_0 \tag{3.2c}$$

for a.a. $t\in(0,T)$ and all $(\bar{\boldsymbol{u}},\bar{\phi})\in\big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d\times H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$, where $\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)$, $\boldsymbol{\varepsilon}_D(t)$ and $\bar{\boldsymbol{\varepsilon}}$ are the small strain tensors corresponding to $\boldsymbol{u}(t)$, $\boldsymbol{u}_D(t)$, $\bar{\boldsymbol{u}}$ and

$$\mathbb{B}_1(\boldsymbol{P})=\begin{pmatrix}\mathbb{C}(\boldsymbol{P}) & \boldsymbol{e}(\boldsymbol{P})^T\\ -\boldsymbol{e}(\boldsymbol{P}) & \boldsymbol{\epsilon}(\boldsymbol{P})\end{pmatrix}, \tag{3.3}$$

$$\begin{aligned}l_{t,\boldsymbol{P}}(\bar{\boldsymbol{u}},\bar{\phi})=&\int_{\boldsymbol{\Omega}}\boldsymbol{f}_1(t)\cdot\bar{\boldsymbol{u}}d\boldsymbol{x}+\int_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}}\boldsymbol{t}(t)\cdot\bar{\boldsymbol{u}}d\boldsymbol{S}-\Big(\int_{\boldsymbol{\Omega}}f_2(t)\bar{\phi}d\boldsymbol{x}+\int_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}}\rho(t)\bar{\phi}d\boldsymbol{S}\Big)\\&-\int_{\boldsymbol{\Omega}}\Big(\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)-\boldsymbol{e}^T(\boldsymbol{P})\nabla\phi_D(t)\Big):\bar{\boldsymbol{\varepsilon}}d\boldsymbol{x}\\&+\int_{\boldsymbol{\Omega}}\Big(\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)-\boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi_D(t)+\boldsymbol{P}\Big)\cdot\nabla\bar{\phi}d\boldsymbol{x},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathcal{H}(t,\boldsymbol{u},\phi,\boldsymbol{P})=&\int_{\boldsymbol{\Omega}}H\big(t,\varepsilon(\boldsymbol{u}),\nabla\phi,\boldsymbol{P}\big)+\omega(\boldsymbol{P})+\frac{\kappa}{2}|\nabla\boldsymbol{P}|^2d\boldsymbol{x}\\&-\Big(\int_{\boldsymbol{\Omega}}\boldsymbol{f}_3(t)\cdot\boldsymbol{P}d\boldsymbol{x}+\int_{\partial\boldsymbol{\Omega}}\boldsymbol{\pi}(t)\cdot\boldsymbol{P}d\boldsymbol{S}\Big),\end{aligned} \tag{3.5}$$

$$\Psi_\beta(\boldsymbol{P})=\Psi_1(\boldsymbol{P})+\frac{1}{2}\beta\|\boldsymbol{P}\|^2_{L^2}=:\Psi_1(\boldsymbol{P})+\Psi_{2,\beta}(\boldsymbol{P}) \tag{3.6}$$

with

$$\begin{aligned}H(t,\boldsymbol{\varepsilon},\nabla\phi,\boldsymbol{P})=&\frac{1}{2}\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big):\big(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)\\&+\Big(\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big)-\frac{1}{2}\boldsymbol{\epsilon}(\boldsymbol{P})\big(\nabla\phi+\nabla\phi_D(t)\big)+\boldsymbol{P}\Big)\cdot\big(\nabla\phi+\nabla\phi_D(t)\big),\end{aligned} \tag{3.7}$$

$$\omega(\boldsymbol{P})=\begin{cases}\psi_0+\psi_1(\boldsymbol{P}_1^2+\boldsymbol{P}_2^2)+\psi_2(\boldsymbol{P}_1^4+\boldsymbol{P}_2^4)+\psi_3\boldsymbol{P}_1^2\boldsymbol{P}_2^2+\psi_4(\boldsymbol{P}_1^6+\boldsymbol{P}_2^6), & \text{if } d=2;\\[2ex] \psi_0+\psi_1(\boldsymbol{P}_1^2+\boldsymbol{P}_2^2+\boldsymbol{P}_3^2)+\psi_2(\boldsymbol{P}_1^4+\boldsymbol{P}_2^4+\boldsymbol{P}_3^4)\\ \quad+\psi_3(\boldsymbol{P}_1^2\boldsymbol{P}_2^2+\boldsymbol{P}_2^2\boldsymbol{P}_3^2+\boldsymbol{P}_1^2\boldsymbol{P}_3^2)+\psi_4(\boldsymbol{P}_1^6+\boldsymbol{P}_2^6+\boldsymbol{P}_3^6), & \text{if } d=3,\end{cases} \tag{3.8}$$

where $\kappa,\beta>0$ and ψ_0 to ψ_4 are given constants with $\psi_4>0$.[1]

[1]The expression of ω is originally given in the paper [55].

3.2 Replacement of the gradient energy

We point out that the proof of showing the existence of viscous solutions of (3.2) is based on certain Rothe's method. More precisely, we first construct certain time discrete interpolant solutions and show a priori estimates for such solutions; then from the boundedness of the sequence given by the time discrete interpolant solutions, one infers the weak convergence (up to a subsequence) of the sequence of the time discrete interpolant solutions in some Sobolev space, which leads to strong convergence in some Lebesgue space due to the Sobolev's compact embedding. However, we point out that ω is a polynomial of sixth order, and it is well known that the embedding $H^1(\mathbf{\Omega}) \hookrightarrow L^6(\mathbf{\Omega})$ is only continuous but not compact in three dimensional space. Due to this reason we have to replace our gradient energy term $\|\nabla \boldsymbol{P}\|_{L^2}^2$ by a fractional term of higher order in the three dimensional case.

Assumption 3.1. *Let* $s \in [\max\{1, \frac{d}{2}\}, 2)$*. The gradient energy* $\mathcal{G}_{2,grad}(\boldsymbol{P}) = \frac{\kappa}{2}\|\nabla \boldsymbol{P}\|_{L^2}^2$ *appearing in* (3.5) *is replaced by*

- $\mathcal{G}_{2,grad,s}(\boldsymbol{P}) = \frac{\kappa}{2}\|\nabla \boldsymbol{P}\|_{L^2}^2$*, if* $s = 1$*;*
- $\mathcal{G}_{2,grad,s}(\boldsymbol{P}) = \frac{\kappa}{2}\int_{\mathbf{\Omega}\times\mathbf{\Omega}} \frac{|\nabla \boldsymbol{P}(\boldsymbol{x}) - \nabla \boldsymbol{P}(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s \rfloor)}} d\boldsymbol{x} d\boldsymbol{y}$*, if* $s \neq 1$*.*

Remark 3.2. At the first glance, such replacement of gradient energy is made artificially. However, we believe that such replacement also makes sense from a physical viewpoint, since the gradient energy term describes the interaction effect between the bound charge of the ferroelectric material, whose strength should be inversely proportional to the distance between the bound charge, while by the original gradient energy term, such inverse proportional relationship is neglected. △

The Assumption 3.1 is kept for the rest of Chapter 3 and (3.5) is replaced by

$$\begin{aligned} \mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) = &\int_{\mathbf{\Omega}} H(t, \boldsymbol{\varepsilon}, \nabla\phi, \boldsymbol{P}) d\boldsymbol{x} + \mathcal{G}_{2,grad,s}(\boldsymbol{P}) \\ &- \Big(\int_{\mathbf{\Omega}} \boldsymbol{f}_3(t) \cdot \boldsymbol{P} d\boldsymbol{x} + \int_{\partial\mathbf{\Omega}} \boldsymbol{\pi}(t) \cdot \boldsymbol{P} d\boldsymbol{S} \Big) \end{aligned} \tag{3.9}$$

and the underlying space for $\boldsymbol{P}$ is replaced by $\big(H^s(\mathbf{\Omega})\big)^d$.

3.3 Assumptions for the existence results

In order to guarantee the well-definedness of the main problem, we still need to solve following problems:

- Existence of $\boldsymbol{u}_D$, ϕ_D given by (3.1);
- Well definedness of the integrals appearing in (3.2) to (3.4) and in (3.9);
- Well definedness of the Gâteaux-differential of $\mathcal{H}$ defined by (3.9) w.r.t. $\boldsymbol{P}$.

We give the following assumptions, which guarantee that the above mentioned prerequisites are fulfilled. Meanwhile, these assumptions will also be used to characterize the preliminary conditions for the existence results given in Sections 3.7 and 3.8.

A1 $\mathbf{\Omega} \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, $\partial\mathbf{\Omega}_{\boldsymbol{u}} \dot{\cup} \partial\mathbf{\Omega}_{\boldsymbol{\sigma}} = \partial\mathbf{\Omega}_\phi \dot{\cup} \partial\mathbf{\Omega}_{\boldsymbol{D}} = \partial\mathbf{\Omega}$, $\partial\mathbf{\Omega}_{\boldsymbol{u}}, \partial\mathbf{\Omega}_\phi$ are $(d-1)$-sets, $\mathbf{\Omega} \cup \partial\mathbf{\Omega}_{\boldsymbol{u}}, \mathbf{\Omega} \cup \partial\mathbf{\Omega}_\phi$ are G1-regular (c.f. Section 2.1).

A2 $\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ (c.f. Section 2.4) defined on $\mathbb{R}^d$ are differentiable and uniformly Lipschitzian, their derivatives are also uniformly Lipschitzian.

A3 There exists some $\alpha > 0$ such that

$$\mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \geq \alpha|\boldsymbol{\varepsilon}|^2,$$
$$\boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{D} \cdot \boldsymbol{D} \geq \alpha|\boldsymbol{D}|^2,$$
$$\sup_{\boldsymbol{P}\in\mathbb{R}^d} \Big(|\mathbb{C}(\boldsymbol{P})| + |\boldsymbol{e}(\boldsymbol{P})| + |\boldsymbol{\varepsilon}^0(\boldsymbol{P})| + |\boldsymbol{\epsilon}(\boldsymbol{P})|\Big) < \infty$$

uniformly for all $\boldsymbol{P} \in \mathbb{R}^d,\ \boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d),\ \boldsymbol{D} \in \mathbb{R}^d$.

A4 There exists some $p^* \in (2, \infty)$ such that

$$\begin{aligned}
\boldsymbol{f}_1 &\in C^{1,1}([0,T]; \big(L^{\frac{p^*d}{d+p^*}}(\boldsymbol{\Omega})\big)^d),\\
\boldsymbol{t} &\in C^{1,1}([0,T]; \big(L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}})\big)^d),\\
f_2 &\in C^{1,1}([0,T];, L^{\frac{p^*d}{d+p^*}}(\boldsymbol{\Omega})),\\
\rho &\in C^{1,1}([0,T]; L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{D}})),\\
\boldsymbol{u}_D &\in C^{1,1}([0,T]; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^d),\\
\phi_D &\in C^{1,1}([0,T]; B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\phi})).
\end{aligned}$$

A5 There exists some $q^* \in (1, \infty)$ such that

$$\begin{aligned}
\boldsymbol{f}_3 &\in C^{1,1}([0,T]; \big(L^{q^*}(\boldsymbol{\Omega})\big)^d),\\
\boldsymbol{\pi} &\in C^{1,1}([0,T]; \big(L^{q^*}(\partial\boldsymbol{\Omega})\big)^d).
\end{aligned}$$

A6 There exist $C > 0, C' \in \mathbb{R}$ such that the polynomial ω defined by (3.8) satisfies

$$\omega(\boldsymbol{P}) \geq C|\boldsymbol{P}|^2 + C'$$

for all $\boldsymbol{P} \in \mathbb{R}^d$.

A7 $\Psi_1 : \big(H^s(\boldsymbol{\Omega})\big)^d \to [0, \infty)$ is convex, positively 1-homogeneous, weakly lower semicontinuous in $\big(H^s(\boldsymbol{\Omega})\big)^d$ and there exist $d_1, d_2 > 0$ such that for all $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$

$$d_1\|\boldsymbol{P}\|_{L^1} \leq \Psi_1(\boldsymbol{P}) \leq d_2\|\boldsymbol{P}\|_{L^1}. \tag{3.11}$$

Remark 3.3. The uniform Lipschitz continuity of the coefficients given by Assumption A2 also implies the uniform boundedness of their derivatives. △

Remark 3.4. Note that $\boldsymbol{t}$ and ρ from Assumption A4 are of class $\big(L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega})\big)^d$ and $L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega})$ by extending $\boldsymbol{t}$, ρ to $\boldsymbol{t}(\boldsymbol{x}) = \boldsymbol{0}$, $\rho(\boldsymbol{y}) = 0$ for $\boldsymbol{x} \in \partial\boldsymbol{\Omega} \setminus \partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$ and $\boldsymbol{y} \in \partial\boldsymbol{\Omega} \setminus \partial\boldsymbol{\Omega}_{\boldsymbol{D}}$ respectively. With this trivial extension one infers that the embedding Lemma 3.5 below can be applied to $\boldsymbol{t}$ and ρ. △

First we present the Sobolev's embedding Lemma 3.5. From this lemma we are able to infer the following facts:

- The functionals given by Assumption A4 (except $\boldsymbol{u}_D$, ϕ_D, which are to be handled by Lemma 3.8 below) are of class $C^{1,1}([0,T]; W_{\boldsymbol{\Gamma}}^{-1,p^*})$ (where $\boldsymbol{\Gamma}$ corresponds to different Dirichlet boundary parts given by Assumption A1);
- The functionals from Assumption A5 are in the space $C^{1,1,}([0,T];\mathcal{Y}^*)$, where $\mathcal{Y}^*$ is the dual space of some Banach space $\mathcal{Y}$ such that the underlying set $\big(H^s(\boldsymbol{\Omega})\big)^d$ of $\boldsymbol{P}$ is compactly embedded to $\mathcal{Y}$. Here, we have $\mathcal{Y} = L^{\frac{q^*}{q^*-1}}$ and $\mathcal{Y}^* = L^{q^*}$ corresponding to Assumption A5.

The above mentioned statements will be essential for Proposition 3.18 and Corollary 3.24 given below.

Lemma 3.5. *Let $d \in \{2,3\}$. Let $\boldsymbol{\Omega} \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and $\boldsymbol{\Gamma} \subset \partial\boldsymbol{\Omega}$ be a $(d-1)$-set such that $\boldsymbol{\Omega} \cup \boldsymbol{\Gamma}$ is G1-regular (see Section 2.1). Let also $s \in [\max\{1, \frac{d}{2}\}, 2)$. Then the following embeddings hold:*

1. *$L^{\mathfrak{p}}(\boldsymbol{\Omega}) \hookrightarrow W_{\boldsymbol{\Gamma}}^{-1,p}(\boldsymbol{\Omega})$ for $p \in (2,\infty)$ and $\mathfrak{p} \in [\frac{pd}{p+d}, \infty]$;*
2. *$L^{\mathfrak{p}}(\partial\boldsymbol{\Omega}) \hookrightarrow W_{\boldsymbol{\Gamma}}^{-1,p}(\boldsymbol{\Omega})$ for $p \in (2,\infty)$ and $\mathfrak{p} \in [\frac{p(d-1)}{d}, \infty]$;*
3. *$H^s(\boldsymbol{\Omega}) \hookrightarrow\hookrightarrow L^{\mathfrak{p}}(\boldsymbol{\Omega})$ for all $\mathfrak{p} \in [1,\infty)$;*
4. *$H^s(\boldsymbol{\Omega}) \hookrightarrow\hookrightarrow L^{\mathfrak{p}}(\partial\boldsymbol{\Omega})$ for all $\mathfrak{p} \in [1,\infty)$.*

Proof. Let $p \in (2,\infty)$. Then the Hölder conjugate p' of p is in the interval $(1,2)$. In particular, $p' < 2 \leq d$. Next, we obtain the following Sobolev's embedding relation

$$q \in [1, \frac{dp'}{d-p'}] \Rightarrow 1 - \frac{d}{p'} \geq 0 - \frac{d}{q}.$$

Thus

$$q \in [1, \frac{dp'}{d-p'}] \Rightarrow W_{\boldsymbol{\Gamma}}^{1,p'}(\boldsymbol{\Omega}) \subset W^{1,p'}(\boldsymbol{\Omega}) \hookrightarrow L^q(\boldsymbol{\Omega}).$$

On the other hand, one also obtains that

$$q \in [1, \frac{dp'}{d-p'}] \Leftrightarrow q' \in [\frac{pd}{p+d}, \infty].$$

Therefore using dual relation we obtain that

$$\mathfrak{p} \in [\frac{pd}{p+d}, \infty] \Rightarrow L^{\mathfrak{p}}(\boldsymbol{\Omega}) \hookrightarrow W_{\boldsymbol{\Gamma}}^{-1,p}(\boldsymbol{\Omega}),$$

which completes the proof of the first statement. For the second statement, we obtain from the Sobolev's trace embedding [51, Chap. 2, Thm. 4.2] that

$$q \in [1, \frac{p'(d-1)}{d-p'}] \Rightarrow W_{\boldsymbol{\Gamma}}^{1,p'}(\boldsymbol{\Omega}) \subset W^{1,p'}(\boldsymbol{\Omega}) \hookrightarrow L^q(\partial\boldsymbol{\Omega}).$$

On the other hand,

$$q \in [1, \frac{p'(d-1)}{d-p'}] \Leftrightarrow q' \in [\frac{p(d-1)}{d}, \infty].$$

Thus

$$\mathfrak{p} \in [\frac{p(d-1)}{d}, \infty] \Rightarrow L^{\mathfrak{p}}(\partial\mathbf{\Omega}) \hookrightarrow W_{\mathbf{\Gamma}}^{-1,p}(\mathbf{\Omega}),$$

which completes the proof of the second statement. Now since $s - \frac{d}{2}$ is always nonnegative due to the definition of s, the third and fourth statements are evident. This completes the desired proof. □

Next, we obtain from Assumption A3 that $\boldsymbol{\epsilon}(\boldsymbol{P}) \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$ has a matrix inverse $\boldsymbol{\epsilon}^{-1}(\boldsymbol{P}) \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$ for all $\boldsymbol{P} \in \mathbb{R}^d$. We point out that since the the terms $\mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$ and $\boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi \cdot \nabla\phi$ appearing in the functional $\mathcal{H}$ in (3.7) have opposite signs, variational formulations based on $\mathcal{H}$ can not be coercive w.r.t. $(\boldsymbol{\varepsilon}, \nabla\phi)$. To solve this problem, we will apply certain Legendre transform to formulate a new equivalent variational problem with symmetric coefficient tensor $\mathbb{B}_2(\boldsymbol{P})$ given by (3.19) below. We refer to Section 3.4 for details. It should also be indicated that the bottom right entry of $\mathbb{B}_2(\boldsymbol{P})$ is $\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})$ but not $\boldsymbol{\epsilon}(\boldsymbol{P})$ anymore. Hence similar statements as the ones given by the Assumptions A2 and A3 should be formulated for the matrix $\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})$. We show that this is indeed true as long as the Assumptions A2 and A3 hold.

Lemma 3.6. *Let $\boldsymbol{\epsilon}$ be a matrix that satisfies the Assumptions A2 and A3. Then $\boldsymbol{\epsilon}^{-1}$ also satisfies the Assumptions A2 and A3, with some elliptic constant $\alpha^* > 0$.*

Proof. From the Assumptions A2 and A3 and basic linear algebra, one obtains immediately that $\boldsymbol{\epsilon}(\boldsymbol{P})$ is an invertible matrix for all $\boldsymbol{P} \in \mathbb{R}^d$ and $\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})$ is uniformly elliptic (with elliptic constant $\alpha^* > 0$) and bounded for all $\boldsymbol{P} \in \mathbb{R}^d$. Since

$$\boldsymbol{E}_d = \boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{\epsilon}^{-1}(\boldsymbol{P}),$$

where $\boldsymbol{E}_d$ is the d-dimensional unit matrix, we obtain from product rule that

$$\mathbf{0} = \big(D_{\boldsymbol{P}}\boldsymbol{\epsilon}(\boldsymbol{P})\big)\boldsymbol{\epsilon}^{-1}(\boldsymbol{P}) + \boldsymbol{\epsilon}(\boldsymbol{P})\big(D_{\boldsymbol{P}}(\boldsymbol{\epsilon}^{-1})(\boldsymbol{P})\big)$$

or

$$D_{\boldsymbol{P}}(\boldsymbol{\epsilon}^{-1})(\boldsymbol{P}) = -\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})D_{\boldsymbol{P}}\boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{\epsilon}^{-1}(\boldsymbol{P}).$$

Since $\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})$ and $D_{\boldsymbol{P}}\boldsymbol{\epsilon}(\boldsymbol{P})$ are uniformly bounded and Lipschitz continuous in $\boldsymbol{P} \in \mathbb{R}^d$, we infer that $D_{\boldsymbol{P}}\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})$ is also uniformly bounded and Lipschitz continuous in $\boldsymbol{P} \in \mathbb{R}^d$. Thus $\boldsymbol{\epsilon}^{-1}$ satisfies the Assumptions A2 and A3. □

Remark 3.7. It is clear that the Assumption A3 holds for $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^{-1}$ if we replace α and α^* by $\min\{\alpha, \alpha^*\}$. Thus we can w.l.o.g. assume that $\alpha = \alpha^*$. △

Now we show that the three conditions discussed at the beginning of Section 3.3 are satisfied. The well-definedness of the integrals over $\mathbf{\Omega}$ or subsets $\mathbf{\Gamma}$ of $\partial\mathbf{\Omega}$ given in (3.2) to (3.4) and in (3.9) are trivially obtained by the Assumptions A2 to A5 and the fact that $H^s(\mathbf{\Omega})$ is continuously embedded to $L^r(\mathbf{\Omega})$ and $L^r(\partial\mathbf{\Omega})$ for all $r \in [1, \infty)$. The existence of $\boldsymbol{u}_D$ and ϕ_D is ensured by the following lemma:

Lemma 3.8 ([34, Chap. 7])**.** *Let $\boldsymbol{F} \subset \mathbb{R}^d$ be a $(d-1)$-set. Then there exists a continuous extension from the Besov space $B_{p,p}^{1-\frac{1}{p}}(\boldsymbol{F})$ to $W^{1,p}(\mathbb{R}^d)$ for all $p \in (1, \infty)$.*

From Lemma 3.8 we infer the embedding

$$B^{1-\frac{1}{p}}_{p,p}(\boldsymbol{F}) \hookrightarrow W^{1,p}(\mathbb{R}^d) \hookrightarrow W^{1,p}(\boldsymbol{\Omega}),$$

where the second embedding is the canonical restriction operator. This ensures the existence of $\boldsymbol{u}_D$ and ϕ_D in the space $\left(W^{1,p^*}(\boldsymbol{\Omega})\right)^d \times W^{1,p^*}(\boldsymbol{\Omega})$ with trace value $\boldsymbol{u}_D$, ϕ_D on $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}$ and $\partial\boldsymbol{\Omega}_\phi$ respectively, where p^* is the number given by the Assumption A4. In particular, using Hölder's inequality one obtains that the tensors $\boldsymbol{\varepsilon}(\boldsymbol{u}_D)$ and $\nabla\phi_D$ are of class W^{-1,p^*} via the realization

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_D)[\bar{\boldsymbol{u}}] = \int_{\boldsymbol{\Omega}} \boldsymbol{\varepsilon}(\boldsymbol{u}_D) : \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) d\boldsymbol{x},$$
$$\nabla\phi_D[\bar{\phi}] = \int_{\boldsymbol{\Omega}} \nabla\phi_D \cdot \nabla\bar{\phi} d\boldsymbol{x}$$

for test function $(\bar{\boldsymbol{u}}, \bar{\phi})$ of class $W^{1,(p^*)'}$ with $(p^*)' = \frac{p^*}{p^*-1}$.

At the end, we point out that the Gâteaux-differential $D_{\boldsymbol{P}}\mathcal{H}$ is not always well-defined for all $(\boldsymbol{u}, \phi)$ of class H^1, since $\boldsymbol{P}$ is not necessarily essentially bounded on $\boldsymbol{\Omega}$ for $s = \frac{d}{2}$. However, under the above given conditions, it can be shown that the solutions given by (3.2a) are of class $W^{1,q}$ for some $q > 2$ (see Lemma 3.19 below). Using this property, the Gâteaux-differentiability of $\mathcal{H}$ w.r.t. $\boldsymbol{P}$ follows from Lemma A.8.

3.4 Equivalent energetic formulation and reduced energy

In order to apply the viscous method given in [39], our first step is to formulate a time discrete minimization problem which is derived from the inclusion (3.2b). Therefore, one expects that the solution $(\boldsymbol{u}, \phi)$ of (3.2a) can be seen as some kind of "minimizer" of a (quadratic in $(\boldsymbol{u}, \phi)$) energy functional. However, it seems unreasonable to take $(\boldsymbol{u}, \phi)$ as a minimizer due to the following reasons: First, the tensor $\mathbb{B}_1(\boldsymbol{P})$ given by (3.3) has antidiagonals, this means that $(\boldsymbol{u}, \phi)$ can not be the minimizer induced by the elliptic problem (3.2a), since by setting the test function $(\bar{\boldsymbol{u}}, \bar{\phi})$ equal to $(\boldsymbol{u}, \phi)$ therein, the coupling terms will cancel out; On the other hand, the terms $\mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$ and $\boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi \cdot \nabla\phi$ appearing in the functional $\mathcal{H}$ in (3.7) have opposite signs, and therefore one obtains no coercivity of the to be minimized functional w.r.t. $\boldsymbol{P}$. We will apply certain Legendre transform to formulate an equivalent problem such that the coercivity of the new formulated energy functional is present. First, we define the function $\boldsymbol{D}_\nu : [0,T] \to \left(L^2(\boldsymbol{\Omega})\right)^d$ as follows: using Poincare's inequality (see Lemma A.1) and Lax-Milgram we know that there exists a uniquely determined $\phi_\nu : [0,T] \to H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ such that

$$\int_{\boldsymbol{\Omega}} \nabla\phi_\nu(t) \cdot \nabla\bar{\phi} d\boldsymbol{x} = \int_{\boldsymbol{\Omega}} f_2(t)\bar{\phi} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}} \rho(t)\bar{\phi} d\boldsymbol{S} \tag{3.12}$$

for a.a. $t \in (0,T)$ and all $\bar{\phi} \in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$. Then we define $\boldsymbol{D}_\nu := \nabla\phi_\nu$. In particular, using Assumption A4 and applying Proposition 3.18 given below to the equation (3.12), we infer that there exists some $\mathfrak{q}^* \in (2, p^*]$ such that $\boldsymbol{D}_\nu \in C^{1,1}([0,T]; \left(L^{\mathfrak{q}^*}(\boldsymbol{\Omega})\right)^d)$ and

$$\begin{aligned}&\|\boldsymbol{D}_\nu\|_{C^{1,1}([0,T],L^{\mathfrak{q}^*}(\boldsymbol{\Omega}))}\\ \le& C\Big(\|f_2\|_{C^{1,1}([0,T],L^{\frac{\mathfrak{q}^* d}{d+\mathfrak{q}^*}}(\boldsymbol{\Omega}))} + \|\rho\|_{C^{1,1}([0,T],L^{\frac{\mathfrak{q}^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{D}}))}\Big).\end{aligned} \tag{3.13}$$

Here, p^* is the number given by Assumption A4 and C is some positive constant depending only on $\boldsymbol{\Omega}$ and $\partial\boldsymbol{\Omega}_\phi$, where the dependence is deduced from Proposition 3.18. Next, we define the energy functional $\mathcal{E} = \mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ by

$$\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) = \mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) + \mathcal{E}_2(\boldsymbol{P}) - l_3(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}), \tag{3.14}$$

where

$$\begin{aligned}\mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) &:= \int_{\boldsymbol{\Omega}} \frac{1}{2}\mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} d\boldsymbol{x} \\ &=: \int_{\boldsymbol{\Omega}} U_1\big(t, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{D}, \boldsymbol{P}\big) d\boldsymbol{x},\end{aligned} \tag{3.15}$$

$$\mathcal{E}_2(\boldsymbol{P}) := \mathcal{G}_{2,grad,s}(\boldsymbol{P}) + \int_{\boldsymbol{\Omega}} \omega(\boldsymbol{P}) d\boldsymbol{x}, \tag{3.16}$$

$$\begin{aligned}l_3(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) &:= \Big(\int_{\boldsymbol{\Omega}} \boldsymbol{f}_1(t) \cdot \boldsymbol{u} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} \boldsymbol{t}(t) \cdot \boldsymbol{u} d\boldsymbol{S} - \int_{\boldsymbol{\Omega}} \nabla\phi_D(t) \cdot \boldsymbol{D} d\boldsymbol{x} \Big) \\ &\quad + \Big(\int_{\boldsymbol{\Omega}} \boldsymbol{f}_3(t) \cdot \boldsymbol{P} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}} \boldsymbol{\pi}(t) \cdot \boldsymbol{P} d\boldsymbol{S} \Big) \\ &=: l_3^1(t, \boldsymbol{u}, \boldsymbol{D}) + l_3^2(t, \boldsymbol{P})\end{aligned} \tag{3.17}$$

for $\boldsymbol{u} \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d$, $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and

$$\boldsymbol{D} \in M_{\boldsymbol{D}} := \Big\{ \boldsymbol{D} \in \big(L^2(\boldsymbol{\Omega})\big)^d : \int_{\boldsymbol{\Omega}} \boldsymbol{D} \cdot \nabla\phi d\boldsymbol{x} = 0 \ \forall \phi \in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega}) \Big\}, \tag{3.18}$$

namely, $\boldsymbol{D}$ is divergence free in distributional sense. Here, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the small strain tensor generated by $\boldsymbol{u}$ and $\mathbb{B}_2(\boldsymbol{P})$ is defined by

$$\mathbb{B}_2(\boldsymbol{P}) = \begin{pmatrix} \mathbb{C}(\boldsymbol{P}) + \boldsymbol{e}^T(\boldsymbol{P})\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})\boldsymbol{e}(\boldsymbol{P}) & -\boldsymbol{e}^T(\boldsymbol{P})\boldsymbol{\epsilon}^{-1}(\boldsymbol{P}) \\ -\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})\boldsymbol{e}(\boldsymbol{P}) & \boldsymbol{\epsilon}^{-1}(\boldsymbol{P}) \end{pmatrix}. \tag{3.19}$$

For the other terms defined in (3.12) to (3.19), see the variable list at the beginning of Section 2.4. For the sake of simplicity we define

$$\tilde{\boldsymbol{D}} := \boldsymbol{D} + \boldsymbol{D}_\nu.$$

Notice also that $\mathbb{B}_2(\boldsymbol{P})$ is of class $L^\infty(\boldsymbol{\Omega}; \mathrm{Lin}(\mathbb{R}^{d\times d} \times \mathbb{R}^d, \mathbb{R}^{d\times d} \times \mathbb{R}^d))$ for each $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ due to Assumption A3.

Lemma 3.9. *Let the Assumptions A1 to A6 be satisfied. Let* $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) : (0, T) \to \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}} \times \big(H^s(\boldsymbol{\Omega})\big)^d$ *be a solution of the inclusion*

$$\boldsymbol{0} \in D_{\boldsymbol{q}}\mathcal{E}\big(t, \boldsymbol{q}(t)\big) + \partial\Psi_\beta\big(\boldsymbol{P}'(t)\big), \ \boldsymbol{P}(0) = \boldsymbol{P}_0 \tag{3.20}$$

for a.a. $t \in (0, T)$. *Then* $\boldsymbol{q}$ *induces a solution* $(\boldsymbol{u}, \phi, \boldsymbol{P})$ *of the laws* (3.2). *On the other hand, if* $(\boldsymbol{u}, \phi, \boldsymbol{P})$ *solves* (3.2), *then* $(\boldsymbol{u}, \phi, \boldsymbol{P})$ *induces a solution* $\boldsymbol{q}$ *which solves* (3.20) *for a.a.* $t \in (0, T)$.

Proof. Let us first suppose that $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ is solution of (3.20). Since Ψ_β is a functional of a single variable $\boldsymbol{P}$, we have

$$\begin{aligned}\mathbf{0} &\in D_{\boldsymbol{u}}\mathcal{E}\big(t, \boldsymbol{q}(t)\big),\\ \mathbf{0} &\in D_{\boldsymbol{D}}\mathcal{E}\big(t, \boldsymbol{q}(t)\big).\end{aligned}$$

From Lemma A.7 we know that for each $t \in [0, T]$ and $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ is Gâteaux-differentiable in $(\boldsymbol{u}, \boldsymbol{D})$. Thus using the Gâteaux-differentiability of $\mathcal{E}$ w.r.t. $\boldsymbol{u}$ and $\boldsymbol{D}$ we know that the subdifferentiability is equivalent to differentiability, namely

$$\begin{aligned}\mathbf{0} &= D_{\boldsymbol{u}}\mathcal{E}\big(t, \boldsymbol{q}(t)\big),\\ \mathbf{0} &= D_{\boldsymbol{D}}\mathcal{E}\big(t, \boldsymbol{q}(t)\big).\end{aligned}$$

By evaluating in test functions, we then easily derive that

$$\begin{aligned}0 =& D_{\boldsymbol{u}}\mathcal{E}\big(t, \boldsymbol{u}(t), \boldsymbol{D}(t), \boldsymbol{P}(t)\big)[\bar{\boldsymbol{u}}]\\ =& \int_{\boldsymbol{\Omega}} \mathbb{C}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big) : \bar{\boldsymbol{\varepsilon}} - \boldsymbol{\epsilon}^{-1}\big(\boldsymbol{P}(t)\big)\Big(\tilde{\boldsymbol{D}}(t) - \boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\\ &+ \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big) - \boldsymbol{P}(t)\Big) \cdot \Big(\boldsymbol{e}\big(\boldsymbol{P}(t)\big)\bar{\boldsymbol{\varepsilon}}\Big) d\boldsymbol{x} - \left(\int_{\boldsymbol{\Omega}} \boldsymbol{f}_1(t) \cdot \bar{\boldsymbol{u}} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} \boldsymbol{t}(t) \cdot \bar{\boldsymbol{u}} d\boldsymbol{S}\right),\end{aligned} \tag{3.21}$$

$$\begin{aligned}0 =& D_{\boldsymbol{D}}\mathcal{E}\big(t, \boldsymbol{u}(t), \boldsymbol{D}(t), \boldsymbol{P}(t)\big)[\bar{\boldsymbol{D}}]\\ =& \int_{\boldsymbol{\Omega}} \Big(\boldsymbol{\epsilon}^{-1}\big(\boldsymbol{P}(t)\big)\Big(\tilde{\boldsymbol{D}}(t) - \boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big) - \boldsymbol{P}(t)\Big) + \nabla\phi_D(t)\Big) \cdot \bar{\boldsymbol{D}} d\boldsymbol{x}\end{aligned} \tag{3.22}$$

for all $\bar{\boldsymbol{u}} \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d$ and $\bar{\boldsymbol{D}} \in M_{\boldsymbol{D}}$. Now recall that $-\mathrm{Div} : \big(L^2(\boldsymbol{\Omega})\big)^d \to H^{-1}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ is defined by

$$-\mathrm{Div}(\boldsymbol{T})[\bar{\phi}] = \int_{\boldsymbol{\Omega}} \boldsymbol{T} \cdot \nabla\bar{\phi} d\boldsymbol{x}$$

for $\bar{\phi} \in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$. Define $G := -\mathrm{Div}$. Then

$$\operatorname{Ker} G = \left\{\boldsymbol{T} \in \big(L^2(\boldsymbol{\Omega})\big)^d : \int_{\boldsymbol{\Omega}} \boldsymbol{T} \cdot \nabla\phi d\boldsymbol{x} = 0 \ \forall\phi \in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})\right\} = M_{\boldsymbol{D}}.$$

For an element $l \in H^{-1}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$, using Poincare's inequality and Lax-Milgram we know that there exists a $\phi_l \in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ such that

$$\int_{\boldsymbol{\Omega}} \nabla\phi_l \cdot \nabla\bar{\phi} d\boldsymbol{x} = \langle l, \bar{\phi}\rangle_{H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})}$$

for all $\bar{\phi} \in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$. Letting $\boldsymbol{T} = \nabla\phi_l$, it follows that G is surjective and therefore $\mathrm{Ran}(G)$ is closed in $H^{-1}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$. Using (3.22) we obtain that

$$\boldsymbol{\epsilon}^{-1}\big(\boldsymbol{P}(t)\big)\Big(\tilde{\boldsymbol{D}}(t) - \boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big) - \boldsymbol{P}(t)\Big) + \nabla\phi_D(t) \in (\operatorname{Ker} G)^\perp.$$

Then from the closed range theorem it follows that

$$\boldsymbol{\epsilon}^{-1}\big(\boldsymbol{P}(t)\big)\Big(\tilde{\boldsymbol{D}}(t)-\boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big)-\boldsymbol{P}(t)\Big)+\nabla\phi_D(t) \\ \in (\operatorname{Ker} G)^{\perp}=\operatorname{Ran} G',$$

where G' is the adjoint of G. Since $H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ is a Hilbert space, we know that $H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega}) = \big(H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})\big)^{**}$ and there exists a $-\phi(t)\in H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ such that

$$\int_{\boldsymbol{\Omega}}\Big(\boldsymbol{\epsilon}^{-1}\big(\boldsymbol{P}(t)\big)\Big(\tilde{\boldsymbol{D}}(t)-\boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big)-\boldsymbol{P}(t)\Big)+\nabla\phi_D(t)\Big)\cdot\bar{\boldsymbol{T}}d\boldsymbol{x} \\ =G'\big(-\phi(t)\big)[\bar{\boldsymbol{T}}]=G(\bar{\boldsymbol{T}})[-\phi(t)]=\int_{\boldsymbol{\Omega}}\big(-\nabla\phi(t)\big)\cdot\bar{\boldsymbol{T}}d\boldsymbol{x}$$

for all $\bar{\boldsymbol{T}}\in\big(L^2(\boldsymbol{\Omega})\big)^d$. This implies that

$$\boldsymbol{\epsilon}^{-1}\big(\boldsymbol{P}(t)\big)\Big(\tilde{\boldsymbol{D}}(t)-\boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big)-\boldsymbol{P}(t)\Big)+\nabla\phi_D(t)=-\nabla\phi(t) \tag{3.23}$$

almost everywhere in $\boldsymbol{\Omega}$. Together with (3.21) and (3.22) we obtain that

$$\int_{\boldsymbol{\Omega}}\mathbb{C}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big):\bar{\boldsymbol{\varepsilon}}+\boldsymbol{e}\big(\boldsymbol{P}(t)\big)^T\big(\nabla\phi(t)+\nabla\phi_D(t)\big):\bar{\boldsymbol{\varepsilon}}d\boldsymbol{x} \\ =\int_{\boldsymbol{\Omega}}\boldsymbol{f}_1(t)\cdot\bar{\boldsymbol{u}}d\boldsymbol{x}+\int_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}}\boldsymbol{t}(t)\cdot\bar{\boldsymbol{u}}d\boldsymbol{S}, \tag{3.24}$$

$$\int_{\boldsymbol{\Omega}}\boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big)\cdot\nabla\bar{\phi}-\boldsymbol{\epsilon}\big(\boldsymbol{P}(t)\big)\big(\nabla\phi(t)+\nabla\phi_D(t)\big)\cdot\nabla\bar{\phi} \\ +\boldsymbol{P}(t)\cdot\nabla\bar{\phi}d\boldsymbol{x} \\ =\int_{\boldsymbol{\Omega}}f_2(t)\bar{\phi}d\boldsymbol{x}+\int_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}}\rho(t)\bar{\phi}d\boldsymbol{S}. \tag{3.25}$$

Subtract (3.25) from (3.24) and rearranging terms, we obtain (3.2a). Next, we utilize the regularity result Lemma 3.19 from Section 3.5 given below to infer that $\boldsymbol{D}$ is of class L^p for some $p>2$: From Lemma 3.19 we obtain that under the Assumptions A1 to A6, the solution $(\boldsymbol{u},\phi)$ is in fact of class $W^{1,p}$ for some $p>2$. Then using (3.23) one infers immediately that $\boldsymbol{D}$ is of class L^p for some $p>2$. Hence, from Lemma A.8 we obtain the Gâteaux-differentiability of $\mathcal{E}(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})$ in $\boldsymbol{P}$. The derivative of $\mathcal{E}(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})$ w.r.t. $\boldsymbol{P}$ reads

$$D_{\boldsymbol{P}}\mathcal{E}(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})[\bar{\boldsymbol{P}}] \\ =\int_{\boldsymbol{\Omega}}D_{\boldsymbol{P}}U_1\big(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{D},\boldsymbol{P}\big)(\bar{\boldsymbol{P}})+D_{\boldsymbol{P}}\omega(\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}+\kappa\langle\boldsymbol{P},\bar{\boldsymbol{P}}\rangle_s-l_3^2(t,\bar{\boldsymbol{P}}) \tag{3.26}$$

for $\bar{\boldsymbol{P}}\in\big(H^s(\boldsymbol{\Omega})\big)^d$. Define $\boldsymbol{E}:=-(\nabla\phi+\nabla\phi_D)$ and

$$\widehat{U}_1(\boldsymbol{\varepsilon},\tilde{\boldsymbol{D}},\boldsymbol{P}):=U_1(t,\boldsymbol{\varepsilon},\boldsymbol{D},\boldsymbol{P}),$$
$$\widehat{H}(\boldsymbol{\varepsilon},\boldsymbol{E},\boldsymbol{P}):=H(t,\boldsymbol{\varepsilon},\nabla\phi,\boldsymbol{P}).$$

Using direct calculation, we obtain from (3.23) and the derivative of $\widehat{U}_1$ w.r.t. $\tilde{\boldsymbol{D}}$ that

$$\boldsymbol{E} = D_{\tilde{\boldsymbol{D}}}\widehat{U}_1(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{D}}, \boldsymbol{P}).$$

Define for $\boldsymbol{E}, \boldsymbol{P} \in \mathbb{R}^d$ and $\boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$ the Legendre transform $\widehat{U}_1^*$ of $\widehat{U}_1$ w.r.t. $\boldsymbol{E}$ by

$$\widehat{U}_1^*(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) := \sup_{\tilde{\boldsymbol{D}} \in \mathbb{R}^d} \{\tilde{\boldsymbol{D}} \cdot \boldsymbol{E} - \widehat{U}_1(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{D}}, \boldsymbol{P})\}$$

From Lemma A.10 it follows

$$\widehat{U}_1^*(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) + \widehat{U}_1(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{D}}, \boldsymbol{P}) = \tilde{\boldsymbol{D}} \cdot \boldsymbol{E}.$$

We obtain from direct calculation that

$$\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) = \widehat{U}_1(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{D}}, \boldsymbol{P}) - \tilde{\boldsymbol{D}} \cdot \boldsymbol{E} = -\widehat{U}_1^*(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}). \tag{3.27}$$

Together with Lemma A.10 it follows

$$\tilde{\boldsymbol{D}} = D_{\boldsymbol{E}}\widehat{U}_1^*(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) = -D_{\boldsymbol{E}}\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}). \tag{3.28}$$

Now, we point out that $\boldsymbol{E}$ can also be seen as a function of $\boldsymbol{\varepsilon}, \tilde{\boldsymbol{D}}, \boldsymbol{P}$ which is differentiable w.r.t. $\boldsymbol{P}$ due to (3.23). Then from the chain rule we obtain that

$$\begin{aligned} & D_{\boldsymbol{P}}U_1(t, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{P}) \\ =& D_{\boldsymbol{P}}\widehat{U}_1(\boldsymbol{\varepsilon}, \tilde{\boldsymbol{D}}, \boldsymbol{P}) \\ =& D_{\boldsymbol{E}}\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) \cdot D_{\boldsymbol{P}}\boldsymbol{E} + \tilde{\boldsymbol{D}} \cdot D_{\boldsymbol{P}}\boldsymbol{E} + D_{\boldsymbol{P}}\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) \\ =& \big(D_{\boldsymbol{E}}\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) + \tilde{\boldsymbol{D}}\big) \cdot D_{\boldsymbol{P}}\boldsymbol{E} + D_{\boldsymbol{P}}\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) \\ =& D_{\boldsymbol{P}}\widehat{H}(\boldsymbol{\varepsilon}, \boldsymbol{E}, \boldsymbol{P}) \\ =& D_{\boldsymbol{P}}H(t, \boldsymbol{\varepsilon}, \nabla\phi, \boldsymbol{P}), \end{aligned} \tag{3.29}$$

where the second equality comes from (3.27) and the chain rule and the fourth equality from (3.28). Using (3.26) and (3.29), we infer that

$$\begin{aligned} & D_{\boldsymbol{P}}\mathcal{E}\big(t, \boldsymbol{q}(t)\big)[\bar{\boldsymbol{P}}] \\ =& \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}H\big(t, \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \nabla\phi(t), \boldsymbol{P}(t)\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\omega\big(\boldsymbol{P}(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x} + \kappa\langle \boldsymbol{P}(t), \bar{\boldsymbol{P}}\rangle_s - l_3^2(t, \bar{\boldsymbol{P}}) \\ =& D_{\boldsymbol{P}}\mathcal{H}\big(t, \boldsymbol{u}(t), \phi(t), \boldsymbol{P}(t)\big)[\bar{\boldsymbol{P}}]. \end{aligned}$$

The law (3.20) implies immediately (3.2b)-(3.2c) and this completes the first part of the proof. Now let $(\boldsymbol{u}, \phi, \boldsymbol{P})$ be a solution of (3.2). Define

$$\tilde{\boldsymbol{D}}(t) := \boldsymbol{e}\big(\boldsymbol{P}(t)\big)\Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}(t)\big)\Big) + \boldsymbol{P}(t) - \boldsymbol{\epsilon}\big(\boldsymbol{P}(t)\big)\big(\nabla\phi(t) + \nabla\phi_D(t)\big). \tag{3.30}$$

(3.25) implies that $\boldsymbol{D}(t) = \tilde{\boldsymbol{D}}(t) - \boldsymbol{D}_\nu(t) \in M_{\boldsymbol{D}}$. (3.24), (3.30) imply (3.21), (3.22). Together with (3.2b)-(3.2c) and (3.29), we see that $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ is a solution of (3.20). □

Remark 3.10. One obtains the following bijective relation: First, let $(\boldsymbol{u}, \phi, \boldsymbol{P})$ be a solution of the problem (3.2). Let $(\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ be the solution of (3.20), which is as constructed in the second part of the proof of Lemma 3.9 and $\boldsymbol{D}$ is given as the difference of $\tilde{\boldsymbol{D}}$ given by (3.30) and $\boldsymbol{D}_\nu = \nabla\phi_\nu$ given by (3.12). Then we define the mapping $\mathfrak{L}$ from the solutions set of (3.2) to the solutions set of (3.20) by

$$\mathfrak{L} : (\boldsymbol{u}, \phi, \boldsymbol{P}) \mapsto (\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}).$$

In a converse way, given a solution $(\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ of (3.20), one obtains a solution $(\boldsymbol{u}, \phi, \boldsymbol{P})$ of (3.2) as constructed in the first part of the proof of Lemma 3.9. We then define the mapping $\mathfrak{J}$ from the solutions set of (3.20) to the solutions set of (3.2) by

$$\mathfrak{J} : (\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) \mapsto (\boldsymbol{u}, \phi, \boldsymbol{P}).$$

It is then easy to verified from the proof of Lemma 3.9 that $\mathfrak{L}$ is a bijective function with inverse $\mathfrak{J}$. △

Remark 3.11. From (3.29) we obtain that if $\boldsymbol{D}$ and $\nabla\phi$ are given via the relation (3.23), then we obtain the identity

$$D_{\boldsymbol{P}} H(t, \boldsymbol{\varepsilon}, \nabla\phi, \boldsymbol{P}) = D_{\boldsymbol{P}} U_1(t, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{P}).$$

Dealing $D_{\boldsymbol{P}} H$ and $D_{\boldsymbol{P}} U_1$ as linear functionals which are evaluated at $(t, \boldsymbol{\varepsilon}, \nabla\phi, \boldsymbol{P})$ and $(t, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{P})$ respectively and map $\bar{\boldsymbol{P}} \in \mathbb{R}^d$ to $\mathbb{R}$, we obtain from the chain rule and simple calculation the following useful identity

$$\begin{aligned} & D_{\boldsymbol{P}} H(t, \boldsymbol{\varepsilon}, \nabla\phi, \boldsymbol{P})(\bar{\boldsymbol{P}}) \\ =& D_{\boldsymbol{P}} U_1(t, \boldsymbol{\varepsilon}, \boldsymbol{D}, \boldsymbol{P})(\bar{\boldsymbol{P}}) \\ =& \frac{1}{2} D_{\boldsymbol{P}} \mathbb{B}_2(\boldsymbol{P}) \bar{\boldsymbol{P}} \begin{pmatrix} \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} \\ & + \mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} -D_{\boldsymbol{P}} \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{P}} \end{pmatrix} \end{aligned}$$

for $\bar{\boldsymbol{P}} \in \mathbb{R}^d$, which will be used several times in the rest of the thesis. △

3.4.1 Reduced energy $\mathcal{I}$

In what follows, we show that one can in fact deal with an equivalent problem to (3.20) which concerns only the variable $(t, \boldsymbol{P})$.

Lemma 3.12. *Let the Assumptions A1 to A6 be satisfied. Then for each* $t \in [0, T]$ *and* $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$*, the functional* $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ *admits a unique minimizer* $\big(\boldsymbol{u}_{\min}(t, \boldsymbol{P}), \boldsymbol{D}_{\min}(t, \boldsymbol{P})\big)$ *in* $\big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$.

Proof. Let $t \in [0, T]$ and $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ be given. We first show the weak lower semi-continuity of $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ w.r.t. $(\boldsymbol{u}, \boldsymbol{D})$. It suffices to show the uniform ellipticity of $\mathbb{B}_2(\boldsymbol{P})$, i.e. to show that there exists a constant $C > 0$ such that

$$\mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} \geq C\big(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{D}|^2\big) \tag{3.31}$$

for all $\boldsymbol{P} \in \mathbb{R}^d$, $\boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$ and $\boldsymbol{D} \in \mathbb{R}^d$. Indeed, the convexity of the integrand U_1 defined by (3.15) in $(\boldsymbol{\varepsilon}, \boldsymbol{D})$ follows from (3.31). Since $\nabla\boldsymbol{u} \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ is linear, U_1 is convex in $(\nabla \boldsymbol{u}, \boldsymbol{D})$. Since U_1 is nonnegative, we obtain from [12, Thm. 3.23] (taking $U_1 = f$, $a = b = c = 0$, $q = 2$ and $\xi = (\nabla \boldsymbol{u}, \boldsymbol{D})$ therein) that the weak lower semi-continuity of $\mathcal{E}_1$ follows from the convexity of the integrand U_1 in $(\nabla \boldsymbol{u}, \boldsymbol{D})$. Since $\mathcal{E}_2$ depends only on $\boldsymbol{P}$ and l_3 is affine in $(\boldsymbol{u}, \boldsymbol{D})$, the weak lower semi-continuity of $\mathcal{E}$ in $(\boldsymbol{u}, \boldsymbol{D})$ follows. We refer to Lemma A.6 for the proof of (3.31). Now due to Lemma A.6 and (3.15) we obtain that

$$\begin{aligned}\mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) &\geq C\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\|_{L^2}^2 + \|\boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P}\|_{L^2}^2\big) \\ &\geq C\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2}^2 + \|\boldsymbol{D}\|_{L^2}^2\big) + \mu_{t,\boldsymbol{P}}\end{aligned}$$

for some positive constant C and some real number $\mu_{t,\boldsymbol{P}}$, where $\mu_{t,\boldsymbol{P}}$ depends on t and $\boldsymbol{P}$. Here, the last inequality is a direct consequence of the Young's inequality. Due to Korn's inequality (Lemma A.2) it follows

$$\begin{aligned}\mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) &\geq C\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2}^2 + \|\boldsymbol{D}\|_{L^2}^2\big) + \mu_{t,\boldsymbol{P}} \\ &\geq C\big(\|\boldsymbol{u}\|_{H^1}^2 + \|\boldsymbol{D}\|_{L^2}^2\big) + \mu_{t,\boldsymbol{P}}.\end{aligned}$$

Since $\mathcal{E}_2$ depends only on $\boldsymbol{P}$ and l_3 is affine in $(\boldsymbol{u}, \boldsymbol{D})$ for each t and $\boldsymbol{P}$, the coercivity of $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ in $(\boldsymbol{u}, \boldsymbol{D})$ follows. Now the existence of a minimizer follows from Tonelli's abstract existence theorem.

It is still left to show the uniqueness of $(\boldsymbol{u}_{\min}, \boldsymbol{D}_{\min})$. First we point out that from (3.31) we can even infer the strict convexity of U_1 in $(\boldsymbol{\varepsilon}, \boldsymbol{D})$. Suppose that there are two minimizers $(\boldsymbol{u}_1, \boldsymbol{D}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{D}_2)$. Then estimating like in [12, Thm. 3.30, Step 2] (where the strict convexity of U_1 is utilized), we have $\boldsymbol{\varepsilon}(\boldsymbol{u}_1) = \boldsymbol{\varepsilon}(\boldsymbol{u}_2)$ and $\boldsymbol{D}_1 = \boldsymbol{D}_2$ for a.a. $\boldsymbol{x} \in \boldsymbol{\Omega}$. But from Korn's inequality we obtain immediately that

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^2} \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2} = 0.$$

Thus $\boldsymbol{u}_1 = \boldsymbol{u}_2$ for a.a. $\boldsymbol{x} \in \boldsymbol{\Omega}$ and the uniqueness of the minimizer follows. □

Now we define the reduced energy $\mathcal{I}$ by

$$\mathcal{I}(t, \boldsymbol{P}) := \mathcal{E}\big(t, \boldsymbol{u}_{\min}(t, \boldsymbol{P}), \boldsymbol{D}_{\min}(t, \boldsymbol{P}), \boldsymbol{P}\big) \tag{3.32}$$

and consider the following problem: Find $\boldsymbol{P} : (0, T) \to \big(H^s(\boldsymbol{\Omega})\big)^d$ such that

$$\boldsymbol{0} \in D_{\boldsymbol{P}}\mathcal{I}\big(t, \boldsymbol{P}(t)\big) + \partial\Psi_\beta\big(\boldsymbol{P}'(t)\big), \ \boldsymbol{P}(0) = \boldsymbol{P}_0 \tag{3.33}$$

for a.a. $t \in (0, T)$.

Lemma 3.13. *Let the Assumptions A1 to A6 be satisfied. Let $\boldsymbol{P}$ be a solution of* (3.33)*. Then $\boldsymbol{P}$ induces a solution $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ of* (3.20)*. On the other hand, if $\boldsymbol{q}$ is a solution of* (3.20)*, then $\boldsymbol{q}$ induces a solution $\boldsymbol{P}$ of* (3.33)*.*

Proof. Let $\boldsymbol{q}$ be a solution of (3.20). Then from the convexity of U_1 in $(\boldsymbol{\varepsilon}, \boldsymbol{D})$ for fixed t and $\boldsymbol{P}$ (deduced from the proof of Lemma 3.12) we also know that $\big(\boldsymbol{u}(t), \boldsymbol{D}(t)\big)$ is the unique minimizer of $\mathcal{E}\big(t, \boldsymbol{u}(t), \boldsymbol{D}(t), \boldsymbol{P}(t)\big)$ (see for instance [12, Thm. 3.37, Step 2]), where the uniqueness follows from Lemma 3.12. It follows that $\boldsymbol{P}$ is a solution of (3.33), which is a direct consequence of (3.49) below. Now suppose that $\boldsymbol{P}$ is a solution of (3.33). Again using [12, Thm. 3.37] it follows that $\big(\boldsymbol{u}_{\min}\big(t, \boldsymbol{P}(t)\big), \boldsymbol{D}_{\min}\big(t, \boldsymbol{P}(t)\big)\big)$ is also a critical point of the Gâteaux-differential of $\mathcal{E}\big(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}(t)\big)$ w.r.t $(\boldsymbol{u}, \boldsymbol{D})$. Therefore, $(\boldsymbol{u}_{\min}, \boldsymbol{D}_{\min}, \boldsymbol{P})$ is also a solution of (3.20). □

Remark 3.14. Analogously as in Remark 3.10, if $\tilde{\mathfrak{L}}$ is the operator from the solutions set of (3.20) to the solutions set of (3.33), and $\tilde{\mathfrak{J}}$ is the one from (3.33) to (3.20), where $\tilde{\mathfrak{L}}, \tilde{\mathfrak{J}}$ are constructed by the proof of Lemma 3.13, then $\tilde{\mathfrak{L}}$ is bijective with inverse $\tilde{\mathfrak{J}}$. △

Due to Lemma 3.13 we have thus formulated an equivalent reduced problem. In what follows, we will investigate the reduced problem given below:

Reduced main problem

Find $\boldsymbol{P} : (0,T) \to \big(H^s(\boldsymbol{\Omega})\big)^d$ such that

$$\boldsymbol{0} \in D_{\boldsymbol{P}}\mathcal{I}\big(t, \boldsymbol{P}(t)\big) + \partial\Psi_\beta\big(\boldsymbol{P}'(t)\big),\ \boldsymbol{P}(0) = \boldsymbol{P}_0 \qquad \text{for a.a. } t \in (0,T). \tag{3.34}$$

Finally, we formulate the following proposition, which indicates that the inclusion (3.34) can be equivalently interpreted as an integral identity or an integral inequality. The idea is to make use of several equivalent formulations in subdifferential calculus involving (3.34) (which are sometimes called the Fenchel's formulas, see Lemma A.10). We refer the details of the proof to [39].

Proposition 3.15 ([39, Prop. 3.1])**.** *Let the Assumptions A1 to A7 be satisfied. Let* $\boldsymbol{P} \in H^1(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d)$. *Then the following statements are equivalent:*

1. $\boldsymbol{P}$ *is a solution of* (3.34).

2. $\boldsymbol{P}$ *fulfills for all* $0 \le u \le t \le T$ *the identity*

$$\begin{aligned}&\int_u^t \Psi_\beta\big(\boldsymbol{P}'(\tau)\big)d\tau + \int_u^t \Psi_\beta^*\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\tau, \boldsymbol{P}(\tau)\big)\Big)d\tau + \mathcal{I}\big(t, \boldsymbol{P}(t)\big)\\ =&\mathcal{I}\big(u, \boldsymbol{P}(u)\big) + \int_u^t \partial_t\mathcal{I}\big(\tau, \boldsymbol{P}(\tau)\big)d\tau.\end{aligned}$$

3. $\boldsymbol{P}$ *fulfills for all* $0 \le t \le T$ *the inequality*

$$\begin{aligned}&\int_0^t \Psi_\beta\big(\boldsymbol{P}'(\tau)\big)d\tau + \int_0^t \Psi_\beta^*\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\tau, \boldsymbol{P}(\tau)\big)\Big)d\tau + \mathcal{I}\big(t, \boldsymbol{P}(t)\big)\\ \le&\mathcal{I}\big(0, \boldsymbol{P}(0)\big) + \int_0^t \partial_t\mathcal{I}\big(\tau, \boldsymbol{P}(\tau)\big)d\tau.\end{aligned}$$

3.5 Preliminary regularity results

As stated in Lemma A.8, for fixed given $t \in [0,T]$ and $(\boldsymbol{u}, \phi, \boldsymbol{D}) \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d \times H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega}) \times M_{\boldsymbol{D}}$, the Gâteaux-differentiability of $\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P})$ and $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ w.r.t. $\boldsymbol{P}$ will be ensured if $(\boldsymbol{u}, \phi, \boldsymbol{D})$ is of class $(W^{1,p})^2 \times L^p$ for some $p > 2$. We show that this statement holds under the Assumptions A1 to A6.

Remark 3.16. In view of the equivalence Lemma 3.9, the following results are mostly formulated only for the pair $(\boldsymbol{u}, \boldsymbol{D})$, since we are able to obtain similar results for the pair $(\boldsymbol{u}, \phi)$ via the relation (3.23), and the energy functional $\mathcal{E}$ having the variable $(\boldsymbol{u}, \boldsymbol{D})$ plays the major role in the rest of this chapter. △

Assumption 3.17 ([28, Def. 4.4])**.** *Let* $\boldsymbol{\Gamma} \subset \partial\boldsymbol{\Omega}$ *be closed. There exists a linear, continuous extension operator* $E : \big(W^{1,1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\big)^d \to \big(W^{1,1}(\mathbb{R}^d)\big)^d$ *which simultaneously defines a continuous extension operator* $E : \big(W^{1,p}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\big)^d \to \big(W^{1,p}(\mathbb{R}^d)\big)^d$ *for all* $p \in (1, \infty)$.

Due to [28] we know that under the condition that $\boldsymbol{\Omega} \cup \boldsymbol{\Gamma}$ is G1-regular and $\boldsymbol{\Gamma}$ is a $(d-1)$-set, the Assumption 3.17 is satisfied.

Based on Assumption 3.17, we introduce a useful regularity result (Proposition 3.18 below) from [28]. For $1 \le i \le m$ and $\boldsymbol{\Gamma}_i \subset \partial\boldsymbol{\Omega}$ we define the space

$$\mathbb{W}^{1,p}_{\boldsymbol{\Gamma}} := \prod_{i=1}^{m} W^{1,p}_{\boldsymbol{\Gamma}_i}(\boldsymbol{\Omega}).$$

For $\mathbb{A} \in L^\infty\big(\boldsymbol{\Omega}, \mathrm{Lin}(\mathbb{R}^m \times \mathbb{R}^{md}, \mathbb{R}^m \times \mathbb{R}^{md})\big)$ we define

$$\langle \mathcal{A}(\boldsymbol{u}), \boldsymbol{v}\rangle := \int_{\boldsymbol{\Omega}} \mathbb{A}\begin{pmatrix}\boldsymbol{u}\\ \nabla \boldsymbol{u}\end{pmatrix} : \begin{pmatrix}\boldsymbol{v}\\ \nabla \boldsymbol{v}\end{pmatrix} d\boldsymbol{x}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{W}^{1,2}_{\boldsymbol{\Gamma}}$.

Proposition 3.18 ([28, Thm. 6.2])**.** *Let the Assumption 3.17 be satisfied for all $\boldsymbol{\Gamma}_i$ and let all $\boldsymbol{\Gamma}_i$ be $(d-1)$-sets. Furthermore, assume that there exists a positive constant η such that*

$$\langle \mathcal{A}(\boldsymbol{v}), \boldsymbol{v}\rangle \ge \eta \|\boldsymbol{v}\|_{H^1}$$

for all $\boldsymbol{v} \in \mathbb{W}^{1,2}_{\boldsymbol{\Gamma}}$. Then there exists some $p^{**} > 2$ such that for all $p \in [2, p^{**}]$, $\mathcal{A} : \boldsymbol{v} \mapsto \langle \mathcal{A}(\boldsymbol{v}), \cdot\rangle$ defines a continuously invertible isomorphism from $\mathbb{W}^{1,p}_{\boldsymbol{\Gamma}}$ to $\mathbb{W}^{-1,p}_{\boldsymbol{\Gamma}}$. In particular, the norm of the inverse isomorphism operator depends only on the ellipticity constant η and the L^∞-norm of the coefficient tensor $\mathbb{A}$.

Using Proposition 3.18 we obtain the following regularity result, which gives a sufficient condition for Lemma A.8 that reveals the Gâteaux-differentiability of $\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P})$ and $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ in $\boldsymbol{P}$:

Lemma 3.19. *Let the Assumptions A1 to A6 be satisfied. Then there exists a constant $q \in (2, \infty)$, such that for each $t \in [0, T]$ and $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, the minimizer $\big(\boldsymbol{u}_{\min}(t, \boldsymbol{P}), \boldsymbol{D}_{\min}(t, \boldsymbol{P})\big)$ given by Lemma 3.12 is of class $W^{1,p} \times L^p$ for all $p \in [2, q]$ and satisfies the inequality*

$$\|\boldsymbol{u}_{\min}(t, \boldsymbol{P})\|_{W^{1,p}} + \|\boldsymbol{D}_{\min}(t, \boldsymbol{P})\|_{L^p} \le C\big(1 + \Lambda + \|\boldsymbol{P}\|_{L^p}\big), \tag{3.35}$$

where

$$\Lambda = \|\boldsymbol{f}_1\|_{C^{1,1}} + \|f_2\|_{C^{1,1}} + \|\boldsymbol{t}\|_{C^{1,1}} + \|\rho\|_{C^{1,1}} + \|\boldsymbol{u}_D\|_{C^{1,1}} + \|\phi_D\|_{C^{1,1}} \tag{3.36}$$

and the norm $\|\cdot\|_{C^{1,1}}$ is defined as the the norm $\|\cdot\|_{C^{1,1}(0,T;\mathcal{X})}$ for corresponding spaces $\mathcal{X}$ given by the Assumption A4. In particular, C is uniform for all $p \in [2, q]$ and independent on the choice of $\boldsymbol{P}$.

Proof. Recall that $l_{t,\boldsymbol{P}}$, the r.h.s. of (3.2a), is defined by

$$\begin{aligned} l_{t,\boldsymbol{P}}(\bar{\boldsymbol{u}}, \bar{\phi}) = &\int_{\boldsymbol{\Omega}} \boldsymbol{f}_1(t) \cdot \bar{\boldsymbol{u}} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} \boldsymbol{t}(t) \cdot \bar{\boldsymbol{u}} d\boldsymbol{S} - \Big(\int_{\boldsymbol{\Omega}} f_2(t)\bar{\phi} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}} \rho(t)\bar{\phi} d\boldsymbol{S}\Big)\\ &- \int_{\boldsymbol{\Omega}} \Big(\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \boldsymbol{e}^T(\boldsymbol{P})\nabla\phi_D(t)\Big) : \bar{\boldsymbol{\varepsilon}} d\boldsymbol{x}\\ &+ \int_{\boldsymbol{\Omega}} \Big(\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi_D(t) + \boldsymbol{P}\Big) \cdot \nabla\bar{\phi} d\boldsymbol{x} \end{aligned}$$

for test functions $(\bar{\boldsymbol{u}}, \bar{\phi})$. We then write

$$l_{t,\boldsymbol{P}} =: \widehat{l}_{t,\boldsymbol{P}} - \mathrm{Div}\boldsymbol{P}.$$

Due to the Assumption A3, $\mathbb{B}_1(\boldsymbol{P})$ is essentially bounded on $\boldsymbol{\Omega}$ and uniformly elliptic, thus Korn's and Poincaré's inequalities imply the coercive condition $\langle \mathcal{A}(\boldsymbol{v}), \boldsymbol{v}\rangle \geq \eta \|\boldsymbol{v}\|_{H^1}$ in Proposition 3.18. From Lemma 3.5 and Lemma 3.8 it follows that $l_{t,\boldsymbol{P}}$ is of class $W^{-1,p}$ for all $2 \leq p \leq q$ for some $q > 2$: Indeed, using standard dual estimation and Hölder's inequality we see for $\bar{\boldsymbol{u}} \in \big(W^{1,p'}(\boldsymbol{\Omega})\big)^d$ and $\bar{\phi} \in W^{1,p'}(\boldsymbol{\Omega})$ that

$$\begin{aligned}
|\widehat{l}_{t,\boldsymbol{P}}(\bar{\boldsymbol{u}}, \bar{\phi})| \leq & C\big(\|\boldsymbol{f}_1\|_{C^{1,1}} + \|\boldsymbol{t}\|_{C^{1,1}} + \|f_2\|_{C^{1,1}} + \|\rho\|_{C^{1,1}}\big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big) \\
& + C\big(\|\boldsymbol{\varepsilon}_D\|_{C^{1,1}} + \|\nabla \phi_D\|_{C^{1,1}}\big)\big(\|\bar{\boldsymbol{\varepsilon}}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big) \\
\leq & C\Lambda\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big).
\end{aligned}$$

On the other hand, $-\mathrm{Div}\boldsymbol{P}$ is of class $W^{-1,p}$ for all $1 \leq p < \infty$ due to Sobolev's embedding. Moreover, from Hölder's inequality it follows that $\| - \mathrm{Div}\boldsymbol{P}\|_{W^{-1,p}}$ is bounded by $\|\boldsymbol{P}\|_{L^p}$. Thus from Proposition 3.18 it follows that the solution $(\boldsymbol{u}, \phi)$ of (3.2a) (for fixed t and $\boldsymbol{P}$) satisfies

$$\|\boldsymbol{u}(t, \boldsymbol{P})\|_{W^{1,p}} + \|\phi(t, \boldsymbol{P})\|_{W^{1,p}} \leq C\big(\Lambda + \|\boldsymbol{P}\|_{L^p}\big) \tag{3.37}$$

for all $2 \leq p \leq q := \min\{p^*, p^{**}, \mathfrak{q}^*\}$ with $q > 2$, where $\mathfrak{q}^*$ is defined by (3.13). Now (3.35) follows from (3.23). The independence of C on $\boldsymbol{P}$ and $p \in [2, q]$ follows directly from the last statement of Proposition 3.18 and Assumption A3. □

Using Proposition 3.18 one also obtains the differentiability of $\mathcal{I}$ w.r.t. t and $\boldsymbol{P}$, which is given in the forthcoming sections. Before stating these differentiability results we will still need the following preliminary regularity lemma:

Lemma 3.20. *Let the Assumptions A1 to A6 be satisfied. Then for all* $t_1, t_2 \in [0, T]$, $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \big(H^s(\boldsymbol{\Omega})\big)^d$ *and* $p \in [2, q)$, *where* q *is given by Lemma 3.19, we have*

$$\begin{aligned}
&\|\boldsymbol{u}_{\min}(t_1, \boldsymbol{P}_1) - \boldsymbol{u}_{\min}(t_2, \boldsymbol{P}_2)\|_{W^{1,p}} + \|\boldsymbol{D}_{\min}(t_1, \boldsymbol{P}_1) - \boldsymbol{D}_{\min}(t_2, \boldsymbol{P}_2)\|_{L^p} \\
\leq & C\big(1 + \Lambda + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big),
\end{aligned} \tag{3.38}$$

where $r := \frac{qp}{q-p}$ *and* Λ *is defined by* (3.36).

Proof. Let $(\boldsymbol{u}_i, \phi_i)$ be the solutions of (3.2a) with given pairs $(t_i, \boldsymbol{P}_i)$, $i = 1, 2$. Denote also that $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}(\boldsymbol{u}_i)$. We obtain from (3.2a) that

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P}_1) \begin{pmatrix} \boldsymbol{\varepsilon}_1 \\ \nabla\phi_1 \end{pmatrix} : \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x} = l_{t_1,\boldsymbol{P}_1}(\bar{\boldsymbol{u}}, \bar{\phi}), \tag{3.39}$$

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P}_2) \begin{pmatrix} \boldsymbol{\varepsilon}_2 \\ \nabla\phi_2 \end{pmatrix} : \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x} = l_{t_2,\boldsymbol{P}_2}(\bar{\boldsymbol{u}}, \bar{\phi}). \tag{3.40}$$

Subtract (3.40) from (3.39) and rearranging terms, we have

$$\begin{aligned}
&\int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P}_1) \begin{pmatrix} \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \\ \nabla\phi_1 - \nabla\phi_2 \end{pmatrix} : \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x} \\
=&(\widehat{l}_{t_1,\boldsymbol{P}_1} - \widehat{l}_{t_2,\boldsymbol{P}_2})(\bar{\boldsymbol{u}}, \bar{\phi}) + \int_{\boldsymbol{\Omega}} (\boldsymbol{P}_1 - \boldsymbol{P}_2)\cdot\nabla\bar{\phi} d\boldsymbol{x} - \int_{\boldsymbol{\Omega}} \big(\mathbb{B}_1(\boldsymbol{P}_1) - \mathbb{B}_1(\boldsymbol{P}_2)\big) \begin{pmatrix} \boldsymbol{\varepsilon}_2 \\ \nabla\phi_2 \end{pmatrix} : \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x} \\
=&: I_1 + I_2 + I_3.
\end{aligned} \tag{3.41}$$

Divide I_1 into two parts:

$$I_1 = (\widehat{l}_{t_1,\boldsymbol{P}_1} - \widehat{l}_{t_2,\boldsymbol{P}_1}) + (\widehat{l}_{t_2,\boldsymbol{P}_1} - \widehat{l}_{t_2,\boldsymbol{P}_2}) =: I_{11} + I_{12}.$$

It follows from standard dual estimation and the Assumption A4 that for $\bar{\boldsymbol{u}} \in \big(W^{1,p'}(\boldsymbol{\Omega})\big)^d$ and $\bar{\phi} \in W^{1,p'}(\boldsymbol{\Omega})$

$$\begin{aligned}|I_{11}| \le& C|t_1 - t_2|\big(\|\boldsymbol{f}_1\|_{C^{1,1}} + \|\boldsymbol{t}\|_{C^{1,1}} + \|f_2\|_{C^{1,1}} + \|\rho\|_{C^{1,1}}\big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big)\\ &+ C|t_1 - t_2|\big(\|\boldsymbol{\varepsilon}_D\|_{C^{1,1}} + \|\nabla\phi_D\|_{C^{1,1}}\big)\big(\|\bar{\varepsilon}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big)\\ \le& C\Lambda|t_1 - t_2|\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big).\end{aligned}$$

For I_{12}, it suffices to consider the following summands

$$\begin{aligned}&\int_{\boldsymbol{\Omega}} \big(\mathbb{C}(\boldsymbol{P}_1) - \mathbb{C}(\boldsymbol{P}_2)\big)\boldsymbol{\varepsilon}_D(t_2) : \bar{\varepsilon} d\boldsymbol{x} =: I_{121},\\ &\int_{\boldsymbol{\Omega}} \big(\mathbb{C}(\boldsymbol{P}_1)\varepsilon^0(\boldsymbol{P}_1) - \mathbb{C}(\boldsymbol{P}_2)\varepsilon^0(\boldsymbol{P}_2)\big) : \bar{\varepsilon} d\boldsymbol{x} =: I_{122}\end{aligned}$$

in I_{12}, estimation of other summands in I_{12} can be deduced analogously. It follows that

$$\begin{aligned}|I_{121}| \le& C\int_{\boldsymbol{\Omega}} |\boldsymbol{P}_1 - \boldsymbol{P}_2||\boldsymbol{\varepsilon}_D(t_2)||\bar{\varepsilon}| d\boldsymbol{x}\\ \le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\|\boldsymbol{\varepsilon}_D(t_2)\|_{L^q}\|\bar{\varepsilon}\|_{L^{p'}}\\ \le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\|\boldsymbol{\varepsilon}_D\|_{C^{1,1}}\|\bar{\varepsilon}\|_{L^{p'}}\\ \le& C\Lambda\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\|\bar{\varepsilon}\|_{L^{p'}},\end{aligned}$$

where $r = \frac{qp}{q-p} > p$ and $\frac{1}{r} + \frac{1}{q} + \frac{1}{p'} = 1$. On the other hand,

$$\begin{aligned}|I_{122}| \le& C\int_{\boldsymbol{\Omega}} |\boldsymbol{P}_1 - \boldsymbol{P}_2||\bar{\varepsilon}| d\boldsymbol{x}\\ \le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\|\bar{\varepsilon}\|_{L^{p'}}\\ \le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\|\bar{\varepsilon}\|_{L^{p'}}.\end{aligned}$$

We also obtain the following estimation:

$$|I_2| \le \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\|\nabla\bar{\phi}\|_{L^{p'}} \le C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\|\nabla\bar{\phi}\|_{L^{p'}}$$

and

$$\begin{aligned}|I_3| \le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big(\|\boldsymbol{\varepsilon}_2\|_{L^q} + \|\nabla\phi_2\|_{L^q}\big)\big(\|\bar{\varepsilon}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big)\\ \le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big(\Lambda + \|\boldsymbol{P}_2\|_{L^q}\big)\big(\|\bar{\varepsilon}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big),\end{aligned}$$

where for the last inequality we have used the relation given by (3.37). Then applying Proposition 3.18 to (3.41) we obtain that

$$\begin{aligned}&\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,p}} + \|\phi_1 - \phi_2\|_{W^{1,p}}\\ \le& C\big(\Lambda + \|\boldsymbol{P}_2\|_{L^q}\big)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big).\end{aligned} \tag{3.42}$$

Now (3.38) follows from (3.23). □

Remark 3.21. Since Λ given by (3.36) is uniquely determined by the Assumption A4, we can w.l.o.g. assume that Λ is bounded by some positive constant C which does not depend on the particular choice of $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$ in various inequalities. Thus in the rest of this chapter, the symbol Λ is replaced by the positive constant C. △

3.5.1 Differentiability of $\mathcal{I}$ w.r.t. t

Lemma 3.22. *Let the Assumptions A1 to A6 be satisfied. Then for every* $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, $t \to \mathcal{I}(t, \boldsymbol{P})$ *is in* $C^{1,1}([0,T])$ *with*

$$\begin{aligned}\partial_t \mathcal{I}(t, \boldsymbol{P}) = &\int_{\boldsymbol{\Omega}} \mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t,\boldsymbol{P})\big) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}_{\min}(t,\boldsymbol{P}) + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}'_D(t) \\ \boldsymbol{D}'_\nu(t) \end{pmatrix} d\boldsymbol{x} \\ &- l'_3\big(t, \boldsymbol{u}_{\min}(t,\boldsymbol{P}), \boldsymbol{D}_{\min}(t,\boldsymbol{P}), \boldsymbol{P}\big).\end{aligned} \tag{3.43}$$

Furthermore, for $t_1, t_2, t \in [0,T]$, $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, $p \in [2,q)$, *where* q *is given by Lemma 3.19, we have the inequalities*

$$|\partial_t \mathcal{I}(t,\boldsymbol{P})| \leq C\big(1 + \|\boldsymbol{P}\|_{L^q} + \|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}}\big), \tag{3.44}$$

where q^* *is defined by Assumption A5,* $(q^*)' = \frac{q^*}{q^*-1}$ *and*

$$\|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}} := \|\boldsymbol{P}\|_{L^{(q^*)'}(\boldsymbol{\Omega})} + \|\boldsymbol{P}\|_{L^{(q^*)'}(\partial\boldsymbol{\Omega})}; \tag{3.45}$$

$$\begin{aligned}&|\partial_t \mathcal{I}(t_1,\boldsymbol{P}_1) - \partial_t \mathcal{I}(t_2,\boldsymbol{P}_2)| \\ \leq &C\alpha(\boldsymbol{P}_1,\boldsymbol{P}_2)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^{r_1}} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|^{\Sigma}_{L^{(q^*)'}}\big),\end{aligned} \tag{3.46}$$

where $r_1 = \frac{q}{q-2}$, $r = \frac{qp}{q-p}$ *and*

$$\alpha(\boldsymbol{P}_1,\boldsymbol{P}_2) = 1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q} + \|\boldsymbol{P}_1\|^{\Sigma}_{L^{(q^*)'}} + \|\boldsymbol{P}_2\|^{\Sigma}_{L^{(q^*)'}}.$$

Proof. Recall from (3.14) that

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 - l_3.$$

We first show the differentiability of $\mathcal{E}(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})$ w.r.t. t for given fixed $(\boldsymbol{u},\boldsymbol{D},\boldsymbol{P}) \in \big(H^1(\boldsymbol{\Omega})\big)^d \times \big(L^2(\boldsymbol{\Omega})\big)^d \times \big(H^s(\boldsymbol{\Omega})\big)^d$ and show that

$$\begin{aligned}\partial_t \mathcal{E}\big(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P}\big) = &\int_{\boldsymbol{\Omega}} \mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}'_D(t) \\ \boldsymbol{D}'_\nu(t) \end{pmatrix} d\boldsymbol{x} \\ &- l'_3\big(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P}\big).\end{aligned}$$

Since $\mathcal{E}_2$ depends only on $\boldsymbol{P}$, we only have to consider $\mathcal{E}_1$ and l_3. It is straightforward to show that the derivative of l_3 is l'_3 and we omit the details here. Now we calculate the derivative of $\mathcal{E}_1$ w.r.t. t. It suffices to consider one side derivative (since the other side can be deduced analogously), thus let $h \in (0,\varepsilon]$ for some small $\varepsilon > 0$. It follows from the chain rule that

$$\begin{aligned}&\frac{1}{h}\big(\mathcal{E}_1(t+h,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P}) - \mathcal{E}_1(t,\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})\big) \\ =&\int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t+\sigma h) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t+\sigma h) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}'_D(t+\sigma h) \\ \boldsymbol{D}'_\nu(t+\sigma h) \end{pmatrix} d\sigma d\boldsymbol{x} \\ =:&\int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P}) J_{1,\sigma,h} : J_{2,\sigma,h} d\sigma d\boldsymbol{x}.\end{aligned}$$

We obtain from Hölder's inequality and Fubini's theorem that

$$\begin{aligned}
&\Big|\int_{\boldsymbol{\Omega}}\Big(\int_0^1 \mathbb{B}_2(\boldsymbol{P})J_{1,\sigma,h} : J_{2,\sigma,h}d\sigma\Big) - \mathbb{B}_2(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u})+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\\ \boldsymbol{D}+\boldsymbol{D}_\nu(t)-\boldsymbol{P}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}_D'(t)\\ \boldsymbol{D}_\nu'(t)\end{pmatrix}d\boldsymbol{x}\Big|\\
=&\Big|\int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P})J_{1,\sigma,h} : J_{2,\sigma,h}d\sigma d\boldsymbol{x} - \int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P})J_{1,0,0} : J_{2,0,0}d\sigma d\boldsymbol{x}\Big|\\
=&\Big|\int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P})(J_{1,\sigma,h}-J_{1,0,0}) : J_{2,\sigma,h}d\sigma d\boldsymbol{x} + \int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P})J_{1,0,0} : (J_{2,\sigma,h}-J_{2,0,0})d\sigma d\boldsymbol{x}\Big|\\
\le& C\Big(\int_0^1 \|J_{1,\sigma,h}-J_{1,0,0}\|_{L^2}\|J_{2,\sigma,h}\|_{L^2} + \|J_{2,\sigma,h}-J_{2,0,0}\|_{L^2}\|J_{1,0,0}\|_{L^2}d\sigma\Big)\\
=:& C\int_0^1 \tilde{J}(\sigma,h)d\sigma.
\end{aligned} \tag{3.47}$$

The Assumption A4 and (3.13) ensure that $\boldsymbol{\varepsilon}_D$, $\boldsymbol{D}_\nu$ are of class $C^{1,1}([0,T];L^2)$, therefore estimating $\tilde{J}(\sigma,h)$ in terms of $\boldsymbol{\varepsilon}_D$ and $\boldsymbol{D}_\nu$ by using certain Hölder's inequality (which is derived from the penultimate line of (3.47), and we omit the details here due to its straightforward but tedious calculation), we point out that $\tilde{J}(\sigma,h)$ is uniformly bounded on $[0,1]\times[0,\varepsilon]$ and goes to zero as $h\to 0$ for each $\sigma\in[0,1]$. From the Lebesgue dominated convergence theorem we infer that (3.47) goes to zero as $h\to 0$, which shows the differentiability of $\mathcal{E}_1$ w.r.t. t. Now we want to utilize the sup-inf arguments given in [39, Lem. 2.4] to show the differentiability of $\mathcal{I}$ w.r.t. t. On the one hand, since $(\boldsymbol{u},\boldsymbol{D}) = (\boldsymbol{u}_{\min},\boldsymbol{D}_{\min})$ is a minimizer, we obtain that

$$\begin{aligned}
&\limsup_{h\searrow 0}\frac{\mathcal{I}(t+h,\boldsymbol{P})-\mathcal{I}(t,\boldsymbol{P})}{h}\\
\le& \lim_{h\searrow 0} h^{-1}\Big(\mathcal{E}\big(t+h,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\big) - \mathcal{E}\big(t,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\big)\Big)\\
=&\partial_t\mathcal{E}\big(t,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\big).
\end{aligned}$$

On the other hand, for $h>0$ we obtain that

$$\begin{aligned}
&\frac{\mathcal{I}(t+h,\boldsymbol{P})-\mathcal{I}(t,\boldsymbol{P})}{h}\\
\ge& h^{-1}\Big(\mathcal{E}\big(t+h,\boldsymbol{u}_{\min}(t+h,\boldsymbol{P}),\boldsymbol{D}_{\min}(t+h,\boldsymbol{P}),\boldsymbol{P}\big) - \mathcal{E}\big(t,\boldsymbol{u}_{\min}(t+h,\boldsymbol{P}),\boldsymbol{D}_{\min}(t+h,\boldsymbol{P}),\boldsymbol{P}\big)\Big)\\
=&\int_{\boldsymbol{\Omega}}\int_0^1 \mathbb{B}_2(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t+h,\boldsymbol{P})\big)+\boldsymbol{\varepsilon}_D(t+\sigma h)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\\ \boldsymbol{D}_{\min}(t+h,\boldsymbol{P})+\boldsymbol{D}_\nu(t+\sigma h)-\boldsymbol{P}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}_D'(t+\sigma h)\\ \boldsymbol{D}_\nu'(t+\sigma h)\end{pmatrix}d\sigma d\boldsymbol{x}.
\end{aligned} \tag{3.48}$$

Now using the similar Hölder's inequalities as the ones for estimating (3.47) we conclude that the last term of (3.48) converges to $\partial_t\mathcal{E}\big(t,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\big)$ as $h\searrow 0$. Therefore we obtain the desired result.

It follows from standard dual estimation, (3.35) and Hölder's inequality that (notice that $1 \le q' \le 2 \le q$)

$$\begin{aligned}
|\partial_t \mathcal{I}(t,\boldsymbol{P})| \le& \Big(C \int_{\boldsymbol{\Omega}} \big(1 + |\boldsymbol{\varepsilon}(t,\boldsymbol{P})| + |\boldsymbol{D}(t,\boldsymbol{P})| + |\boldsymbol{\varepsilon}_D(t)| + |\boldsymbol{D}_\nu(t)| + |\boldsymbol{P}|\big) \\
&\cdot \big(|\boldsymbol{\varepsilon}'_D(t)| + |\boldsymbol{D}'_\nu(t)|\big) d\boldsymbol{x} \Big) + C\big(\|\boldsymbol{u}(t,\boldsymbol{P})\|_{W^{1,q'}} + \|\boldsymbol{D}(t,\boldsymbol{P})\|_{L^{q'}} + \|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\le& \Big(C\big(1 + \|\boldsymbol{\varepsilon}(t,\boldsymbol{P})\|_{L^q} + \|\boldsymbol{D}(t,\boldsymbol{P})\|_{L^q} + \|\boldsymbol{\varepsilon}_D(t)\|_{L^q} + \|\boldsymbol{D}_\nu(t)\|_{L^q} + \|\boldsymbol{P}\|_{L^q}\big) \\
&\cdot \big(\|\boldsymbol{\varepsilon}'_D(t)\|_{L^{q'}} + \|\boldsymbol{D}'_\nu(t)\|_{L^{q'}}\big)\Big) + C\big(\|\boldsymbol{u}(t,\boldsymbol{P})\|_{W^{1,q}} + \|\boldsymbol{D}(t,\boldsymbol{P})\|_{L^q} + \|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\le& \Big(C\big(1 + \|\boldsymbol{\varepsilon}(t,\boldsymbol{P})\|_{L^q} + \|\boldsymbol{D}(t,\boldsymbol{P})\|_{L^q} + \|\boldsymbol{\varepsilon}_D(t)\|_{L^q} + \|\boldsymbol{D}_\nu(t)\|_{L^q} + \|\boldsymbol{P}\|_{L^q}\big) \\
&\cdot \big(\|\boldsymbol{\varepsilon}'_D(t)\|_{L^q} + \|\boldsymbol{D}'_\nu(t)\|_{L^q}\big)\Big) + C\big(\|\boldsymbol{u}(t,\boldsymbol{P})\|_{W^{1,q}} + \|\boldsymbol{D}(t,\boldsymbol{P})\|_{L^q} + \|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\le& C\big(1 + \|\boldsymbol{P}\|_{L^q}\big) + C\big(1 + \|\boldsymbol{P}\|_{L^q} + \|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\le& C\big(1 + \|\boldsymbol{P}\|_{L^q} + \|\boldsymbol{P}\|^{\Sigma}_{L^{(q^*)'}}\big),
\end{aligned}$$

where $\big(\boldsymbol{\varepsilon}(t,\boldsymbol{P}), \boldsymbol{D}(t,\boldsymbol{P})\big) = \Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t,\boldsymbol{P})\big), \boldsymbol{D}_{\min}(t,\boldsymbol{P})\Big)$. Let

$$\begin{aligned}
\partial_t \mathcal{I}(t,\boldsymbol{P}) &=: I(t,\boldsymbol{P}) - l'_3(t, \boldsymbol{u}_{\min}(t,\boldsymbol{P}), \boldsymbol{D}_{\min}(t,\boldsymbol{P}), \boldsymbol{P}) \\
&=: I(t,\boldsymbol{P}) + J(t,\boldsymbol{P}).
\end{aligned}$$

Define

$$I_{i_1,i_2,i_3} := \int_{\boldsymbol{\Omega}} \mathbb{B}_2(\boldsymbol{P}_{i_1}) \begin{pmatrix} \boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t_{i_2},\boldsymbol{P}_{i_2})\big) + \boldsymbol{\varepsilon}_D(t_{i_2}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}_{i_2}) \\ \boldsymbol{D}_{\min}(t_{i_2},\boldsymbol{P}_{i_2}) + \boldsymbol{D}_\nu(t_{i_2}) - \boldsymbol{P}_{i_2} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}'_D(t_{i_3}) \\ \boldsymbol{D}'_\nu(t_{i_3}) \end{pmatrix} d\boldsymbol{x}.$$

It then reads

$$I(t_1,\boldsymbol{P}_1) - I(t_2,\boldsymbol{P}_2) = (I_{111} - I_{211}) + (I_{211} - I_{221}) + (I_{221} - I_{222}).$$

Similarly as done previously, we obtain from (3.35), (3.38) and the Lipschitz continuity of $\mathbb{B}_2(\boldsymbol{P})$ in $\boldsymbol{P}$ that

$$\begin{aligned}
&|I_{111} - I_{211}| \\
\le& C \int_{\boldsymbol{\Omega}} |\boldsymbol{P}_1 - \boldsymbol{P}_2|\big(1 + |\boldsymbol{\varepsilon}_1| + |\boldsymbol{D}_1| + |\boldsymbol{\varepsilon}_D(t_1)| + |\boldsymbol{D}_\nu(t_1)| + |\boldsymbol{P}_1|\big) \cdot \big(|\boldsymbol{\varepsilon}'_D(t_1)| + |\boldsymbol{D}'_\nu(t_1)|\big) d\boldsymbol{x} \\
\le& C\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^{r_1}}\big(1 + \|\boldsymbol{\varepsilon}_1\|_{L^q} + \|\boldsymbol{D}_1\|_{L^q}\| + \|\boldsymbol{\varepsilon}_D(t_1)\|_{L^q} + \|\boldsymbol{D}_\nu(t_1)\|_{L^q} + \|\boldsymbol{P}_1\|_{L^q}\big) \\
&\cdot \big(\|\boldsymbol{\varepsilon}'_D(t_1)\|_{L^q} + \|\boldsymbol{D}'_\nu(t_1)\|_{L^q}\big) \\
\le& C\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^{r_1}} \\
\le& C\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^{r_1}},
\end{aligned}$$

where $r_1 := \frac{q}{q-2}$ with $\frac{1}{r_1} + \frac{1}{q} + \frac{1}{q} = 1$ and $(\boldsymbol{\varepsilon}_i, \boldsymbol{D}_i) = \Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t_i, \boldsymbol{P}_i)\big), \boldsymbol{D}_{\min}(t_i, \boldsymbol{P}_i)\Big)$,

$$\begin{aligned}
|I_{211} - I_{221}| \leq & C \int_{\boldsymbol{\Omega}} \big(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\boldsymbol{D}_1 - \boldsymbol{D}_2| + |\boldsymbol{\varepsilon}_D(t_1) - \boldsymbol{\varepsilon}_D(t_2)| \\
& + |\boldsymbol{D}_\nu(t_1) - \boldsymbol{D}_\nu(t_2)| + |\boldsymbol{P}_1 - \boldsymbol{P}_2|\big) \cdot \big(|\boldsymbol{\varepsilon}'_D(t_1)| + |\boldsymbol{D}'_\nu(t_1)|\big) d\boldsymbol{x} \\
\leq & C\big(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{L^p} + \|\boldsymbol{D}_1 - \boldsymbol{D}_2\|_{L^p} + \|\boldsymbol{\varepsilon}_D(t_1) - \boldsymbol{\varepsilon}_D(t_2)\|_{L^p} \\
& + \|\boldsymbol{D}_\nu(t_1) - \boldsymbol{D}_\nu(t_2)\|_{L^p} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\big)\big(\|\boldsymbol{\varepsilon}'_D(t_1)\|_{L^q} + \|\boldsymbol{D}'_\nu(t_1)\|_{L^q}\big) \\
\leq & C\Big(\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big) \\
& + |t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\Big) \\
\leq & C\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\big) \\
\leq & C\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big),
\end{aligned}$$

$$\begin{aligned}
|I_{221} - I_{222}| \leq & C \int_{\boldsymbol{\Omega}} \big(1 + |\boldsymbol{\varepsilon}_2| + |\boldsymbol{D}_2| + |\boldsymbol{\varepsilon}_D(t_2)| + |\boldsymbol{D}_\nu(t_2)| + |\boldsymbol{P}_2|\big) \\
& \cdot \big(|\boldsymbol{\varepsilon}'_D(t_1) - \boldsymbol{\varepsilon}'_D(t_2)| + |\boldsymbol{D}'_\nu(t_1) - \boldsymbol{D}'_\nu(t_2)|\big) d\boldsymbol{x} \\
\leq & C\big(1 + \|\boldsymbol{\varepsilon}_2\|_{L^q} + \|\boldsymbol{D}_2\|_{L^q} + \|\boldsymbol{\varepsilon}_D(t_2)\|_{L^q} + \|\boldsymbol{D}_\nu(t_2)\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big) \\
& \cdot \big(\|\boldsymbol{\varepsilon}'_D(t_1) - \boldsymbol{\varepsilon}'_D(t_2)\|_{L^q} + \|\boldsymbol{D}'_\nu(t_1) - \boldsymbol{D}'_\nu(t_2)\|_{L^q}\big) \\
\leq & C|t_1 - t_2|\big(1 + \|\boldsymbol{P}_2\|_{L^q}\big) \\
\leq & C|t_1 - t_2|\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big).
\end{aligned}$$

For the part $J(t, \boldsymbol{P})$, we write

$$\begin{aligned}
& J(t_1, \boldsymbol{P}_1) - J(t_2, \boldsymbol{P}_2) \\
= & \big(J(t_1, \boldsymbol{P}_1) - J(t_2, \boldsymbol{P}_1)\big) + \big(J(t_2, \boldsymbol{P}_1) - J(t_2, \boldsymbol{P}_2)\big) \\
=: & J_1 + J_2.
\end{aligned}$$

Similarly, it follows

$$\begin{aligned}
|J_1| \leq & C|t_1 - t_2|\big(\|\boldsymbol{u}_1\|_{W^{1,q'}} + \|\boldsymbol{D}_1\|_{L^{q'}} + \|\boldsymbol{P}_1\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\leq & C|t_1 - t_2|\big(\|\boldsymbol{u}_1\|_{W^{1,q}} + \|\boldsymbol{D}_1\|_{L^q} + \|\boldsymbol{P}_1\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\leq & C|t_1 - t_2|\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q} + \|\boldsymbol{P}_1\|^{\Sigma}_{L^{(q^*)'}} + \|\boldsymbol{P}_2\|^{\Sigma}_{L^{(q^*)'}}\big),
\end{aligned}$$

$$\begin{aligned}
|J_2| \leq & C\big(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,p}} + \|\boldsymbol{D}_1 - \boldsymbol{D}_2\|_{L^p} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\leq & C\big((1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q})\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r}\big) + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|^{\Sigma}_{L^{(q^*)'}}\big) \\
\leq & C\big(1 + \|\boldsymbol{P}_1\|_{L^q} + \|\boldsymbol{P}_2\|_{L^q}\big)\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|^{\Sigma}_{L^{(q^*)'}}\big).
\end{aligned}$$

Sum up all, we obtain the desired result. □

3.5.2 Differentiability of $\mathcal{I}$ w.r.t. $\boldsymbol{P}$

Lemma 3.23. *Let the Assumptions A1 to A6 be satisfied. Let*

$$\tilde{\mathcal{I}}(t, \boldsymbol{P}) := \mathcal{I}(t, \boldsymbol{P}) - \frac{1}{2}\kappa\langle \boldsymbol{P}, \boldsymbol{P}\rangle_s.$$

Then for every $t \in [0,T]$, $\boldsymbol{P} \to \tilde{\mathcal{I}}(t,\boldsymbol{P})$ *is Gâteaux-differentiable in* $\big(H^s(\boldsymbol{\Omega})\big)^d$ *and*

$$\begin{aligned} &D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t,\boldsymbol{P})[\bar{\boldsymbol{P}}] \\ =&\int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}U_1\Big(t,\boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t,\boldsymbol{P})\big),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\Big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\omega(\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x} - l_3^2(t,\bar{\boldsymbol{P}}) \end{aligned} \tag{3.49}$$

for every $\boldsymbol{P},\bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$. *Furthermore, for* $t_1,t_2,t \in [0,T]$, $\boldsymbol{P}_1,\boldsymbol{P}_2,\bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ *and* $p \in (2,q)$*, where* q *is given by Lemma 3.19, we have*

$$\big|D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t,\boldsymbol{P})[\bar{\boldsymbol{P}}]\big| \le C\big(1+\|\boldsymbol{P}\|_{L^q}^2+\|\boldsymbol{P}\|_{L^6}^5\big)\big(\|\bar{\boldsymbol{P}}\|_{L^{r_1}}+\|\bar{\boldsymbol{P}}\|_{L^6}+\|\bar{\boldsymbol{P}}\|_{L^{(q^*)'}}^{\Sigma}\big), \tag{3.50}$$

where $\|\boldsymbol{P}\|_{L^{(q^*)'}}^{\Sigma}$ *is given by* (3.45)*;*

$$\begin{aligned} &\big|D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t_1,\boldsymbol{P}_1)[\bar{\boldsymbol{P}}] - D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t_2,\boldsymbol{P}_2)[\bar{\boldsymbol{P}}]\big| \\ \le& C\gamma(\boldsymbol{P}_1,\boldsymbol{P}_2)\big(|t_1-t_2|+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^r}+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^{r_2}}+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^6}\big) \\ &\cdot\big(\|\bar{\boldsymbol{P}}\|_{L^{r_2}}+\|\bar{\boldsymbol{P}}\|_{L^{r_3}}+\|\bar{\boldsymbol{P}}\|_{L^6}+\|\bar{\boldsymbol{P}}\|_{L^{(q^*)'}}^{\Sigma}\big), \end{aligned} \tag{3.51}$$

where $r = \frac{qp}{q-p}, r_1 = \frac{q}{q-2}, r_2 = \frac{2q}{q-2}, r_3 = \frac{qp}{qp-(q+p)}$ *and*

$$\gamma(\boldsymbol{P}_1,\boldsymbol{P}_2) = 1 + \sum_{i=1}^{2}\|\boldsymbol{P}_i\|_{L^q}^2 + \sum_{i=1}^{2}\|\boldsymbol{P}_i\|_{L^6}^4.$$

Proof. The proof of (3.49) is essentially the same as the proof of (3.43). Since the adoption of the proof is straightforward but tedious, we omit the details here. Writing U_1 in (3.49) explicitly we obtain that

$$\begin{aligned} D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t,\boldsymbol{P})[\bar{\boldsymbol{P}}] =& \int_{\boldsymbol{\Omega}} \Bigg(\frac{1}{2}D_{\boldsymbol{P}}\mathbb{B}_2(\boldsymbol{P})\bar{\boldsymbol{P}}\begin{pmatrix}\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}+\boldsymbol{D}_\nu(t)-\boldsymbol{P}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}+\boldsymbol{D}_\nu(t)-\boldsymbol{P}\end{pmatrix}\Bigg) \\ &+\Bigg(\mathbb{B}_2(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}+\boldsymbol{D}_\nu(t)-\boldsymbol{P}\end{pmatrix} : \begin{pmatrix}-D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0(\boldsymbol{P})\bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{P}}\end{pmatrix}\Bigg) \\ &+\Big(D_{\boldsymbol{P}}\omega(\boldsymbol{P})(\bar{\boldsymbol{P}})\Big)d\boldsymbol{x} + \Big(-l_3^2(t,\bar{\boldsymbol{P}})\Big) \\ =:& I_1+I_2+I_3+I_4, \end{aligned} \tag{3.52}$$

where $(\boldsymbol{\varepsilon},\boldsymbol{D}) = \big(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\min}),\boldsymbol{D}_{\min}\big)$. Then estimating similarly as in Lemma 3.20, we deduce that

$$
\begin{aligned}
|I_1| &\le C \int_{\mathbf{\Omega}} |\bar{\boldsymbol{P}}|\big(1+|\boldsymbol{\varepsilon}|+|\boldsymbol{D}|+|\boldsymbol{\varepsilon}_D(t)|+|\boldsymbol{D}_\nu(t)|+|\boldsymbol{P}|\big)^2 d\boldsymbol{x}\\
&\le C\|\boldsymbol{P}\|_{r_1}\big(1+\|\boldsymbol{\varepsilon}\|_{L^q}+\|\boldsymbol{D}\|_{L^q}+\|\boldsymbol{P}\|_{L^q}\big)^2\\
&\le C\big(1+\|\boldsymbol{P}\|^2_{L^q}\big)\|\boldsymbol{P}\|_{r_1},
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &\le C \int_{\mathbf{\Omega}} |\bar{\boldsymbol{P}}|\big(1+|\boldsymbol{\varepsilon}|+|\boldsymbol{D}|+|\boldsymbol{\varepsilon}_D(t)|+|\boldsymbol{D}_\nu(t)|+|\boldsymbol{P}|\big) d\boldsymbol{x}\\
&\le C\|\boldsymbol{P}\|_{r_1}\big(1+\|\boldsymbol{\varepsilon}\|_{L^q}+\|\boldsymbol{D}\|_{L^q}+\|\boldsymbol{P}\|_{L^q}\big)\\
&\le C\big(1+\|\boldsymbol{P}\|_{L^q}\big)\|\boldsymbol{P}\|_{r_1}\\
&\le C\big(1+\|\boldsymbol{P}\|^2_{L^q}\big)\|\boldsymbol{P}\|_{r_1},
\end{aligned}
$$

$$
\begin{aligned}
|I_3| &\le C \int_{\mathbf{\Omega}} \big(1+|\boldsymbol{P}|^5\big)|\bar{\boldsymbol{P}}| d\boldsymbol{x}\\
&\le C\big(1+\|\boldsymbol{P}\|^5_{L^6}\big)\|\bar{\boldsymbol{P}}\|_{L^6},
\end{aligned}
$$

$$
|I_4| \le C\|\bar{\boldsymbol{P}}\|^{\Sigma}_{L^{(q^*)'}},
$$

where $r_1 = \frac{q}{q-2}$. Summing up we obtain (3.50). To obtain (3.51), we first estimate the difference terms

$$
\begin{aligned}
\Delta_1 &= \big|I_1(t_1,\boldsymbol{P}_1)[\bar{\boldsymbol{P}}] - I_1(t_2,\boldsymbol{P}_2)[\bar{\boldsymbol{P}}]\big|,\\
\Delta_2 &= \big|I_2(t_1,\boldsymbol{P}_1)[\bar{\boldsymbol{P}}] - I_2(t_2,\boldsymbol{P}_2)[\bar{\boldsymbol{P}}]|.
\end{aligned}
$$

It suffices to estimate the following summands

$$
J_1 = \int_{\mathbf{\Omega}} |\bar{\boldsymbol{P}}||\boldsymbol{P}_1-\boldsymbol{P}_2|\big(1+|\boldsymbol{\varepsilon}_1|+|\boldsymbol{D}_1|+|\boldsymbol{\varepsilon}_D(t_1)|+|\boldsymbol{D}_\nu(t_1)|+|\boldsymbol{P}_1|\big)^2 d\boldsymbol{x}
$$

and

$$
\begin{aligned}
J_2 = \int_{\mathbf{\Omega}} &|\bar{\boldsymbol{P}}|\big(1+|\boldsymbol{\varepsilon}_1|+|\boldsymbol{D}_1|+|\boldsymbol{\varepsilon}_D(t_1)|+|\boldsymbol{D}_\nu(t_1)|+|\boldsymbol{P}_1|\big)\cdot\big(|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|\\
&+|\boldsymbol{D}_1-\boldsymbol{D}_2|+|\boldsymbol{\varepsilon}_D(t_1)-\boldsymbol{\varepsilon}_D(t_2)|+|\boldsymbol{D}_\nu(t_1)-\boldsymbol{D}_\nu(t_2)|+|\boldsymbol{P}_1-\boldsymbol{P}_2|\big)d\boldsymbol{x}
\end{aligned}
$$

given in Δ_1 and Δ_2, since the other summands can be estimated analogously. Here we used the notation

$$
(\boldsymbol{\varepsilon}_i,\boldsymbol{D}_i) = \Big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}_{\min}(t_i,\boldsymbol{P}_i)\big),\boldsymbol{D}_{\min}(t_i,\boldsymbol{P}_i)\Big).
$$

We then obtain from Hölder's inequality that

$$
\begin{aligned}
|J_1| \le& C\|\bar{\boldsymbol{P}}\|_{L^{r_2}}\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^{r_2}}\big(1+\|\boldsymbol{\varepsilon}_1\|_{L^q}+\|\boldsymbol{D}_1\|_{L^q}+\|\boldsymbol{P}_1\|_{L^q}\big)^2\\
\le& C\|\bar{\boldsymbol{P}}\|_{L^{r_2}}\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^{r_2}}\big(1+\|\boldsymbol{P}_1\|^2_{L^q}+\|\boldsymbol{P}\|^2_{L^q}\big),
\end{aligned}
$$

$$
\begin{aligned}
|J_2| \le& C\|\bar{\boldsymbol{P}}\|_{L^{r_3}}\big(1+\|\boldsymbol{\varepsilon}_1\|_{L^q}+\|\boldsymbol{D}_1\|_{L^q}+\|\boldsymbol{P}_1\|_{L^q}\big)\cdot\big(\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|_{L^p}\\
&+\|\boldsymbol{D}_1-\boldsymbol{D}_2\|_{L^p}+\|\boldsymbol{\varepsilon}_D(t_1)-\boldsymbol{\varepsilon}_D(t_2)\|_{L^p}+\|\boldsymbol{D}_\nu(t_1)-\boldsymbol{D}_\nu(t_2)\|_{L^p}+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^p}\big)\\
\le& C\|\bar{\boldsymbol{P}}\|_{L^{r_3}}\big(1+\|\boldsymbol{P}_1\|^2_{L^q}+\|\boldsymbol{P}_2\|^2_{L^q}\big)\big(|t_1-t_2|+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^r}+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^p}\big)\\
\le& C\|\bar{\boldsymbol{P}}\|_{L^{r_3}}\big(1+\|\boldsymbol{P}_1\|^2_{L^q}+\|\boldsymbol{P}_2\|^2_{L^q}\big)\big(|t_1-t_2|+\|\boldsymbol{P}_1-\boldsymbol{P}_2\|_{L^r}\big),
\end{aligned}
$$

where $r = \frac{qp}{q-p}$, $r_2 = \frac{2q}{q-2}$ and $r_3 = \frac{qp}{qp-(q+p)}$ with

$$\frac{1}{r_2} + \frac{1}{r_2} + \frac{1}{q} + \frac{1}{q} = 1,$$
$$\frac{1}{r_3} + \frac{1}{q} + \frac{1}{p} = 1.$$

For I_3, I_4, it follows

$$\begin{aligned}
&\big|I_3(t_1, \boldsymbol{P}_1)[\bar{\boldsymbol{P}}] - I_3(t_2, \boldsymbol{P}_2)[\bar{\boldsymbol{P}}]\big| \\
\leq &\int_{\boldsymbol{\Omega}} |D_{\boldsymbol{P}}\omega(\boldsymbol{P}_1) - D_{\boldsymbol{P}}\omega(\boldsymbol{P}_2)||\bar{\boldsymbol{P}}|d\boldsymbol{x} \\
\leq &C \int_{\boldsymbol{\Omega}} \big(1 + |\boldsymbol{P}_1|^4 + |\boldsymbol{P}_2|^4\big)|\boldsymbol{P}_1 - \boldsymbol{P}_2||\bar{\boldsymbol{P}}|d\boldsymbol{x} \\
\leq &C\big(1 + \|\boldsymbol{P}_1\|_{L^6}^4 + \|\boldsymbol{P}_2\|_{L^6}^4\big)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^6}\|\bar{\boldsymbol{P}}\|_{L^6},
\end{aligned}$$

$$\begin{aligned}
&\big|I_4(t_1, \boldsymbol{P}_1)[\bar{\boldsymbol{P}}] - I_4(t_2, \boldsymbol{P}_2)[\bar{\boldsymbol{P}}]\big| \\
\leq &C|t_1 - t_2|\|\bar{\boldsymbol{P}}\|_{L^{(q^*)'}}^{\Sigma}.
\end{aligned}$$

Sum up all, we obtain (3.51). □

Finally, using Lemma 3.22 and Lemma 3.23 we obtain the following sequential convergence result, which will be essential in the analysis of the construction of viscous solutions.

Corollary 3.24. *Let the Assumptions A1 to A6 be satisfied. Let $t_n \to t$ and $\boldsymbol{P}_n \rightharpoonup \boldsymbol{P}$ in $\big(H^s(\boldsymbol{\Omega})\big)^d$ as $n \to \infty$. Then we have*

- $\liminf_{n\to\infty} \mathcal{I}(t_n, \boldsymbol{P}_n) \geq \mathcal{I}(t, \boldsymbol{P})$.
- $\tilde{\mathcal{I}}(t_n, \boldsymbol{P}_n) \to \tilde{\mathcal{I}}(t, \boldsymbol{P})$ *as* $n \to \infty$.
- $\partial_t\mathcal{I}(t_n, \boldsymbol{P}_n) \to \partial_t\mathcal{I}(t, \boldsymbol{P})$ *as* $n \to \infty$.
- $D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t_n, \boldsymbol{P}_n) \to D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t, \boldsymbol{P})$ *strongly in* $[\big(H^s(\boldsymbol{\Omega})\big)^d]^*$ *as* $n \to \infty$,

where the functional $\tilde{\mathcal{I}}$ is defined by Lemma 3.23 and $[\big(H^s(\boldsymbol{\Omega})\big)^d]^*$ is the dual space of $(H^s(\boldsymbol{\Omega}))^d$.

Proof. This is a direct consequence of Lemma 3.22, Lemma 3.23 and the compact embeddings given in Lemma 3.5. □

3.6 Time discrete interpolant solutions

We now construct a sequence of time discrete interpolant solutions which are temporarily piecewise affine interpolants and converge weakly to the desired solution of the main problem (3.34). We first show the coerciveness of the functional $\mathcal{I}$ w.r.t. $\boldsymbol{P}$.

Lemma 3.25. *Let the Assumptions A1 to A6 be satisfied. Then $\mathcal{I}(t, \boldsymbol{P})$ is coercive w.r.t. $\boldsymbol{P}$ for all $t \in [0, T]$, i.e., there exist $C > 0, C' \in \mathbb{R}$ such that*

$$\mathcal{I}(t, \boldsymbol{P}) \geq C\|\boldsymbol{P}\|_{H^s}^2 + C' \tag{3.53}$$

holds for all $t \in [0, T]$.

Proof. Recall that

$$\mathcal{I}(t,\boldsymbol{P}) = \mathcal{E}\big(t,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\big)$$

and

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 - l_3.$$

Since $\mathcal{E}_1$ is nonnegative due to its definition, we have

$$\mathcal{I}(t,\boldsymbol{P}) \geq \mathcal{E}_2(\boldsymbol{P}) - l_3(t,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}). \tag{3.54}$$

Due to the Assumption A6 it follows that there is some $C>0$ and $C' \in \mathbb{R}$ such that

$$\mathcal{E}_2(\boldsymbol{P}) \geq C\big(\langle \boldsymbol{P},\boldsymbol{P}\rangle_s + \|\boldsymbol{P}\|_{L^2}^2\big) + C' = C\|\boldsymbol{P}\|_{H^s}^2 + C', \tag{3.55}$$

since $\sqrt{\langle\cdot,\cdot\rangle_s + \|\cdot\|_{L^2}^2}$ defines an equivalent norm on $\big(H^s(\boldsymbol{\Omega})\big)^d$, see Lemma C.12. On the other hand, it follows from (3.35), the Assumption A5 and the definition of l_3 given by (3.17) that

$$|l_3\big(t,\boldsymbol{u}_{\min}(t,\boldsymbol{P}),\boldsymbol{D}_{\min}(t,\boldsymbol{P}),\boldsymbol{P}\big)| \leq C\big(1+\|\boldsymbol{P}\|_{H^s}\big) \leq C + \varepsilon\|\boldsymbol{P}\|_{H^s}^2 \tag{3.56}$$

for arbitrary $\varepsilon > 0$, where the second inequality of (3.56) is deduced from Young's inequality. Combining (3.55) and (3.56), we obtain that

$$\mathcal{I}(t,\boldsymbol{P}) \geq (C-\varepsilon)\|\boldsymbol{P}\|_{H^s}^2 + C'.$$

The coercivity follows immediately by choosing sufficiently small ε. □

The following lemma will be essential for a Gronwall-type inequality (Lemma 3.28) later on.

Lemma 3.26. *Let the Assumptions A1 to A6 be satisfied. Then there exist constants $c_0, c_1 > 0$ independent on $\boldsymbol{P}$ such that for all $t \in [0,T]$ and $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$*

$$|\partial_t \mathcal{I}(t,\boldsymbol{P})| \leq c_1\big(\mathcal{I}(t,\boldsymbol{P}) + c_0\big). \tag{3.57}$$

Proof. We obtain that

$$|\partial_t \mathcal{I}(t,\boldsymbol{P})| \leq C\big(1+\|\boldsymbol{P}\|_{H^s}\big) \leq C\big(1+\|\boldsymbol{P}\|_{H^s}^2\big) \leq C\big(1+\mathcal{I}(t,\boldsymbol{P})\big),$$

where the first inequality comes from (3.44) and Sobolev's embedding (see Lemma 3.5), the second from Young's inequality and the last from (3.53). □

Lemma 3.27. *Let the Assumptions A1 to A7 be satisfied and let Ψ_β be the functional defined by (3.6). Then for each $t \in [0,T]$, $\tau > 0$ and $\bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, the functional $\mathcal{I}(t,\boldsymbol{P}) + \tau\Psi_\beta(\frac{\boldsymbol{P}-\bar{\boldsymbol{P}}}{\tau})$ has a minimizer $\boldsymbol{P}$ in $\big(H^s(\boldsymbol{\Omega})\big)^d$.*

Proof. Since Ψ_β is nonnegative, the functional $\mathcal{I}(t,\boldsymbol{P}) + \tau\Psi_\beta(\frac{\boldsymbol{P}-\bar{\boldsymbol{P}}}{\tau})$ is coercive w.r.t. $\boldsymbol{P}$ due to Lemma 3.25. From Corollary 3.24 and the definition of Ψ_β we know that the functional $\mathcal{I}(t,\boldsymbol{P}) + \tau\Psi_\beta(\frac{\boldsymbol{P}-\bar{\boldsymbol{P}}}{\tau})$ is weakly lower semi-continuous. Then the existence of a minimizer follows from Tonelli's abstract existence theorem. □

In what follows, we will use certain Rothe's method (analogous to the one given in [39]) to construct a viscous solution of (3.34) for every $\beta > 0$. Given $\beta > 0$, $\boldsymbol{P}_0 \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and a partition $\{0 = t_0 < ... < t_N = T\}$ of $[0, T]$ with fineness $\tau = \sup_{0 \le k \le N-1}(t_{k+1} - t_k)$, the sequence $(\boldsymbol{P}_k^\tau)_{0 \le k \le N}$ is defined inductively by $\boldsymbol{P}_0^\tau = \boldsymbol{P}_0$ and

$$\boldsymbol{P}_{k+1}^\tau \in \mathrm{Argmin}_{\boldsymbol{P} \in (H^s(\boldsymbol{\Omega}))^d} \{\mathcal{I}(t_{k+1}^\tau, \boldsymbol{P}) + \tau_k \Psi_\beta\Big(\frac{\boldsymbol{P} - \boldsymbol{P}_k^\tau}{\tau_k}\Big)\}$$

with $\tau_k := t_{k+1}^\tau - t_k^\tau$, $k = 0, 1, ..., N-1$ (the existence of $\boldsymbol{P}_{k+1}^\tau$ is guaranteed by Lemma 3.27). Since $\boldsymbol{P}_{k+1}^\tau$ are chosen to be minimizers, it reads that

$$\mathbf{0} \in \partial\Psi_1\Big(\frac{\boldsymbol{P}_{k+1}^\tau - \boldsymbol{P}_k^\tau}{\tau_k}\Big) + \beta\frac{\boldsymbol{P}_{k+1}^\tau - \boldsymbol{P}_k^\tau}{\tau_k} + D_{\boldsymbol{P}}\mathcal{I}(t_{k+1}^\tau, \boldsymbol{P}_{k+1}^\tau). \tag{3.58}$$

We define

$$\begin{aligned}
&\bar{\boldsymbol{P}}_\tau(t) = \boldsymbol{P}_{k+1}^\tau, && \text{for } t \in (t_k^\tau, t_{k+1}^\tau],\\
&\underline{\boldsymbol{P}}_\tau(t) = \boldsymbol{P}_k^\tau, && \text{for } t \in [t_k^\tau, t_{k+1}^\tau),\\
&\hat{\boldsymbol{P}}_\tau(t) = \boldsymbol{P}_k^\tau + \frac{t - t_k^\tau}{\tau_k}(\boldsymbol{P}_{k+1}^\tau - \boldsymbol{P}_k^\tau), && \text{for } t \in [t_k^\tau, t_{k+1}^\tau],\\
&\tau(t) = \tau_k, && \text{for } t \in (t_k^\tau, t_{k+1}^\tau),\\
&\bar{t}_\tau(t) = t_{k+1}^\tau, && \text{for } t \in (t_k^\tau, t_{k+1}^\tau],\\
&\underline{t}_\tau(t) = t_k^\tau, && \text{for } t \in [t_k^\tau, t_{k+1}^\tau).
\end{aligned} \tag{3.59}$$

Then (3.58) can be rewritten to

$$\mathbf{0} \in \partial\Psi_1\big(\hat{\boldsymbol{P}}_\tau'(t)\big) + \beta\hat{\boldsymbol{P}}_\tau'(t) + D_{\boldsymbol{P}}\mathcal{I}\big(\bar{t}_\tau(t), \bar{\boldsymbol{P}}_\tau(t)\big) \qquad \text{for a.a } t \in (0, T). \tag{3.60}$$

Lemma 3.28. *Let the Assumptions A1 to A7 be satisfied. Denote by* $e_k := \mathcal{I}(t_k, \boldsymbol{P}_k^\tau)$, $k = 1, ..., N$. *Then we have*

$$\mathcal{I}(t_k, \boldsymbol{P}_k^\tau) \le (e_0 + c_0)\exp(c_1 t_k) - c_0 \tag{3.61}$$

and

$$\sum_{j=1}^{k} \tau_{j-1}\Psi_\beta\Big(\frac{\boldsymbol{P}_j^\tau - \boldsymbol{P}_{j-1}^\tau}{\tau_{j-1}}\Big) \le (e_0 + c_0)\exp(c_1 t_k), \tag{3.62}$$

where c_0, c_1 *are defined by Lemma 3.26. In particular,* $\|\boldsymbol{P}_k^\tau\|_{H^s} \le C$ *with some positive constant* C *for all* $k = 1, ..., N$ *and all* $\tau > 0$.

Proof. The idea is to apply a discrete type Gronwall inequality to the inequality (3.57), which is similarly done as in the proof of [19, Thm 3.2]. More precisely, one only has to replace the dissipation functional $\mathcal{D}(z_{j-1}, z_j)$ in [19, Thm 3.2] by $\tau_j\Psi_\beta\Big(\frac{\boldsymbol{P}_{j-1} - \boldsymbol{P}_j}{\tau_j}\Big)$ and the proof of [19, Thm 3.2] still remains true. Since the adoption of the proof is trivial, we refer the details to [19]. □

Lemma 3.29. *Let the Assumptions A1 to A7 be satisfied. Let* $\hat{\boldsymbol{P}}_\tau$, $\bar{\boldsymbol{P}}_\tau$, $\underline{\boldsymbol{P}}_\tau$ *be given as in* (3.59). *Suppose also that* $D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)$ *is of class* L^2. *Then there exist constants* $C, C' > 0$

such that

$$\max\{\|\hat{\boldsymbol{P}}_\tau\|_{L^\infty(0,T;(H^s(\boldsymbol{\Omega}))^d)}, \|\bar{\boldsymbol{P}}_\tau\|_{L^\infty(0,T;(H^s(\boldsymbol{\Omega}))^d)}, \|\underline{\boldsymbol{P}}_\tau\|_{L^\infty(0,T;(H^s(\boldsymbol{\Omega}))^d)}\} \le C, \tag{3.63}$$

$$\|\hat{\boldsymbol{P}}'_\tau\|_{L^2(0,T;(L^2(\boldsymbol{\Omega}))^d)} \le \frac{C}{\sqrt{\beta}}, \tag{3.64}$$

$$\begin{aligned}&\max\{\|\bar{\boldsymbol{P}}_\tau - \hat{\boldsymbol{P}}_\tau\|_{L^\infty(0,T;(H^s(\boldsymbol{\Omega}))^d)}, \|\underline{\boldsymbol{P}}_\tau - \hat{\boldsymbol{P}}_\tau\|_{L^\infty(0,T;(H^s(\boldsymbol{\Omega}))^d)}\} \\ \le& C\sqrt{\tau}\|\hat{\boldsymbol{P}}'_\tau\|_{L^2(0,T;(H^s(\boldsymbol{\Omega}))^d)},\end{aligned} \tag{3.65}$$

$$\int_0^T \Psi_1(\hat{\boldsymbol{P}}'_\tau(t))dt + \Psi_{2,\beta}(\hat{\boldsymbol{P}}'_\tau(t))dt \le C, \tag{3.66}$$

$$\|\hat{\boldsymbol{P}}'_\tau\|_{L^2(0,T;(H^s(\boldsymbol{\Omega}))^d)} \le \sqrt{\frac{C}{\beta} + \frac{1}{\beta}\|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|^2_{L^2} + C'}. \tag{3.67}$$

Proof. (3.63) follows directly from Lemma 3.28, the coercivity of $\mathcal{I}$ and the definition of $\hat{\boldsymbol{P}}_\tau, \bar{\boldsymbol{P}}_\tau, \underline{\boldsymbol{P}}_\tau$. Using the definition of $\hat{\boldsymbol{P}}_\tau$, we see that (3.62) implies that

$$\int_0^T \Psi_1\big(\hat{\boldsymbol{P}}'_\tau(t)\big)dt + \Psi_{2,\beta}\big(\hat{\boldsymbol{P}}'_\tau(t)\big)dt \le C,$$

from which we obtain (3.66). Bounding the Ψ_1-term in (3.66) from below by zero, we see that

$$C \ge \int_0^T \Psi_{2,\beta}\big(\hat{\boldsymbol{P}}'_\tau(t)\big)dt = \beta\int_0^T \|\hat{\boldsymbol{P}}'_\tau(t)\|^2_{L^2}dt,$$

from which (3.64) follows. Next we show (3.65). Let $t \in [t_{k-1}, t_k)$ and $\boldsymbol{x} \in \boldsymbol{\Omega}$. From the definition of the interpolant functions given by (3.59) we obtain that

$$\begin{aligned}&\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x}) - \hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x}) \\ =&\boldsymbol{P}^\tau_k(\boldsymbol{x}) - \boldsymbol{P}^\tau_{k-1}(\boldsymbol{x}) - \frac{t-t_{k-1}}{\tau_{k-1}}\big(\boldsymbol{P}^\tau_k(\boldsymbol{x}) - \boldsymbol{P}^\tau_{k-1}(\boldsymbol{x})\big) \\ =&\big(\tau_{k-1} - (t-t_{k-1})\big)\hat{\boldsymbol{P}}'_\tau(t,\boldsymbol{x})\end{aligned}$$

and

$$\begin{aligned}&\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x}) - \nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x}) \\ =&\nabla\boldsymbol{P}^\tau_k(\boldsymbol{x}) - \nabla\boldsymbol{P}^\tau_{k-1}(\boldsymbol{x}) - \frac{t-t_{k-1}}{\tau_{k-1}}\big(\nabla\boldsymbol{P}^\tau_k(\boldsymbol{x}) - \nabla\boldsymbol{P}^\tau_{k-1}(\boldsymbol{x})\big) \\ =&\big(\tau_{k-1} - (t-t_{k-1})\big)\nabla\hat{\boldsymbol{P}}'_\tau(t,\boldsymbol{x}).\end{aligned}$$

Then for $t \in [t_{k-1}, t_k)$ we have

$$\begin{aligned}&|\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x}) - \hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x})| \\ \le&\tau_{k-1}|\hat{\boldsymbol{P}}'_\tau(t,\boldsymbol{x})| = \int_{t_{k-1}}^{t_k} |\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{x})|d\sigma \le \sqrt{\tau}\Big(\int_{t_{k-1}}^{t_k} |\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{x})|^2 d\sigma\Big)^{\frac{1}{2}}\end{aligned} \tag{3.68}$$

and

$$\begin{aligned}&|\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x})-\nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x})|\\ \le&\tau_{k-1}|\nabla\hat{\boldsymbol{P}}'_\tau(t,\boldsymbol{x})|=\int_{t_{k-1}}^{t_k}|\nabla\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{x})|d\sigma\le\sqrt{\tau}\Big(\int_{t_{k-1}}^{t_k}|\nabla\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{x})|^2d\sigma\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.69}$$

Replace $\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x})$ and $\nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x})$ in (3.69) by $\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x})-\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{y})$ and $\nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x})-\nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{y})$ with $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{\Omega}$ respectively and then divide them by $|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s\rfloor)}$, we obtain from the first inequality of (3.69) that

$$\begin{aligned}&\frac{\big|\big(\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{x})-\nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{x})\big)-\big(\nabla\bar{\boldsymbol{P}}_\tau(t,\boldsymbol{y})-\nabla\hat{\boldsymbol{P}}_\tau(t,\boldsymbol{y})\big)\big|}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s\rfloor)}}\\ \le&\tau_{k-1}\frac{|\nabla\hat{\boldsymbol{P}}'_\tau(t,\boldsymbol{x})-\nabla\hat{\boldsymbol{P}}'_\tau(t,\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s\rfloor)}}\\ =&\int_{t_{k-1}}^{t_k}\frac{|\nabla\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{x})-\nabla\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s\rfloor)}}d\sigma\\ \le&\sqrt{\tau}\Big(\int_{t_{k-1}}^{t_k}\Big(\frac{|\nabla\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{x})-\nabla\hat{\boldsymbol{P}}'_\tau(\sigma,\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s\rfloor)}}\Big)^2d\sigma\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.70}$$

Now squaring (3.68) to (3.70), integrating (3.68) and (3.69) over $\boldsymbol{\Omega}$ w.r.t. $\boldsymbol{x}$ and (3.70) over $\boldsymbol{\Omega}^2$ w.r.t. $(\boldsymbol{x},\boldsymbol{y})$, summing them up ((3.68) and (3.69) in the case $s=1$, or (3.68) and (3.70) in the case $s\neq 1$), and taking supremum over $t\in[0,T]$, we conclude that

$$\sup_{t\in[0,T]}\|\bar{\boldsymbol{P}}_\tau(t)-\hat{\boldsymbol{P}}_\tau(t)\|^2_{H^s}\le\tau\int_0^T\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s}d\sigma.$$

For $\underline{\boldsymbol{P}}_\tau(t)$ one can show the same estimate similarly. Taking finally square roots on both sides, it follows (3.65). It is left to show (3.67). First we state the following inequality from [39, Prop. 4.1, (4.11)]:

$$\begin{aligned}\frac{\beta}{2}\|\hat{\boldsymbol{P}}'_\tau(t)\|^2_{L^2}+\int_0^{\bar{t}_\tau(t)}\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s}d\sigma&\\ \le\frac{1}{2\beta}\|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|^2_{L^2}&+\int_0^{\bar{t}_\tau(t)}\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^2}d\sigma\\ &-\int_0^{\bar{t}_\tau(t)}\frac{1}{\tau(\sigma)}\Big\langle\Delta_{\tau(\sigma)}D_{\boldsymbol{P}}\tilde{\mathcal{I}}\big(\bar{t}_\tau(\sigma),\bar{\boldsymbol{P}}_\tau(\sigma)\big),\hat{\boldsymbol{P}}'_\tau(\sigma)\Big\rangle_{H^s}d\sigma,\end{aligned} \tag{3.71}$$

where for a time partition τ and a function b which is constant on the intervals (t^τ_i,t^τ_{i+1}), the operator $\Delta_{\tau(\cdot)}b(\cdot)$ is defined by

$$\Delta_{\tau(\sigma)}b(\sigma)=b(\sigma)-b(\sigma')$$

for $\sigma\in(t^\tau_k,t^\tau_{k+1})$ and $\sigma'\in(t^\tau_{k-1},t^\tau_k)$, and for $k=0$, $\Delta_{\tau(\sigma)}b(\sigma)=b(\sigma)-b(0)$, and $\langle\cdot,\cdot\rangle_{H^s}$ denotes the dual product in $\big(H^s(\boldsymbol{\Omega})\big)^d$; see also the notation given by (3.59). Instead of giving the complete proof of (3.71), we would rather refer to [39, Prop. 4.1] for details and introduce here only the basic idea of showing the inequality (3.71): writing

$$\bar{h}_\tau(t):=\beta\hat{\boldsymbol{P}}'_\tau(t)+D_{\boldsymbol{P}}\mathcal{I}\big(\bar{t}_\tau(t),\bar{\boldsymbol{P}}_\tau(t)\big),$$

we obtain from (3.60) that

$$-\bar{h}_\tau(t) \in \partial\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(t)\big).$$

Using the 1-homogeneity of Ψ_1 we deduce that

$$\begin{aligned}
&\forall\, t \in (t^\tau_k, t^\tau_{k+1}): && -\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(t)\big) = \langle \bar{h}_\tau(t), \hat{\boldsymbol{P}}'_\tau(t)\rangle_{H^s};\\
&\forall\, r \in [0,T] \setminus \{t^\tau_0, ..., t^\tau_N\}: && \Psi_1\big(\hat{\boldsymbol{P}}'_\tau(t)\big) \ge \langle -\bar{h}_\tau(r), \hat{\boldsymbol{P}}'_\tau(t)\rangle_{H^s}.
\end{aligned}$$

Adding the both relations and then divide it by τ_i^{-1} whence

$$0 \ge \tau_i^{-1}\langle \bar{h}_\tau(\rho) - \bar{h}_\tau(\sigma), \hat{\boldsymbol{P}}'_\tau(\rho)\rangle_{H^s} \tag{3.72}$$

for $\rho \in (t^\tau_i, t^\tau_{i+1})$ and $\sigma \in (t^\tau_{i-1}, t^\tau_i)$. Summing (3.72) over $i = 1, ..., k$ and estimating in a subtle way (we omit the details here due to the complexity of calculation) we arrive finally at (3.71). Next, we prove that

$$\begin{aligned}
&\Big| \int_0^{\bar{t}_\tau(t)} \frac{1}{\tau(\sigma)} \Big\langle \Delta_{\tau(\sigma)} D_{\boldsymbol{P}}\tilde{\mathcal{I}}\big(\bar{t}_\tau(\sigma), \bar{\boldsymbol{P}}_\tau(\sigma)\big), \hat{\boldsymbol{P}}'_\tau(\sigma)\Big\rangle_{H^s} d\sigma\Big|\\
&\le C\int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^2} d\sigma + \frac{1}{2}\int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s} d\sigma + C'.
\end{aligned} \tag{3.73}$$

We make use of the Ehrling's lemma (see [67, Thm. 7.3, p.114]) to show (3.73), which is given as follows: let X_1, X_2, X_3 be normed spaces, $A: X_1 \to X_2$ compact, $T: X_2 \to X_3$ a continuous injection. Then for every $\rho > 0$ there exists a constant C_ρ with

$$\|Ax\|_2 \le \rho\|x\|_1 + C_\rho\|TAx\|_3$$

for all $x \in X_1$. Taking $2 < p < \infty$, $X_1 = H^s(\boldsymbol{\Omega})$, $X_2 = L^p(\boldsymbol{\Omega})$, $X_3 = L^2(\boldsymbol{\Omega})$ and A, T the identity embeddings, we obtain from the Ehrling's lemma (together with the fact that $H^s(\boldsymbol{\Omega})$ is compactly embedded to $L^2(\boldsymbol{\Omega})$) that

$$\|\boldsymbol{P}\|^2_{L^p} \le C_\rho\|\boldsymbol{P}\|^2_{L^2} + \rho\|\boldsymbol{P}\|^2_{H^s} \tag{3.74}$$

for all $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and all $\rho > 0$. Here $C_\rho > 0$ is some positive constant depending on ρ. By (3.51) and Sobolev's embedding we obtain that

$$\begin{aligned}
&\big|\langle D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t_1, \boldsymbol{P}_1) - D_{\boldsymbol{P}}\tilde{\mathcal{I}}(t_2, \boldsymbol{P}_2), \bar{\boldsymbol{P}}\rangle_{H^s}\big|\\
\le& C\|\bar{\boldsymbol{P}}\|_{H^s}\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^r} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^{r_2}} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^6}\big)\\
\le& C\|\bar{\boldsymbol{P}}\|_{H^s}\big(|t_1 - t_2| + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^{p_1}}\big)
\end{aligned} \tag{3.75}$$

for $t_1, t_2 \in [0,T]$, $\boldsymbol{P}_1, \boldsymbol{P}_2, \bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, where $p_1 := \max\{r, r_2, 6\} > 2$ and r, r_2 are given by Lemma 3.23. Taking $\boldsymbol{P}_1 = \boldsymbol{P}^\tau_{k+1}$, $\boldsymbol{P}_2 = \boldsymbol{P}^\tau_k$, $\bar{\boldsymbol{P}} = \hat{\boldsymbol{P}}'_\tau(\sigma)$, $t_1 = t^\tau_{k+1}$, $t_2 = t^\tau_k$ and divide (3.75) by $\tau(\sigma)$ with $\sigma \in (t^\tau_k, t^\tau_{k+1})$, we obtain from (3.59) that

$$\begin{aligned}
&\Big|\frac{1}{\tau(\sigma)}\Big\langle \Delta_{\tau(\sigma)} D_{\boldsymbol{P}}\tilde{\mathcal{I}}\big(\bar{t}_\tau(\sigma), \bar{\boldsymbol{P}}_\tau(r)\big), \hat{\boldsymbol{P}}'_\tau(\sigma)\Big\rangle_{H^s}\Big|\\
\le& C\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|_{H^s}\big(1 + \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|_{L^{p_1}}\big)\\
\le& \rho\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s} + C_\rho\big(1 + \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^{p_1}}\big)\\
\le& C_\rho + C_\rho C_{\rho'}\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^2} + (\rho + C_\rho\rho')\|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s}
\end{aligned} \tag{3.76}$$

for any $\rho, \rho' > 0$, where $C_\rho, C_{\rho'} > 0$ are some positive constants depending on ρ, ρ' respectively. Here, the last inequality follows directly from (3.74) by setting $p = p_1$ in (3.74). Then (3.73) follows by choosing $\rho + C_\rho \rho' = \frac{1}{2}$ in (3.76) and then integrating both sides of (3.76) over $[0, \bar{t}_\tau(t)]$ w.r.t. σ. Now estimating the last term of (3.71) by (3.73) and neglecting $\frac{\beta}{2}\|\hat{\boldsymbol{P}}'_\tau(t)\|^2_{L^2}$ in (3.71), we obtain that

$$\int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s} d\sigma \leq C \int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^2} d\sigma + \frac{1}{\beta}\|D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)\|^2_{L^2} + C'. \tag{3.77}$$

Finally, taking $t = T$ in $\bar{t}_\tau(t)$ and using (3.64) to bound the first term in the r.h.s. of (3.77), it follows (3.67). □

3.7 Existence of viscous solutions

In the following we show the existence of viscous solution of (3.34) for each given $\beta > 0$. The proof follows the lines of [39, Thm. 4.2].

Theorem 3.30. *Let the Assumptions A1 to A7 be satisfied and $\beta > 0$ be an arbitrary positive constant. Let $\{\hat{\boldsymbol{P}}_{\tau^j}\}_{j\in\mathbb{N}} \subset H^1(0,T; (H^s(\boldsymbol{\Omega}))^d)$ be a sequence of piecewise affine interpolants defined by (3.59) corresponding to β and with $\tau^j \to 0$ as $j \to \infty$. Suppose also that $\boldsymbol{P}_0 \in (H^s(\boldsymbol{\Omega}))^d$ and $D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)$ is of class L^2. Then there exist a subsequence $\{\tau^{j_k}\}_{k\in\mathbb{N}}$ of $\{\tau^j\}_{j\in\mathbb{N}}$ and some $\boldsymbol{P} \in H^1(0,T; (H^s(\boldsymbol{\Omega}))^d)$ such that*

- *$\hat{\boldsymbol{P}}_{\tau^{j_k}} \rightharpoonup \boldsymbol{P}$ weakly in $H^1(0,T; (H^s(\boldsymbol{\Omega}))^d)$ as $k \to \infty$;*
- *$\boldsymbol{0} \in D_{\boldsymbol{P}}\mathcal{I}(t, \boldsymbol{P}(t)) + \partial\Psi_\beta(\boldsymbol{P}'(t))$ for a.a. $t \in (0,T)$;*
- *$\int_u^t \Psi_1(\boldsymbol{P}'(\sigma))d\sigma = \lim_{k\to\infty} \int_u^t \Psi_1(\hat{\boldsymbol{P}}'_{\tau^{j_k}}(\sigma))d\sigma$ for all $0 \leq u < t \leq T$.*

Proof. We first make the convention that since we need to apply several times subsequence argument, we do not relabel the indices of the subsequences which appear in the proof. Due to (3.63), we know that $\{\hat{\boldsymbol{P}}_{\tau^j}\}_{j\in\mathbb{N}}$ are uniformly bounded in $L^\infty(0,T; (H^s(\boldsymbol{\Omega}))^d)$ for all $j \in \mathbb{N}$, and therefore also in $L^2(0,T; (H^s(\boldsymbol{\Omega}))^d)$, and due to (3.67), $\{\hat{\boldsymbol{P}}'_{\tau^j}\}_{j\in\mathbb{N}}$ are also uniformly bounded in $L^2(0,T; (H^s(\boldsymbol{\Omega}))^d)$ for all $j \in \mathbb{N}$. It follows that $\{\hat{\boldsymbol{P}}_{\tau^j}\}_{j\in\mathbb{N}}$ are uniformly bounded in $H^1(0,T; (H^s(\boldsymbol{\Omega}))^d)$ for all $j \in \mathbb{N}$ and hence there exist a subsequence $\{\tau^j\}_{j\in\mathbb{N}}$ and some $\boldsymbol{P} \in H^1(0,T; (H^s(\boldsymbol{\Omega}))^d)$ such that

$$\hat{\boldsymbol{P}}_{\tau^j} \rightharpoonup \boldsymbol{P} \text{ in } H^1(0,T; (H^s(\boldsymbol{\Omega}))^d) \text{ as } j \to \infty. \tag{3.78}$$

Due to Lemma B.3 and (3.64) we infer that

$$\hat{\boldsymbol{P}}_{\tau^j}(t) \to \boldsymbol{P}(t) \text{ in } \mathcal{X} \text{ for all } t \in [0,T] \text{ as } j \to \infty, \tag{3.79}$$

where $\mathcal{X}$ is an arbitrary Banach space with $(H^s(\boldsymbol{\Omega}))^d \hookrightarrow\hookrightarrow \mathcal{X} \hookrightarrow (L^2(\boldsymbol{\Omega}))^d$. Due to (3.66) and Lemma B.1 we know that there exist a subsequence $\{\tau^j\}_{j\in\mathbb{N}}$ and some function $\boldsymbol{P}_1 : (0,T) \to (H^s(\boldsymbol{\Omega}))^d$ such that

$$\hat{\boldsymbol{P}}_{\tau^j}(t) \rightharpoonup \boldsymbol{P}_1(t) \text{ in } (H^s(\boldsymbol{\Omega}))^d \text{ for all } t \in [0,T] \text{ as } j \to \infty. \tag{3.80}$$

Due to the compact embeddings given by Lemma 3.5 we conclude that

$$\hat{\boldsymbol{P}}_{\tau^j}(t) \to \boldsymbol{P}_1(t) \text{ in } \big(L^2(\boldsymbol{\Omega})\big)^d \text{ for all } t \in [0,T] \text{ as } j \to \infty. \tag{3.81}$$

Taking $\mathcal{X} = \big(L^2(\boldsymbol{\Omega})\big)^d$, we infer from (3.79) and (3.81) that $\boldsymbol{P}_1 = \boldsymbol{P}$. In particular,

$$\int_u^t \Psi_1\big(\boldsymbol{P}'(\sigma)\big)d\sigma \leq \liminf_{j\to\infty} \int_u^t \Psi_1\big(\hat{\boldsymbol{P}}'_{\tau^j}(\sigma)\big)d\sigma \tag{3.82}$$

for all $0 \leq u < t \leq T$ due to Fatou's lemma. On the other hand, (3.66) also implies that there exist a subsequence $\{\tau^j\}_{j\in\mathbb{N}}$ and some $\boldsymbol{P}_2 \in L^2(0,T;\big(L^2(\boldsymbol{\Omega})\big)^d)$ such that

$$\hat{\boldsymbol{P}}'_{\tau^j} \rightharpoonup \boldsymbol{P}_2 \text{ in } L^2(0,T;\big(L^2(\boldsymbol{\Omega})\big)^d) \text{ as } j \to \infty.$$

We also know that $\hat{\boldsymbol{P}}'_{\tau^j} \rightharpoonup \boldsymbol{P}'$ in $L^2(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d)$ as $j \to \infty$ by (3.78). Since $\big(H^s(\boldsymbol{\Omega})\big)^d$ is compactly embedded into $\big(L^2(\boldsymbol{\Omega})\big)^d$, we deduce that

$$L^2(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d) \hookrightarrow L^2(0,T;\big(L^2(\boldsymbol{\Omega})\big)^d),$$

see [17, Thm. 7.1.23]. From this we obtain that $\boldsymbol{P}_2 = \boldsymbol{P}'$. Using the lower semi-continuity of the L^2-norm and Fatou's lemma we infer that

$$\int_u^t \Psi_{2,\beta}\big(\boldsymbol{P}'(\sigma)\big)d\sigma \leq \liminf_{j\to\infty} \int_u^t \Psi_{2,\beta}\big(\hat{\boldsymbol{P}}'_{\tau^j}(\sigma)\big)d\sigma \tag{3.83}$$

for all $0 \leq u < t \leq T$. Using (3.65) we also conclude that

$$\bar{\boldsymbol{P}}_{\tau^j}(t), \underline{\boldsymbol{P}}_{\tau^j}(t) \rightharpoonup \boldsymbol{P}(t) \text{ in } \big(H^s(\boldsymbol{\Omega})\big)^d \tag{3.84}$$

for all $t \in [0,T]$ as $j \to \infty$. Having all these estimates and weak convergence results in hand, we are now able to utilize a variational identity from [49, Thm. 4.10] for our problem, which is given by (3.85) below and motivates the use of the integral identity and inequality given by Proposition 3.15. We shall give here a sketch of the the proof for the identity (3.85): applying the Fenchel's formula (more precisely, the identity

$$f(x) + f^*(x^*) = \langle x^*, x\rangle_X$$

given in Lemma A.10) to (3.60) with

$$\begin{aligned} f &= \Psi_\beta, \\ x &= \hat{\boldsymbol{P}}'_{\tau^j}(\sigma), \\ x^* &= -D_{\boldsymbol{P}}\mathcal{I}\big(\bar{t}_{\tau^j}(\sigma), \bar{\boldsymbol{P}}_{\tau^j}(\sigma)\big), \end{aligned}$$

then integrating over $[\underline{t}_{\tau^j}(u), \bar{t}_{\tau^j}(t)]$ w.r.t. σ and applying the chain rule and integration by parts to rearrange the terms, we obtain that for $0 \leq u < t \leq T$ we have

$$I_1^l + I_2^l = I_1^r + I_2^r + I_3^r, \tag{3.85}$$

where

$$\begin{aligned} &I_1^l + I_2^l \\ :=& \Big[\int_{\underline{t}_{\tau^j}(u)}^{\bar{t}_{\tau^j}(t)} \Psi_\beta\big(\hat{\boldsymbol{P}}'_{\tau^j}(\sigma)\big) + \Psi_\beta^*\Big(- D_{\boldsymbol{P}}\mathcal{I}\big(\bar{t}_{\tau^j}(\sigma), \hat{\boldsymbol{P}}_{\tau^j}(\sigma)\big)\Big)d\sigma\Big] \\ &+ \Big[\mathcal{I}\big(\bar{t}_{\tau^j}(t), \bar{\boldsymbol{P}}_{\tau^j}(t)\big)\Big] \end{aligned}$$

and

$$\begin{aligned} &I_1^r + I_2^r + I_3^r \\ :=&\Big[\mathcal{I}\big(\underline{t}_{\tau^j}(u), \underline{\boldsymbol{P}}_{\tau^j}(u)\big)\Big] + \Big[\int_{\underline{t}_{\tau^j}(u)}^{\bar{t}_{\tau^j}(t)} \partial_t \mathcal{I}\big(\sigma, \underline{\boldsymbol{P}}_{\tau^j}(\sigma)\big) d\sigma\Big] \\ &+ \Big[-\int_{\underline{t}_{\tau^j}(u)}^{\bar{t}_{\tau^j}(t)} \frac{1}{\tau(\sigma)} R\big(\bar{t}_{\tau^j}(\sigma); \underline{\boldsymbol{P}}_{\tau^j}(\sigma), \bar{\boldsymbol{P}}_{\tau^j}(\sigma)\big) d\sigma\Big], \end{aligned}$$

where

$$R(t; \boldsymbol{P}_1, \boldsymbol{P}_2) := \mathcal{I}(t, \boldsymbol{P}_1) - \mathcal{I}(t, \boldsymbol{P}_2) + \langle D_{\boldsymbol{P}} \mathcal{I}(t, \boldsymbol{P}_2), \boldsymbol{P}_2 - \boldsymbol{P}_1 \rangle_{H^s}$$

for $t \in [0, T]$, $\boldsymbol{P}_1, \boldsymbol{P}_2 \in \big(H^s(\boldsymbol{\Omega})\big)^d$. Using (3.75) we obtain that

$$\begin{aligned} |R(t; \boldsymbol{P}_1, \boldsymbol{P}_2)| &= \Big| \int_0^1 \langle D_{\boldsymbol{P}} \mathcal{I}(t, \boldsymbol{P}_2) - D_{\boldsymbol{P}} \mathcal{I}\big(t, (1-\theta)\boldsymbol{P}_1 + \theta \boldsymbol{P}_2\big), \boldsymbol{P}_2 - \boldsymbol{P}_1 \rangle_{H^s} d\theta \Big| \\ &\le \frac{1}{2}\big(\|\boldsymbol{P}_2 - \boldsymbol{P}_1\|_{H^s}^2 + C\|\boldsymbol{P}_2 - \boldsymbol{P}_1\|_{L^w} \|\boldsymbol{P}_2 - \boldsymbol{P}_1\|_{H^s}\big) \\ &\le C\|\boldsymbol{P}_2 - \boldsymbol{P}_1\|_{H^s}^2 \end{aligned}$$

for some $2 < w < \infty$. Then

$$\begin{aligned} &\Big| \int_{\underline{t}_{\tau^j}(u)}^{\bar{t}_{\tau^j}(t)} \frac{1}{\tau(\sigma)} R\big(\bar{t}_{\tau^j}(\sigma); \underline{\boldsymbol{P}}_{\tau^j}(\sigma), \bar{\boldsymbol{P}}_{\tau^j}(\sigma)\big) d\sigma \Big| \\ \le& C\Big(\sup_{t \in [0,T]} \|\bar{\boldsymbol{P}}_{\tau^j}(t) - \underline{\boldsymbol{P}}_{\tau^j}(t)\|_{H^s}\Big) \int_{\underline{t}_{\tau^j}(\sigma)}^{\bar{t}_{\tau^j}(t)} \|\hat{\boldsymbol{P}}'_{\tau^j}(\sigma)\|_{H^s} d\sigma. \end{aligned} \tag{3.86}$$

Now setting the initial time point u in (3.85) to $u = 0$. Then we immediately infer that $I_1^r = \mathcal{I}(0, \boldsymbol{P}_0)$. Using Corollary 3.24 and (3.80) we obtain that

$$\liminf_{j \to \infty} I_2^l \ge \mathcal{I}\big(t, \boldsymbol{P}(t)\big).$$

Using Corollary 3.24 and (3.80) we also know that

$$D_{\boldsymbol{P}} \mathcal{I}\big(\bar{t}_{\tau^j}(\sigma), \hat{\boldsymbol{P}}_{\tau^j}(\sigma)\big) \rightharpoonup D_{\boldsymbol{P}} \mathcal{I}\big(\sigma, \boldsymbol{P}(\sigma)\big) \text{ in } [\big(H^s(\boldsymbol{\Omega})\big)^d]^*$$

for all $\sigma \in (0, T)$ as $j \to \infty$, where $[\big(H^s(\boldsymbol{\Omega})\big)^d]^*$ is the dual space of $(H^s(\boldsymbol{\Omega}))^d$. Then using (3.82), (3.83), (3.84), the weak lower semi-continuity of Ψ_β^* (see Lemma A.9) and Fatou's lemma we obtain that

$$\liminf_{j \to \infty} I_1^l \ge \int_0^t \Psi_\beta\big(\boldsymbol{P}'(\sigma)\big) + \Psi_\beta^*\Big(-D_{\boldsymbol{P}} \mathcal{I}\big(\sigma, \boldsymbol{P}(\sigma)\big)\Big) d\sigma.$$

Using Corollary 3.24, (3.84) and Lebesgue dominated convergence theorem we also conclude that

$$\lim_{j \to \infty} I_2^r = \int_0^t \partial_t \mathcal{I}\big(\sigma, \boldsymbol{P}(\sigma)\big) d\sigma.$$

In view of (3.67), (3.65) and (3.86), we infer that $\lim_{j \to \infty} I_3^r = 0$. Sum up all, we obtain that

$$\begin{aligned} &\int_0^t \Psi_\beta\big(\boldsymbol{P}'(\sigma)\big) + \Psi_\beta^*\Big(-D_{\boldsymbol{P}} \mathcal{I}\big(\sigma, \boldsymbol{P}(\sigma)\big)\Big) d\sigma + \mathcal{I}\big(t, \boldsymbol{P}(t)\big) \\ \le& \mathcal{I}(0, \boldsymbol{P}_0) + \int_0^t \partial_t \mathcal{I}\big(\sigma, \boldsymbol{P}(\sigma)\big) d\sigma \end{aligned} \tag{3.87}$$

for all $t \in [0,T]$. But due to Proposition 3.15, this inequality is equivalent to the statement that $\boldsymbol{P}$ is a solution of (3.34). This completes the proof of the first and second statements of Theorem 3.30. Again due to Proposition 3.15, the inequality (3.87) holds if and only if it holds as an equality for all $t \in [0,T]$. However, taking $u = 0$ in (3.82), if the inequality (3.82) is strict, then the inequality (3.87) must also be strict in view of previous analysis. Thus (3.82) must hold as an equality in the case $u = 0$ and $t \in [0,T]$. But since $t \in [0,T]$ is arbitrary chosen, by difference argument we infer that (3.82) holds as an equality for all $0 \le u < t \le T$. To summarize, we actually have

$$\begin{aligned}&\int_u^t \Psi_\beta\big(\boldsymbol{P}'(\sigma)\big) + \Psi_\beta^*\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\sigma,\boldsymbol{P}(\sigma)\big)\Big)d\sigma + \mathcal{I}\big(t,\boldsymbol{P}(t)\big)\\ =&\mathcal{I}\big(u,\boldsymbol{P}(u)\big) + \int_u^t \partial_t\mathcal{I}\big(\sigma,\boldsymbol{P}(\sigma)\big)d\sigma\end{aligned}$$

and

$$\int_u^t \Psi_1\big(\boldsymbol{P}'(\sigma)\big)d\sigma = \liminf_{j\to\infty}\int_u^t \Psi_1\big((\hat{\boldsymbol{P}}^3_{\tau^j})'(\sigma)\big)d\sigma$$

for all $0 \le u < t \le T$. Thus up to a subsequence we can even conclude that

$$\int_u^t \Psi_1\big(\boldsymbol{P}'(\sigma)\big)d\sigma = \lim_{j\to\infty}\int_u^t \Psi_1\big((\hat{\boldsymbol{P}}^3_{\tau^j})'(\sigma)\big)d\sigma \tag{3.88}$$

for all $0 \le u < t \le T$. This shows the validity of the third statement of Theorem 3.30. □

Remark 3.31. In fact, arguing similarly as for (3.88), we can also obtain from (3.83) the equality

$$\int_u^t \Psi_{2,\beta}\big(\boldsymbol{P}'(\sigma)\big)d\sigma = \lim_{j\to\infty}\int_u^t \Psi_{2,\beta}\big(\hat{\boldsymbol{P}}'_{\tau^j}(\sigma)\big)d\sigma$$

and together with (3.88) also

$$\int_u^t \Psi_{\beta}\big(\boldsymbol{P}'(\sigma)\big)d\sigma = \lim_{j\to\infty}\int_u^t \Psi_{\beta}\big(\hat{\boldsymbol{P}}'_{\tau^j}(\sigma)\big)d\sigma$$

for all $0 \le u < t \le T$. However, these equalities are useless for the vanishing viscosity analysis given in the next section, since we will push the parameter β to zero and the above mentioned identities involving β make no contribution in the limiting procedure. △

Now having obtained a viscosity solution $\boldsymbol{P}$, we obtain from Lemma 3.19 the existence of the solution $(\boldsymbol{u},\phi)$ of the equation (3.2a). Consequently, we obtain from Lemma 3.9 the existence of the solution $(\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})$ of (3.20). In particular, we have the following regularity result for the functions $\boldsymbol{u}$, ϕ, $\boldsymbol{D}$:

Proposition 3.32. *Let the Assumptions A1 to A7 be satisfied and $\beta > 0$ be a given positive constant. Suppose also that $\boldsymbol{P}_0 \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and $D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)$ is of class L^2. Let $\boldsymbol{P}$ be the viscosity solution obtained from Theorem 3.30. Then the differential system* (3.2) *and* (3.20) *admit a solution $(\boldsymbol{u},\phi,\boldsymbol{P})$ and $(\boldsymbol{u},\boldsymbol{D},\boldsymbol{P})$ ($\boldsymbol{u}$ being identical in former and latter) respectively such that*

$$\begin{aligned}\boldsymbol{u} &\in H^1\big(0,T;\big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d\big),\\ \phi &\in H^1\big(0,T;W^{1,q}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})\big),\\ \boldsymbol{D} &\in H^1\big(0,T;M_{\boldsymbol{D}}\cap\big(L^q(\boldsymbol{\Omega})\big)^d\big),\\ \boldsymbol{P} &\in H^1\big(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d\big),\end{aligned} \tag{3.89}$$

where $M_{\boldsymbol{D}}$ is the space defined by (3.18) *and $q \in (2, \infty)$ is the number given by Lemma 3.19.*

Proof. The existence of $\boldsymbol{P}$ with the claimed regularity given in (3.89) is deduced from Theorem 3.30. Now inserting this $\boldsymbol{P}$ to (3.2a), we obtain from (3.37) given in the proof of Lemma 3.19 that for all $t \in [0, T]$ we have

$$\|\boldsymbol{u}\big(t, \boldsymbol{P}(t)\big)\|_{W^{1,q}} + \|\phi\big(t, \boldsymbol{P}(t)\big)\|_{W^{1,q}} \leq C\big(\Lambda + \|\boldsymbol{P}(t)\|_{L^q}\big),$$

where Λ is defined by (3.36). From the Sobolev's embedding

$$L^q \hookrightarrow H^s$$

we obtain that

$$\|\boldsymbol{u}\big(t, \boldsymbol{P}(t)\big)\|_{W^{1,q}} + \|\phi\big(t, \boldsymbol{P}(t)\big)\|_{W^{1,q}} \leq C\big(\Lambda + \|\boldsymbol{P}(t)\|_{H^s}\big). \tag{3.90}$$

Notice that C and Λ are independent on $t \in [0, T]$, where the former independence is due to Lemma 3.19 and the latter is due to the fact that all external loadings given in Assumption A4 have temporal $C^{1,1}$-regularity. Then the claimed regularity of $(\boldsymbol{u}, \phi)$ given in (3.89) follows immediately from (3.90) and the regularity of $\boldsymbol{P}$ given in (3.89). Now the regularity of $\boldsymbol{D}$ follows immediately from the relation (3.23). This completes the proof. □

3.8 Vanishing viscosity solution

In what follows, we investigate the behavior of the viscous solutions in the limiting case $\beta \to 0$. The following lemma will be crucial for our analysis.

Lemma 3.33. *Let the Assumptions A1 to A7 be satisfied. Let β be a given positive constant in $(0, 1)$ and $\{\tau\}_{i=0,\ldots,N-1}$ be an equidistant partition of $[0, T]$. Let $\hat{\boldsymbol{P}}_\tau$ be as defined by* (3.59) *corresponding to the given number β and the partition $\{\tau\}_{i=0,\ldots,N-1}$. Suppose also that $\boldsymbol{P}_0 \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and $D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)$ is of class L^2. Then there exists some $c > 0$ independent on β such that for all $\tau \leq c\beta$ we have*

$$\int_0^T \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|_{H^s} d\sigma \leq C < \infty \tag{3.91}$$

for some constant $C > 0$ which is independent on β.

Proof. Choosing appropriate ρ, ρ' in (3.76) such that $\rho + C_\rho \rho' = \frac{1}{2}$ in (3.76) and integrate (3.76) over $[0, \bar{t}_\tau(t)]$ w.r.t. σ, we obtain that

$$\begin{aligned} &\int_0^{\bar{t}_\tau(t)} \Big| \frac{1}{\tau(\sigma)} \Big\langle \Delta_{\tau(\sigma)} D_{\boldsymbol{P}} \tilde{\mathcal{I}}\big(\bar{t}_\tau(\sigma), \bar{\boldsymbol{P}}_\tau(r)\big), \hat{\boldsymbol{P}}'_\tau(\sigma) \Big\rangle_{H^s} \Big| d\sigma \\ \leq & C \int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^2} d\sigma + \frac{1}{2} \int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{H^s} d\sigma + C'. \end{aligned} \tag{3.92}$$

Together with (3.71) we conclude that

$$\frac{\beta}{2} \|\hat{\boldsymbol{P}}'_\tau(\frac{t_1}{2})\|^2_{L^2} \leq C \int_0^{\bar{t}_\tau(t)} \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|^2_{L^2} d\sigma + \frac{1}{2\beta} \|D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)\|^2_{L^2} + C'. \tag{3.93}$$

Now choose $t = \frac{t_1}{2} = \frac{\tau}{2}$ and $\tau \le \frac{\beta}{4C}$. Since $\hat{\boldsymbol{P}}'_\tau$ is piecewise constant on $(0, t_1)$, we obtain after rearranging terms that

$$\begin{aligned} &\beta\|\hat{\boldsymbol{P}}'_\tau(\frac{t_1}{2})\|_{L^2}^2 \le \frac{2}{\beta}\|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|_{L^2}^2 + C' \\ \Rightarrow &\beta\|\hat{\boldsymbol{P}}'_\tau(\frac{t_1}{2})\|_{L^2} \le C\big(1 + \|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|_{L^2}\big). \end{aligned} \tag{3.94}$$

Define $m_k := \frac{1}{2}(t^\tau_{k-1} + t^\tau_k), k \in \{1, ..., N\}$. Using (3.74) and (3.76) (in this case, choose $X_3 = L^1$ in the Ehrling's lemma and replace the index L^2 in (3.74) to L^1) we obtain that

$$\begin{aligned} &\Big|\frac{1}{\tau}\Big\langle D_{\boldsymbol{P}}\tilde{\mathcal{I}}\big(t_k, \bar{\boldsymbol{P}}_\tau(m_k)\big) - D_{\boldsymbol{P}}\tilde{\mathcal{I}}\big(t_{k-1}, \bar{\boldsymbol{P}}_\tau(m_{k-1})\big), \hat{\boldsymbol{P}}'_\tau(m_k)\Big\rangle_{H^s}\Big| \\ \le &C_\rho + C_\rho C_{\rho'}\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^1}^2 + (\rho + C_\rho\rho')\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s}^2 \\ \le &\frac{1}{2}\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s}^2 + C\big(1 + \|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^1}^2\big) \\ \le &\frac{1}{2}\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s}^2 + C\Big(1 + \|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^1}\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(m_k)\big)\Big) \\ \le &\frac{1}{2}\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s}^2 + C\Big(1 + \|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^2}\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(m_k)\big)\Big), \end{aligned} \tag{3.95}$$

where we have chosen appropriate ρ, ρ' such that $\rho + C_\rho\rho' = \frac{1}{2}$ and we have used the fact that the norm $\|\cdot\|_{L^1}$ is equivalent to $\Psi_1(\cdot)$ due to the Assumption A7 to deduce the third inequality from the second inequality; notice also that we have used the fact that $\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^1}$ is bounded by $\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^2}$ (up to a prefactor) to deduce the last inequality. We point out that (3.95) is exactly the same inequality as the second inequality given in [39, p.600]. Thus, arguing similarly as in [39] (more precisely, from [39, (4.35)] to [39, (4.36)]), we obtain that

$$\begin{aligned} &2\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^2}\big(\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^2} - \|\hat{\boldsymbol{P}}'_\tau(m_{k-1})\|_{L^2}\big) + \frac{\tau}{\beta}\big(\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^2}^2) + \|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s}^2\big) \\ \le &\frac{4C\tau}{\beta}\Big(1 + \|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{L^2}\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(m_k)\big)\Big) \end{aligned} \tag{3.96}$$

is valid for all $2 \le k \le N$. (3.96) plays exactly the same role as [39, (4.36)], thus using a refined Gronwall inequality of discrete type given in [39] (which makes use of (4.36) and is used to derive the inequality (4.47) therein), and by setting $\tau < c\beta$ with

$$c := \min\{\frac{1}{4C}, 2\},$$

we obtain from [39, Prop. 4.3, (4.47)] that

$$\sum_{k=2}^N \tau\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s} \le C_{\frac{\tau}{\beta}}\Big(T + \beta\|\hat{\boldsymbol{P}}'_\tau(m_1)\|_{L^2} + \sum_{k=2}^N \tau\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(m_k)\big)\Big). \tag{3.97}$$

Here, we point out that the constant $C_{\frac{\tau}{\beta}}$ in (3.98) is in general depending on the quantity $\frac{\tau}{\beta}$, but due to the fact that $\tau < c\beta$, we see that $\frac{\tau}{\beta}$ is bounded by the constant $c = \min\{\frac{1}{4C}, 2\}$ which is independent of β. Thus one may replace the constant $C_{\frac{\tau}{\beta}}$ by some universal constant C which is independent of β and (3.97) becomes

$$\sum_{k=2}^N \tau\|\hat{\boldsymbol{P}}'_\tau(m_k)\|_{H^s} \le C\Big(T + \beta\|\hat{\boldsymbol{P}}'_\tau(m_1)\|_{L^2} + \sum_{k=2}^N \tau\Psi_1\big(\hat{\boldsymbol{P}}'_\tau(m_k)\big)\Big). \tag{3.98}$$

Now taking $t = \frac{t_1}{2}$ in (3.77) we obtain that

$$\begin{aligned}\tau\|\hat{\boldsymbol{P}}'_\tau(m_1)\|^2_{H^s} &\leq C\tau\|\hat{\boldsymbol{P}}'_\tau(m_1)\|^2_{L^2} + \frac{1}{\beta}\|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|^2_{L^2} + C' \\ \Rightarrow \tau\|\hat{\boldsymbol{P}}'_\tau(m_1)\|_{H^s} &\leq C\Big(1 + \tau\|\hat{\boldsymbol{P}}'_\tau(m_1)\|_{L^2} + \sqrt{\frac{\tau}{\beta}}\|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|_{L^2}\Big) \\ &\leq C\Big(1 + \tau\|\hat{\boldsymbol{P}}'_\tau(m_1)\|_{L^2} + \|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|_{L^2}\Big).\end{aligned} \tag{3.99}$$

Adding this to (3.98) and using (3.94), we infer that

$$\int_0^T \|\hat{\boldsymbol{P}}'_\tau(\sigma)\|_{H^s} d\sigma \leq C\Big(1 + \|D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0)\|_{L^2} + \int_0^T \Psi_1\big(\hat{\boldsymbol{P}}'_\tau(\sigma)\big)d\sigma\Big) \leq C, \tag{3.100}$$

where we have also used the fact that

$$\int_0^T \Psi_1\big(\hat{\boldsymbol{P}}'_\tau(\sigma)\big)d\sigma \leq C$$

due to (3.66). □

Now let $\boldsymbol{P}_\beta \in H^1(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d)$ be the solution given by Theorem 3.30 which is approximated by the time discrete interpolant solutions with equidistant grids given in Lemma 3.33. Due to Lemma 3.33 and the fact that $\hat{\boldsymbol{P}}_\tau$ converges to $\boldsymbol{P}_\beta$ weakly in $H^1(0,T;\big(H^s(\boldsymbol{\Omega})\big)^d)$ as $\tau \to 0$ we conclude from (3.91) and Fatou's lemma that

$$\sup_{\beta\in(0,1)} \int_0^T \|\boldsymbol{P}'_\beta(\sigma)\|_{H^s} d\sigma \leq C < \infty \tag{3.101}$$

for some $C > 0$ which is independent on β. This motivates the so called arclength parametrization, which is given precisely in the following.

First, we define the quantity $s_\beta(t)$ by

$$s_\beta(t) := t + \int_0^t \|\boldsymbol{P}'_\beta(\sigma)\|_{H^s} d\sigma \tag{3.102}$$

for $t \in [0,T]$. Let $S_\beta := s_\beta(T)$. Define $\tilde{t}_\beta : [0,S_\beta] \to [0,T]$ and $\tilde{\boldsymbol{P}}_\beta : [0,S_\beta] \to \big(H^s(\boldsymbol{\Omega})\big)^d$ by

$$\begin{aligned}\tilde{t}_\beta(\sigma) &:= s_\beta^{-1}(\sigma), \\ \tilde{\boldsymbol{P}}_\beta(\sigma) &:= \boldsymbol{P}_\beta\big(\tilde{t}_\beta(\sigma)\big).\end{aligned} \tag{3.103}$$

Furthermore, using the chain rule we obtain that

$$\tilde{t}'_\beta(\sigma) + \|\tilde{\boldsymbol{P}}'_\beta(\sigma)\|_{H^s} = 1 \text{ for a.a. } \sigma \in [0,S_\beta]. \tag{3.104}$$

According to (3.101) we infer that $\sup_{\beta\in(0,1)} S_\beta < \infty$. On the other hand, $S_\beta \geq T$ for all $\beta \in (0,1)$ due to the definition of S_β. Thus the family $\{S_\beta\}_{\beta\in(0,1)}$ is bounded and therefore has a converging subsequence $\{S_{\beta_j}\}_{j\in\mathbb{N}}$. Denote this limit by S. Then $S_{\beta_j} \to S$ as $j \to \infty$ with $S \geq T$. Hence w.l.o.g. we may consider the parameterized trajectories on the fixed time interval $[0, S]$, in viewpoint of the scaling trick [16, Rem. 2.3]. Recall that $\boldsymbol{P}_\beta$ fulfills the equation

$$\begin{aligned}&\int_u^t \Psi_\beta\big(\boldsymbol{P}'_\beta(\sigma)\big) + \Psi^*_\beta\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\sigma, \boldsymbol{P}_\beta(\sigma)\big)\Big)d\sigma + \mathcal{I}\big(t, \boldsymbol{P}_\beta(t)\big)\\ =&\mathcal{I}\big(u, \boldsymbol{P}_\beta(u)\big) + \int_s^t \partial_t\mathcal{I}\big(\sigma, \boldsymbol{P}_\beta(\sigma)\big)d\sigma\end{aligned} \tag{3.105}$$

for all $0 \le u < t \le T$. From the inf-sup convolution formula [30, Thm. 2.3.2] (see also [49, Rem. 2.4]) one obtains that

$$\Psi^*_\beta(\boldsymbol{P}^*) = \frac{1}{2\beta}\min_{\boldsymbol{K}^*\in\partial\Psi_1(\mathbf{0})}\|\boldsymbol{P}^* - \boldsymbol{K}^*\|^2_{L^2} \tag{3.106}$$

for $\boldsymbol{P}^* \in [\big(H^s(\boldsymbol{\Omega})\big)^d]^*$, where $[\big(H^s(\boldsymbol{\Omega})\big)^d]^*$ is the dual space of $(H^s(\boldsymbol{\Omega}))^d$. This motivates the definition

$$\operatorname{dist}_2\big(\boldsymbol{P}^*, \partial\Psi_1(\mathbf{0})\big) := \min_{\boldsymbol{K}^*\in\partial\Psi_1(\mathbf{0})}\|\boldsymbol{P}^* - \boldsymbol{K}^*\|_{L^2} \tag{3.107}$$

for $\boldsymbol{P}^* \in [\big(H^s(\boldsymbol{\Omega})\big)^d]^*$ and (3.105) can be rewritten to

$$\begin{aligned}&\int_u^t \Psi_\beta\big(\boldsymbol{P}'_\beta(\sigma)\big) + \frac{1}{2\beta}\operatorname{dist}^2_2\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\sigma, \boldsymbol{P}_\beta(\sigma)\big), \partial\Psi_1(\mathbf{0})\Big)d\sigma + \mathcal{I}\big(t, \boldsymbol{P}_\beta(t)\big)\\ =&\mathcal{I}\big(u, \boldsymbol{P}_\beta(u)\big) + \int_s^t \partial_t\mathcal{I}\big(\sigma, \boldsymbol{P}_\beta(\sigma)\big)d\sigma.\end{aligned} \tag{3.108}$$

Let (τ_1, τ_2) be an arbitrary subinterval of $[0, S]$. Applying variable transformation to (3.103) we obtain that

$$\begin{aligned}\int_{\tau_1}^{\tau_2} &\Psi_1\big(\tilde{\boldsymbol{P}}'_\beta(\sigma)\big) + \frac{\beta}{2\tilde{t}'_\beta(\sigma)}\|\tilde{\boldsymbol{P}}'_\beta(\sigma)\|^2_{L^2}\\ &+ \frac{\tilde{t}'_\beta(\sigma)}{2\beta}\operatorname{dist}^2_2\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\tilde{t}_\beta(\sigma), \tilde{\boldsymbol{P}}_\beta(\sigma)\big), \partial\Psi_1(\mathbf{0})\Big)d\sigma + \mathcal{I}\big(\tilde{t}_\beta(\tau_2), \tilde{\boldsymbol{P}}_\beta(\tau_2)\big)\\ &= \mathcal{I}\big(\tilde{t}_\beta(\tau_1), \tilde{t}_\beta(\tau_1)\big) + \int_{\tau_1}^{\tau_2} \partial_t\mathcal{I}\big(\tilde{t}_\beta(\sigma), \tilde{\boldsymbol{P}}_\beta(\sigma)\big)\tilde{t}'_\beta(\sigma)d\sigma.\end{aligned} \tag{3.109}$$

The equation (3.109) suggests the following definition: define

$$M_\beta : [0,\infty) \times \big(L^2(\boldsymbol{\Omega})\big)^d \times [0,\infty) \to [0,\infty]$$

by

$$M_\beta(\alpha, \boldsymbol{v}, \eta) = \begin{cases} \Psi_1(\boldsymbol{v}) + \frac{\beta}{2\alpha}\|\boldsymbol{v}\|^2_{L^2} + \frac{\alpha}{2\beta}\eta^2, & \alpha > 0,\\ 0, & \alpha = 0,\ \boldsymbol{v} = 0,\ \eta \in [0,\infty),\\ \infty, & \text{otherwise.}\end{cases} \tag{3.110}$$

Then (3.109) can be rewritten to

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} M_\beta\Big(\tilde{t}'_\beta(\sigma), \tilde{\boldsymbol{P}}'_\beta(\sigma), \operatorname{dist}_2\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\tilde{t}_\beta(\sigma), \tilde{\boldsymbol{P}}_\beta(\sigma)\big), \partial\Psi_1(\mathbf{0})\Big)\Big)d\sigma + \mathcal{I}\big(\tilde{t}_\beta(\sigma_2), \tilde{\boldsymbol{P}}_\beta(\sigma_2)\big)\\ =&\mathcal{I}\big(\tilde{t}_\beta(\sigma_1), \tilde{\boldsymbol{P}}_\beta(\sigma_1)\big) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}\big(\tilde{t}_\beta, \tilde{\boldsymbol{P}}_\beta(\sigma)\big)\tilde{t}'_\beta(\sigma)d\sigma\end{aligned} \tag{3.111}$$

for all $0 \le \sigma_1 < \sigma_2 \le S$. This equation can be understood as an energy balance equality: the term M_β interprets the total dissipation of the system, the energy $\mathcal{I}$ at time points σ_1 and σ_2 stand for initial and end states of the evolution respectively, and $\partial_t \mathcal{I}$ denotes the power induced by the temporal changes in the system.

Our goal is to figure out what will happen to the system when the viscosity constant β is getting smaller and smaller. We expect that both the arclength parameterized solution $(\tilde{t}_\beta, \tilde{\boldsymbol{P}}_\beta)$ and the functional M_β would "converge" to some function $(\tilde{t}, \tilde{\boldsymbol{P}})$ and some functional M_0 respectively. In particular, the limiting function and functional should satisfy certain variation equality which is similar to the one given by (3.109). A general framework for such study is the so called Γ-convergence theory. We make it precise in the following. First we introduce the following lemma, which gives the precise expression of a reasonable limiting functional M_0 and its corresponding properties:

Lemma 3.34 ([48, Lem. 3.1]). *Define* $M_0 : [0,\infty) \times \left(L^2(\boldsymbol{\Omega})\right)^d \times [0,\infty) \to [0,\infty]$ *by*

$$M_0(\alpha, \boldsymbol{v}, \eta) = \begin{cases} \Psi_1(\boldsymbol{v}) + \eta \|\boldsymbol{v}\|_{L^2}, & \alpha = 0, \\ \Psi_1(\boldsymbol{v}) + \mathrm{I}_0(\eta), & \alpha > 0, \end{cases} \tag{3.112}$$

where I_0 *denotes the indicator function of the singleton* $\{0\}$, *i.e.,* $\mathrm{I}_0(0) = 0$ *and* $\mathrm{I}_0(\eta) = \infty$ *if* $\eta \neq 0$. *Then* M_β Γ*-converges to* M_0 *as* $\beta \to 0$ *in the following sense:*

- *If* $(\alpha_\beta, \boldsymbol{v}_\beta, \eta_\beta) \to (\alpha, \boldsymbol{v}, \eta)$ *in* $[0,\infty) \times \left(L^2(\boldsymbol{\Omega})\right)^d \times [0,\infty)$, *then*

$$M_0(\alpha, \boldsymbol{v}, \eta) \le \liminf_{\beta \to 0} M_\beta(\alpha_\beta, \boldsymbol{v}_\beta, \eta_\beta).$$

- *For all* $(\alpha, \boldsymbol{v}, \eta)$ *in* $[0,\infty) \times \left(L^2(\boldsymbol{\Omega})\right)^d \times [0,\infty)$ *there exist* $\{(\alpha_\beta, \boldsymbol{v}_\beta, \eta_\beta)\}_{\beta>0}$ *such that*

$$(\alpha_\beta, \eta_\beta) \to (\alpha, \eta) \text{ in } [0,\infty)^2 \ , \ \boldsymbol{v}_\beta \rightharpoonup \boldsymbol{v} \text{ in } \left(L^2(\boldsymbol{\Omega})\right)^d$$

  and

$$M_0(\alpha, \boldsymbol{v}, \eta) \ge \limsup_{\beta \to 0} M_\beta(\alpha_\beta, \boldsymbol{v}_\beta, \eta_\beta)$$

  as $\beta \to 0$.

Furthermore, if for $[a,b] \subset [0,S]$ *we have*

$$\alpha_\beta \to \bar{\alpha} \text{ in } L^1(a,b), \ \boldsymbol{v}_{\boldsymbol{\beta}} \rightharpoonup \bar{\boldsymbol{v}} \text{ in } L^1(a,b; \left(L^2(\boldsymbol{\Omega})\right)^d)$$

and

$$\liminf_{\beta \to 0} \eta_\beta(\sigma) \ge \bar{\eta}(\sigma) \text{ for a.a. } \sigma \in [a,b],$$

then

$$\int_a^b M_0\big(\bar{\alpha}(\sigma), \bar{\boldsymbol{v}}(\sigma), \bar{\eta}(\sigma)\big) d\sigma \le \liminf_{\beta \to 0} \int_a^b M_\beta\big(\alpha_\beta(\sigma), \boldsymbol{v}_\beta(\sigma), \eta_\beta(\sigma)\big) d\sigma. \tag{3.113}$$

Remark 3.35. The proof of Lemma 3.34 is originally formulated in [48, Lem. 3.1], where all underlying sets therein are assumed to be subsets of finite dimensional spaces and the situation differs from the setting stated in Lemma 3.34. However, using the Ioffe's theorem [32], the arguments given in [48, Lem. 3.1] can easily be developed to the ones for an infinite dimensional setting, from which an infinite dimensional version of [48, Lem. 3.1] follows, namely the Lemma 3.34 stated here. See also [64] for more details about such adoption. △

Next, we introduce the concept of a *parameterized solution*. We will show that the previously constructed arclength parameterized viscous solutions converge (up to a subsequence) to a parameterized solution within certain weak-$*$-topology, see Theorem 3.38 below.

Definiton 3.36. *A pair* $(\tilde{t}, \tilde{\boldsymbol{P}}) \in \mathrm{Lip}([0,S];[0,T] \times \big(H^s(\boldsymbol{\Omega})\big)^d)$ *is called a parameterized solution of the main problem if for all* $0 \le \sigma_1 < \sigma_2 \le T$

$$\begin{aligned}&\int_{\sigma_1}^{\sigma_2} M_0\Big(\tilde{t}'(\sigma), \tilde{\boldsymbol{P}}'(\sigma), \mathrm{dist}_2\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\tilde{t}'(\sigma), \tilde{\boldsymbol{P}}'(\sigma)\big), \partial\Psi_1(\boldsymbol{0})\Big)\Big)d\sigma + \mathcal{I}\big(\tilde{t}(\sigma_2), \tilde{\boldsymbol{P}}(\sigma_2)\big)\\ =&\mathcal{I}\big(\tilde{t}(\sigma_1), \tilde{\boldsymbol{P}}(\sigma_1)\big) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}\big(\tilde{t}(\sigma), \tilde{\boldsymbol{P}}(\sigma)\big)\tilde{t}'(\sigma)d\sigma.\end{aligned} \tag{3.114}$$

Here, the space $\mathrm{Lip}(X;Y)$ denotes the set of all Lipschitz continuous functions from X to Y for metric spaces X and Y.

Analogously to Proposition 3.15, the following lemma provides an equivalent formulation for (3.114), in the sense of an inequality:

Lemma 3.37 ([39, Lem. 5.2])**.** *Let the Assumptions A1 to A7 be satisfied. Then* (3.114) *holds if and only if the "=" symbol is replaced by the "≤" symbol and* σ_1 *is replaced by* 0.

Having all these definitions and lemmas in hand, we are finally able to give the main theorem involving the existence of a vanishing viscosity solution.

Theorem 3.38. *Let the Assumptions A1 to A7 be satisfied. Then for every sequence* $\beta_n \to 0$ *there exist a (not-relabeled) subsequence* $\{\beta_n\}_{n\in\mathbb{N}}$ *and some*

$$(\tilde{t}, \tilde{\boldsymbol{P}}) \in W^{1,\infty}(0,S;[0,T] \times \big(H^s(\boldsymbol{\Omega})\big)^d)$$

such that

$$(\tilde{t}_{\beta_n}, \tilde{\boldsymbol{P}}_{\beta_n}) \overset{*}{\rightharpoonup} (\tilde{t}, \tilde{\boldsymbol{P}}) \qquad \text{in } W^{1,\infty}(0,S;[0,T] \times \big(H^s(\boldsymbol{\Omega})\big)^d); \tag{3.115}$$

$$\tilde{t}_{\beta_n} \to \tilde{t} \qquad \text{in } C([0,S];[0,T]); \tag{3.116}$$

$$\tilde{\boldsymbol{P}}_{\beta_n}(\sigma) \rightharpoonup \tilde{\boldsymbol{P}}(\sigma) \qquad \text{in } \big(H^s(\boldsymbol{\Omega})\big)^d \text{ for all } \sigma \in [0,S]; \tag{3.117}$$

$$\tilde{t}'(\sigma) + \|\tilde{\boldsymbol{P}}'(\sigma)\|_{H^s} \le 1 \qquad \text{for a.a. } \sigma \in [0,S]. \tag{3.118}$$

Moreover, $(\tilde{t}, \tilde{\boldsymbol{P}})$ *is a parameterized solution which fulfills* (3.114).

Proof. From the relation (3.104) we know that the sequence $\{(\tilde{t}_{\beta_n}, \tilde{\boldsymbol{P}}_{\beta_n})\}_{n\in\mathbb{N}}$ is bounded in $W^{1,\infty}(0,S;[0,T] \times \big(H^s(\boldsymbol{\Omega})\big)^d)$. Then up to a subsequence, $(\tilde{t}_{\beta_n}, \tilde{\boldsymbol{P}}_{\beta_n})$ weak-$*$-converges to some $(\tilde{t}, \tilde{\boldsymbol{P}}) \in W^{1,\infty}(0,S;[0,T \times \big(H^s(\boldsymbol{\Omega})\big)^d])$ as $n \to \infty$ and the corresponding convergence relations (3.116) and (3.118) follow immediately. Now (3.117) can be obtained from Lemma B.1 and (3.101). To see that $(\tilde{t}, \tilde{\boldsymbol{P}})$ is a parameterized solution, we only need to insert $(\tilde{t}_\beta, \tilde{\boldsymbol{P}}_\beta)$ into (3.111), then using the continuity properties given by Corollary 3.24 and the properties deduced from the Γ-convergence theory given by Lemma 3.34 to push β to zero and to obtain that

$$\begin{aligned}&\int_0^s M_0\Big(\tilde{t}'(\sigma), \tilde{\boldsymbol{P}}'(\sigma), \mathrm{dist}_2\Big(-D_{\boldsymbol{P}}\mathcal{I}\big(\tilde{t}'(\sigma), \tilde{\boldsymbol{P}}'(\sigma)\big), \partial\Psi_1(\boldsymbol{0})\Big)\Big)d\sigma + \mathcal{I}\big(\tilde{t}(s), \tilde{\boldsymbol{P}}(s)\big)\\ \le&\mathcal{I}\big(\tilde{t}(0), \tilde{\boldsymbol{P}}(0)\big) + \int_0^s \partial_t\mathcal{I}\big(\tilde{t}(\sigma), \tilde{\boldsymbol{P}}(\sigma)\big)\tilde{t}'(\sigma)d\sigma\end{aligned} \tag{3.119}$$

is valid for all $s \in [0, S]$, which is similarly done as in the proof of Theorem 3.30 (for a complete proof of (3.119), we refer to [39, Thm 5.1]). Now using Lemma 3.37 we infer that $(\tilde{t}, \tilde{\boldsymbol{P}})$ is a parameterized solution. □

In view of Lemma 3.9 and Lemma 3.13 we are able to define the corresponding parameterized solutions $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{D}}$, $\tilde{\phi}$ as follows: the parameterized solution $(\tilde{\boldsymbol{u}}, \tilde{\phi})$ defined on $[0, S]$ is observed as the solution of (3.2a) by inserting $(t, \boldsymbol{P}) = \big(\tilde{t}(\sigma), \tilde{\boldsymbol{P}}(\sigma)\big)$ in (3.2a) (the existence of $\big(\tilde{\boldsymbol{u}}(\sigma), \tilde{\phi}(\sigma)\big)$ is at least pointwise ensured for every $\sigma \in [0, S]$ in view of Lemma 3.19), and $\tilde{\boldsymbol{D}}$ is then deduced from (3.23) by setting $(\boldsymbol{u}, \phi) = (\tilde{\boldsymbol{u}}, \tilde{\phi})$ (and of course also setting $t = \tilde{t}(\sigma)$ and $\boldsymbol{P} = \tilde{\boldsymbol{P}}(\sigma)$) therein. We have the following regularity result w.r.t $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{D}}$, $\tilde{\phi}$:

Proposition 3.39. *Let the Assumptions A1 to A7 be satisfied. Let $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{D}}$, $\tilde{\phi}$ be the parameterized solutions which are defined previously from the parameterized solution $(\tilde{t}, \tilde{\boldsymbol{P}})$, where $(\tilde{t}, \tilde{\boldsymbol{P}})$ is the parameterized solution given by Theorem 3.38. Let $q > 2$ be the number given by Lemma 3.19. Then for all $p \in [2, q)$ we have*

$$\begin{aligned} \tilde{\boldsymbol{u}} &\in W^{1,\infty}(0, S; \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d), \\ \tilde{\boldsymbol{D}} &\in W^{1,\infty}(0, S; M_{\boldsymbol{D}} \cap \big(L^p(\boldsymbol{\Omega})\big)^d), \\ \tilde{\phi} &\in W^{1,\infty}(0, S; W^{1,p}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})), \end{aligned}$$

where the space $M_{\boldsymbol{D}}$ is given by (3.18)*.*

Proof. We first point out that for any open interval $I \subset \mathbb{R}$ and reflexive Banach spaces X, it is well known that

$$W^{1,\infty}(I; X) = \mathrm{Lip}(I; X),$$

where $\mathrm{Lip}(I; X)$ is the space of all functions $f : I \to X$ which are Lipschitz continuous w.r.t. $t \in I$. Thus to show the claimed regularity of $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{D}}$, $\tilde{\phi}$, we only need to verify the Lipschitz continuity of $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{D}}$, $\tilde{\phi}$ w.r.t. $\sigma \in (0, S)$. Let $\sigma_1, \sigma_2 \in (0, S)$. Taking $t_1 = \tilde{t}(\sigma_1)$, $t_2 = \tilde{t}(\sigma_2)$, $\boldsymbol{P}_1 = \tilde{\boldsymbol{P}}(\sigma_1)$, $\boldsymbol{P}_2 = \tilde{\boldsymbol{P}}(\sigma_2)$ in (3.38), we obtain that

$$\begin{aligned} &\|\tilde{\boldsymbol{u}}(\sigma_1) - \tilde{\boldsymbol{u}}(\sigma_2)\|_{W^{1,p}} + \|\tilde{\phi}(\sigma_1) - \tilde{\phi}(\sigma_2)\|_{W^{1,p}} \\ \le & C\big(|\tilde{t}(\sigma_1) - \tilde{t}(\sigma_2)| + \|\tilde{\boldsymbol{P}}(\sigma_1) - \tilde{\boldsymbol{P}}(\sigma_2)\|_{H^s}\big) \\ \le & C|\sigma_1 - \sigma_2|, \end{aligned} \tag{3.120}$$

where the first inequality comes exactly from (3.38) and for the second inequality we have used the fact from Theorem 3.38 that $(\tilde{t}, \tilde{\boldsymbol{P}})$ has temporal regularity $W^{1,\infty}$ on $(0, S)$ (and is thus Lipschitz continuous on $(0, S)$). From this we obtain the claimed regularity of $\tilde{\boldsymbol{u}}$ and $\tilde{\phi}$. The claimed regularity of $\tilde{\boldsymbol{D}}$ then follows directly from (3.23). This completes the proof. □

3.9 Interpretation of a non-degenerate parameterized solution

In the final section of this chapter we want to discuss the physical interpretation of a vanishing viscosity solution. For the sake of simplicity we focus here only on the parameterized polarization $\tilde{\boldsymbol{P}}$. Similar results can be easily extended to $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{D}}$, $\tilde{\phi}$ in a trivial way, in view of Lemma 3.9 and Lemma 3.13. We first introduce the concept of a *non-degenerate* Lipschitz pair:

Definiton 3.40. *A Lipschitz pair* $(\tilde{t}, \tilde{\boldsymbol{P}}) \in \mathrm{Lip}(0, S; [0, T] \times (H^s(\boldsymbol{\Omega}))^d)$ *is called non-degenerate, if*

$$\tilde{t}'(\sigma) + \|\tilde{\boldsymbol{P}}'(\sigma)\|_{H^s} > 0$$

for a.a. $s \in (0, S)$.

If a parameterized solution is non-degenerate, then we obtain the following very useful expression of a non-degenerate parameterized solution:

Proposition 3.41 ([49, Prop. 5.3, Cor. 5.4])**.** *A non-degenerate Lipschitz pair* $(\tilde{t}, \tilde{\boldsymbol{P}}) \in \mathrm{Lip}(0, S; [0, T] \times (H^s(\boldsymbol{\Omega}))^d)$ *is a parameterized solution of* (3.114) *if and only if there exists a Borel function* $\lambda : (0, S) \to [0, \infty)$ *such that*

$$\begin{aligned} &\boldsymbol{0} \in \partial\Psi_1\big(\tilde{\boldsymbol{P}}'(\sigma)\big) + \lambda(\sigma)\tilde{\boldsymbol{P}}'(\sigma) + D_{\boldsymbol{P}}\mathcal{I}\big(\tilde{t}(\sigma), \tilde{\boldsymbol{P}}(\sigma)\big), \\ &0 = \tilde{t}'(\sigma)\lambda(\sigma) \end{aligned}$$

for a.a. $\sigma \in (0, S)$.

The proof of Proposition 3.41 is again based on several equivalent formulations of the variational identity (3.114), which is similar to the proof of Lemma 3.37. Particularly, the non-degeneracy is essential, since the positivity of the time derivatives given by the inequality

$$\tilde{t}'(\sigma) + \|\tilde{\boldsymbol{P}}'(\sigma)\|_{H^s} > 0$$

implies certain monotonicity of the parameterized solution. However, such monotonicity can in general not be obtained if the above inequality is replaced by equality.

From Proposition 3.41 we obtain the following interpretation of a non-degenerate parameterized solution:

- $\tilde{t}' > 0, \tilde{\boldsymbol{P}}' = \boldsymbol{0}$: this interprets that the polarization will not change as the time increases, which corresponds to a *sticking process*.
- $\tilde{t}' > 0, \tilde{\boldsymbol{P}}' \neq \boldsymbol{0}$: in this case, we must have $\lambda = 0$, therefore we obtain that

  $$\boldsymbol{0} \in \partial\Psi_1\big(\tilde{\boldsymbol{P}}'(\sigma)\big) + D_{\boldsymbol{P}}\mathcal{I}\big(\tilde{t}(\sigma), \tilde{\boldsymbol{P}}(\sigma)\big),$$

  which corresponds to a *rate-independent* evolution.
- $\tilde{t}' = 0, \tilde{\boldsymbol{P}}' \neq \boldsymbol{0}$: The system has switched to a *viscous regime*. Since $\tilde{t}'(\sigma) = 0$, the external time is frozen, which can be understood that the external time scale is much slower than the internal time scale, and since $\tilde{\boldsymbol{P}}'(\sigma) \neq \boldsymbol{0}$, it can be seen that a jump occurs in the external time scale.

It remains an open question whether the solution given by Theorem 3.38 is non-degenerate. In fact, it is shown in [49] that if the underlying set is finite dimensional, then the parameterized solution is always non-degenerate. The idea of the proof is based on the fact that all norms of a finite dimensional normed space are equivalent. Since this does not hold for the infinite dimensional case, one can not obtain the same result by using the method given by [49]. In order to get a non-degenerate parameterized solution, it is suggested in [39] that one possibility would be to apply an alternative reparameterization technique to obtain the non-degenerate property. Since this is out of scope of this thesis, we will not discuss any details here.

Chapter 4

Existence results for dissipation functional of quadratic growth

In this chapter, we focus on the model with quadratic dissipation. In this case, since the dissipation functional is differentiable in $\boldsymbol{P}$, the problem (2.14e) reduces to a semilinear parabolic type equation and the situation becomes totally different to the case with dissipation functional of mixed type. Roughly speaking, we will give local and global existence results for the problem (2.14) with quadratic dissipation functional, which are briefly summarized as follows:

- We will utilize the idea given in [46] to obtain local solutions. More precisely, we will use the Banach fixed point theorem given in [10] to construct local solutions. At the end, we should infer that under the condition

$$\boldsymbol{P}_0 \in \big(\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d, \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d\big)_{\frac{1}{r},r},$$

  the equation (2.14e) admits a unique local solution $\boldsymbol{P}$ with

$$\boldsymbol{P} \in W^{1,r}(0,\hat{T}; \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \cap L^r(0,\hat{T}; \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d)$$

  for some $\hat{T} \in (0,T]$, where $p > d$ and $r > \frac{2p}{p-d}$ are some appropriate given constants. Inserting the local solution $\boldsymbol{P}$ into (2.14), we obtain from certain elliptic existence theory the existence of the solution $(\boldsymbol{u}, \phi)$ of the piezo-system (2.14a) to (2.14d). The corresponding regularity of $(\boldsymbol{u}, \phi)$ is different from case to case due to the different imposed boundary conditions, thus we will make this precise later on in the main theorems respectively. Nevertheless we give a brief introduction to different cases that we will deal with in the following.

- For two dimensional local existence result, we will be dealing with the piezo-system (2.14a) to (2.14d) with mixed boundary conditions. Thus the uniform boundedness of the coefficients will be needed, which guarantees the applicability of Proposition 3.18. For details we refer to Section 4.2

- For three dimensional case, the regularity result given by Proposition 3.18 for the piezo-system with mixed boundary conditions is insufficient. In order to solve this problem, we assume that the underlying domain $\boldsymbol{\Omega}$ has C^1-boundary and the boundary parts corresponding to $(\boldsymbol{u}, \phi)$ are overall Dirichlet. In this case, the regularity result from [14] plays the role of Proposition 3.18, but gives a better regularity of $(\boldsymbol{u}, \phi)$ compared to the one given by Proposition 3.18. As a consequence, the coefficient tensor $\mathbb{B}_1$ need not be uniformly bounded on $\mathbb{R}^3$ anymore, which is a significant improvement compared to the two dimensional case with mixed boundary conditions. We refer to Section 4.3.4 for details.

- We should also point out that unlike in the Proposition 3.18, in three dimensional case the inverse norm of the piezo-operator is not uniform for all admissible $\boldsymbol{P}$ from the underlying space. In fact, the inverse norm will also depend on the modulus of continuity of the function $\boldsymbol{P}$ in an implicit way, see for instance [21, Chap. 7]. We will utilize the continuity arguments given in [46] to overcome such difficulty.

- We are also interested in existence results for less regular domains. Using the regularity results from [1], the existence result for domains with C^1-boundary can be extended to a cuboid (Section 4.3.5) in a natural way. For general polyhedrons and the special piezo-system given by [55], we will utilize the regularity results given in [44] to obtain existence results. See Section 4.4 for more details.

- We also want to give a comparison between the model of this thesis and the thermistor model studied in [46]. The models deal with variables $(\boldsymbol{u}, \phi, \boldsymbol{P})$ and (φ, θ) respectively. The strategies for both models are the same, namely, we reduce the problem to a semilinear parabolic equation with single variable in $\boldsymbol{P}$ (and in θ in the latter case), and then solve the semilinear parabolic equation using the result from [10]. The main difference is that the variable $(\boldsymbol{u}, \phi)$ in this thesis is given as the solution of the piezo-system (vector-wise), while the variable φ in [46] is given as the solution of an elliptic equation (scalar-wise), thus the analysis of this thesis is significantly more complicated than the one given in [46]. We also point out that certain continuity arguments are applied in [46] to obtain the contraction condition in the Banach fixed point theorem from [10]. However, since the external loadings related to the piezo-system given in this thesis will also depend on the variable $\boldsymbol{P}$, a direct application of the continuity arguments given in [46] to our case is difficult (notice that the external loadings related to the elliptic equation corresponding to the variable φ in [46] are independent of θ). Instead, we will use a direct difference comparison method to conclude such contraction condition, see Section 4.2 for details. We should also point out that it is still possible to modify the method in [46] to fit our setting here, but the calculation is not getting simpler due to the complicated expressions of the functionals given in this thesis.

- For the global case, we will apply the Rothe's method given for the case with dissipation functional of mixed type to obtain global results, by setting $\Psi_1 \equiv 0$ in (3.6). Therefore boundedness of the coefficients and their derivatives and replacement of the gradient energy (see Assumption 3.1) will be essential for our analysis.

In the following we will also make extensive use of certain interpolation theory between different function spaces. For the corresponding interpolation theory appearing in this chapter, we refer to Appendix C for details.

4.1 Preliminaries

Let us first introduce the physical model. Due to [55], since the dissipation functional

$$\Psi_\beta(\boldsymbol{P}) = \frac{\beta}{2}\|\boldsymbol{P}\|^2_{L^2(\boldsymbol{\Omega})}$$

is differentiable in $\boldsymbol{P}$, one obtains the following local equations:

$$\boldsymbol{\sigma} = \mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) + \boldsymbol{e}(\boldsymbol{P})^T\nabla\phi \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.1a}$$
$$\boldsymbol{D} = \boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi + \boldsymbol{P} \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.1b}$$
$$\boldsymbol{\Sigma} = \kappa\nabla\boldsymbol{P} \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.1c}$$
$$\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.1d}$$
$$\operatorname{div}\boldsymbol{D} = f_2 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.1e}$$
$$\beta\boldsymbol{P}' = \operatorname{div}\boldsymbol{\Sigma} - D_{\boldsymbol{P}}H - D_{\boldsymbol{P}}\omega + \boldsymbol{f}_3 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.1f}$$

where H, ω are defined by (3.7) and (3.8). We also impose the following boundary and initial value conditions:

$$\boldsymbol{u}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}} = \boldsymbol{u}_D \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \tag{4.2a}$$
$$\boldsymbol{\sigma n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} = \boldsymbol{t} \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}, \tag{4.2b}$$
$$\phi|_{\partial\boldsymbol{\Omega}_{\phi}} = \phi_D \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\phi}, \tag{4.2c}$$
$$\boldsymbol{D}\cdot\boldsymbol{n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}} = \rho \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{D}}, \tag{4.2d}$$
$$\boldsymbol{P}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \boldsymbol{P}_D \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{P}}, \tag{4.2e}$$
$$\boldsymbol{\Sigma n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}} = \boldsymbol{\pi} \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}, \tag{4.2f}$$
$$\boldsymbol{P}(0) = \boldsymbol{P}_0, \tag{4.2g}$$

where $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}, \partial\boldsymbol{\Omega}_{\phi}, \partial\boldsymbol{\Omega}_{\boldsymbol{D}}, \partial\boldsymbol{\Omega}_{\boldsymbol{P}}, \partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}$ are subsets of $\partial\boldsymbol{\Omega}$, see also the notation given at the beginning of Section 2.4.

4.1.1 Fixed point theorem

All of our local results will base on the fixed point theorem formulated in [10] (see also [53]). To introduce the fixed point theorem, we need the following definition.

Definiton 4.1 ([10], [53, Def. 1.1], maximal parabolic regularity)**.** *Let $J = (T_0, T_1)$ be a time interval. Let X be a Banach space and $A : \operatorname{dom}(A) \to X$ be a closed operator with dense domain $\operatorname{dom}(A) \subset X$. Suppose $\tau \in (1,\infty)$. Then we say that A has maximal parabolic $L^\tau(J;X)$-regularity if and only if for every $f \in L^\tau(J;X)$ there is a unique function $w \in W^{1,\tau}(J;X) \cap L^\tau\big(J;\operatorname{dom}(A)\big)$ which satisfies*

$$w'(t) + Aw(t) = f(t) \quad \text{in } X,$$
$$w(T_0) = \mathbf{0}$$

for a.a. $t \in J$.

Remark 4.2. It is shown in [15] that if A has maximal parabolic $L^{\tau_0}(J;X)$-regularity for some $\tau_0 \in (1,\infty)$, then it has maximal parabolic $L^\tau(J;X)$-regularity for all $\tau \in (1,\infty)$. Thus in the following we will only speak of that A has maximal parabolic regularity. △

The maximal parabolic regularity property motivates the following fixed point theorem, which plays the main role in the analysis of local existence results:

Theorem 4.3 ([10], [53, Thm. 3.1])**.** *Let Y, X be Banach spaces, $Y \hookrightarrow X$ densely and let $\tau \in (1,\infty)$. Suppose that $A : J \times (Y,X)_{\frac{1}{\tau},\tau} \to L(Y,X)$ is continuous (where $L(Y,X)$ is defined in Section 2.3) and $A(0,w_0)$ satisfies maximal parabolic regularity on X with $\operatorname{dom}\big(A(0,w_0)\big) = Y$ for some $w_0 \in (Y,X)_{\frac{1}{\tau},\tau}$. Let $S : J \times (Y,X)_{\frac{1}{\tau},\tau} \to X$ be a Carathéodory map, i.e., $S(\cdot,x)$ is measurable for each $x \in (Y,X)_{\frac{1}{\tau},\tau}$ and $S(t,\cdot)$ is continuous for a.a. $t \in J$. Moreover, let $S(\cdot,\mathbf{0})$ be from $L^\tau(J;X)$ and the following assumptions be satisfied:*

(A) For every $M > 0$, there exists a positive constant $L(M)$ such that for all $t \in J$ and all $w, \bar{w} \in (Y,X)_{\frac{1}{\tau},\tau}$ with $\max(\|w\|_{(Y,X)_{\frac{1}{\tau},\tau}}, \|\bar{w}\|_{(Y,X)_{\frac{1}{\tau},\tau}}) \leq M$, we have

$$\|A(t,w) - A(t,\bar{w})\|_{L(Y,X)} \leq L(M)\|w - \bar{w}\|_{(Y,X)_{\frac{1}{\tau},\tau}};$$

(S) For every $M > 0$ there exists a function $h_M \in L^\tau(J)$ such that for all $w, \bar{w} \in (Y,X)_{\frac{1}{\tau},\tau}$ with $\max(\|w\|_{(Y,X)_{\frac{1}{\tau},\tau}}, \|\bar{w}\|_{(Y,X)_{\frac{1}{\tau},\tau}}) \leq M$, it is true that

$$\|S(t,w) - S(t,\bar{w})\|_X \leq h_M(t)\|w - \bar{w}\|_{(Y,X)_{\frac{1}{\tau},\tau}}$$

for a.a. $t \in J$.

Then there exists some $T_{\max} \in J$ such that the problem

$$\begin{aligned} &w'(t) + A\big(t, w(t)\big)w(t) = S\big(t, w(t)\big) \quad \text{in } J \times X, \\ &w(T_0) = w_0 \end{aligned}$$

admits a unique solution $w \in W^{1,\tau}(T_0,\hat{T};X) \cap L^\tau(T_0,\hat{T};Y)$ on $(T_0,\hat{T})$ for every $\hat{T} \in (T_0,T_{\max})$.

Here, the measurability of $S(\cdot,x)$ is understood as the Bochner-measurability, which is given in the following definition:

Definiton 4.4. *Let $J \subset \mathbb{R}$ be an interval and X be a Banach space. A function $u : J \to X$ is called a simple function, if there are measurable sets $A_j \subset J$ and constant values $\mu_j \in X$ with $j = 1,...,n$ for some $n \in \mathbb{N}$ such that*

$$u = \sum_{j=1}^{n} \mu_j \mathbb{1}_{A_j},$$

where $\mathbb{1}_{A_j}$ is the indicator function of the set A_j. A function $u : J \to X$ is called measurable, if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ of simple functions from J to X such that

$$u_n(t) \to u(t) \quad \text{in } X \text{ for a.a. } t \in J.$$

4.1.2 Maximal parabolic regularity and resolvent estimates

For a closed and densely defined operator $A : \mathrm{dom}(A) \to X$, the resolvent set $\rho(A) \subset \mathbb{C}$ is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ such that $\lambda - A$ is bijective from $\mathrm{dom}(A)$ to X and $(\lambda - A)^{-1}$ is a linear and continuous operator on X, i.e., $(\lambda - A)^{-1} \in L(X)$. We obtain from a maximal parabolic regular operator the following very useful resolvent property:

Theorem 4.5 ([15, Thm. 2.2])**.** *Let X be some Banach space and $A : \mathrm{dom}(A) \to X$ be a closed and densely defined linear operator. Assume also that A has maximal parabolic regularity in a finite interval $J = (T_0, T_1)$. Then there exists some $\mathfrak{q} \geq 0$ and some $C > 0$ such that*

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \mathfrak{q}\} \subset \rho(-A)$$

and

$$\mathrm{Re}\lambda \geq \mathfrak{q} \Rightarrow \|(\lambda + A)^{-1}\|_{L(X)} \leq \frac{C}{1 + |\lambda|}, \tag{4.3}$$

where $\rho(-A)$ is the resolvent set of $-A$. In particular, $-A$ generates an analytic semigroup.

For the definition of an analytic semigroup, we refer to [52, Chap. 2.5].

In what follows, we introduce the concept of the so called *positive operator* from [63, 1.14.1]. The definition formulated here is equivalent but slightly different to the original one given in [63], which is due to the purpose of adjusting the model setting of this thesis.

Definiton 4.6 ([63, 1.14.1], positive operator)**.** *Let X be a Banach space and $A : \mathrm{dom}(A) \to X$ be a linear closed operator with dense domain $\mathrm{dom}(A) \subset X$. The operator A is said to be positive, if $[0, \infty)$ belongs to the resolvent set of $-A$ and there exists a number $C \geq 0$ such that*

$$\forall \lambda \in [0, \infty) : \|(\lambda + A)^{-1}\|_{L(X)} \leq \frac{C}{1 + \lambda}.$$

From Corollary 4.9 below we will see that the underlying operator A of the main model (which is the vector-valued Laplace operator $-\Delta$) is a positive operator (more precisely, this is achieved by taking the coefficient matrix $\boldsymbol{\mu}$ in Corollary 4.9 equal to the identity matrix). Such operators provide certain useful interpolation property involving the fractional power of the operator and will be essential for the analysis of local results in the following. For details, we refer to Lemma 4.10 below.

4.1.3 G2-regular set and maximal parabolic regularity

Here we want to give an explanation of how to apply the maximal parabolic regularity to our model. For our case, the operator A is the three dimensional Laplace operator $-\Delta$ defined for the polarization $\boldsymbol{P}$, which motivates the application of the results given in [29]. We first set up the system formulated in [29]. Let $\boldsymbol{\Omega} \subset \mathbb{R}^d$ be the underlying bounded domain and $\boldsymbol{\Gamma} \subset \partial\boldsymbol{\Omega}$ be a relatively closed part of the boundary with positive surface measure. The to be investigated operator A is defined by

$$A = -\nabla \cdot \boldsymbol{\mu} \nabla : H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}) \to H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}).$$

Here, $\boldsymbol{\mu}$ is a symmetric, real coefficient function $\boldsymbol{\mu} : \boldsymbol{\Omega} \to \mathbb{R}^{d \times d}$ which is measurable and essentially bounded in $\boldsymbol{\Omega}$. In particular, $\boldsymbol{\mu}$ is uniformly elliptic, i.e., there exists a constant $c > 0$ such that

$$\boldsymbol{y}^T \boldsymbol{\mu}(\boldsymbol{x}) \boldsymbol{y} \geq c|\boldsymbol{y}|^2$$

for all $\boldsymbol{y} \in \mathbb{R}^d$ and a.a. $\boldsymbol{x} \in \boldsymbol{\Omega}$. We obtain the maximal parabolic regularity of A from the following theorem:

Theorem 4.7 ([29, Thm. 5.4])**.** *Let $\boldsymbol{\Omega} \cup \boldsymbol{\Gamma}$ be G2-regular (c.f. Section 2.1). Let $\tilde{q} := \sup \mathcal{M}$, where*

$$\mathcal{M} := \{q \in [2, \infty) : -\nabla \cdot \boldsymbol{\mu} \nabla + 1 : W^{1,q}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}) \to W^{-1,q}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}) \text{ is linear, continuous and invertible}\}.$$

Then $A = -\nabla \cdot \boldsymbol{\mu} \nabla$ has maximal parabolic regularity for $X = W^{-1,q}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ and $\mathrm{dom}(A) = W^{1,q}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})$ for all $q \in [2, \tilde{q}^*)$, where

$$\tilde{q}^* = \begin{cases} \infty, & \text{if } \tilde{q} \geq d, \\ (\frac{1}{\tilde{q}} - \frac{1}{d})^{-1}, & \text{if } \tilde{q} \in [1, d). \end{cases} \tag{4.4}$$

Due to [25] we know that if $\boldsymbol{\Omega} \cup \boldsymbol{\Gamma}$ is G2-regular, then $\tilde{q} > 2$. Thus from the definition of $\tilde{q}^*$ from Theorem 4.7 we obtain immediately the following result:

Corollary 4.8. *Let $\mathbf{\Omega} \cup \mathbf{\Gamma}$ be G2-regular and $d \in \{2, 3\}$. Then $A = -\nabla \cdot \boldsymbol{\mu} \nabla$ has maximal parabolic regularity with $X = W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})$ and $\mathrm{dom}(A) = W^{1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})$ for all*

$$\begin{cases} q \in [2, \infty), & \text{if } d = 2; \\ q \in [2, 6], & \text{if } d = 3. \end{cases} \tag{4.5}$$

Proof. For $d = 2$ the result is evident. For $\tilde{q} > 2$ in Theorem 4.7 and $d = 3$ we obtain that $\tilde{q}^* = (\frac{1}{\tilde{q}} - \frac{1}{d})^{-1} > 6$. This completes the proof. □

We have already introduced the concept of positive operators (Definition 4.6). In what follows, we show that the maximal parabolic regular operator $A = -\nabla \cdot \boldsymbol{\mu} \nabla$ is indeed a positive operator, which follows from the estimate (4.6) below:

Corollary 4.9. *Let $\mathbf{\Omega} \cup \mathbf{\Gamma}$ be G2-regular and $d \in \{2, 3\}$. Let q be a number satisfying (4.5). Then there exists some $C_q > 0$ depending on q such that*

$$\|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1}\|_{L\left(W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})\right)} \le \frac{C_q}{1 + \lambda} \tag{4.6}$$

for all $\lambda \ge 0$.

Proof. Since q is a number that satisfies (4.5), the operator $-\nabla \cdot \boldsymbol{\mu} \nabla$ is maximal parabolic regular due to Corollary 4.8, and we deduce from Theorem 4.5 that there exists some $\mathfrak{q} \ge 0$ such that

$$\|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1}\|_{L\left(W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})\right)} \le \frac{C_q}{1 + |\lambda|}$$

for all $\mathrm{Re}\lambda \ge \mathfrak{q}$. If $\mathfrak{q} = 0$, then we are done. Otherwise let $\mathfrak{q} > 0$ and consider $\lambda \in \mathbb{R}$ with $0 \le \lambda \le \mathfrak{q}$ (here we consider the closed interval $[0, \mathfrak{q}]$ but not $[0, \mathfrak{q})$, since the compactness of the interval $[0, \mathfrak{q}]$ will be utilized in the remaining part of the proof). From the uniform ellipticity of $\boldsymbol{\mu}$ one deduces immediately from Lax-Milgram that for every $\lambda \ge 0$, the operator

$$-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda : H^1_{\mathbf{\Gamma}}(\mathbf{\Omega}) \to H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega})$$

is in $LH\big(H^1_{\mathbf{\Gamma}}(\mathbf{\Omega}), H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega})\big)$ (see Section 2.3 for the definition of the set $LH(X, Y)$ for Banach spaces X and Y). One easily verifies from Sobolev's embedding that

$$H^1_{\mathbf{\Gamma}}(\mathbf{\Omega}) \hookrightarrow W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega}) \hookrightarrow H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega}),$$

as long as q satisfies (4.5). Thus

$$\begin{aligned} &\|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1}\|_{L\left(W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})\right)} \\ =& \sup_{\boldsymbol{f} \in W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1} \boldsymbol{f}\|_{W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})} / \|\boldsymbol{f}\|_{W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})} \\ \le& C \sup_{\boldsymbol{f} \in W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1} \boldsymbol{f}\|_{W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega})} / \|\boldsymbol{f}\|_{H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega})} \\ \le& C \sup_{\boldsymbol{f} \in W^{-1,q}_{\mathbf{\Gamma}}(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1} \boldsymbol{f}\|_{H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})} / \|\boldsymbol{f}\|_{H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega})} \\ \le& C \sup_{\boldsymbol{f} \in H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega}) \setminus \{\mathbf{0}\}} \|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1} \boldsymbol{f}\|_{H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})} / \|\boldsymbol{f}\|_{H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega})} \\ =& C \|(-\nabla \cdot \boldsymbol{\mu} \nabla + \lambda)^{-1}\|_{L\left(H^{-1}_{\mathbf{\Gamma}}(\mathbf{\Omega}), H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})\right)}. \end{aligned} \tag{4.7}$$

On the other hand, due to [58, CH.III.8] we know that the mapping $LH(X,Y) \ni B \mapsto B^{-1} \in LH(Y,X)$ is continuous. Thus the mapping

$$\lambda \mapsto (-\nabla \cdot \boldsymbol{\mu}\nabla + \lambda)^{-1}$$

is as composition of

$$[0,\infty) \ni \lambda \mapsto \\ -\nabla \cdot \boldsymbol{\mu}\nabla + \lambda \in LH\big(H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}), H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\big)$$

and

$$LH\big(H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}), H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\big) \ni -\nabla \cdot \boldsymbol{\mu}\nabla + \lambda \mapsto \\ (-\nabla \cdot \boldsymbol{\mu}\nabla + \lambda)^{-1} \in LH\big(H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}), H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\big)$$

continuous from $[0,\infty)$ to $LH\big(H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}), H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\big)$. Since $[0,\mathfrak{q}]$ is a compact set, we obtain that

$$\max_{\lambda\in[0,\mathfrak{q}]} \big(1+\lambda\big)\|(-\nabla \cdot \boldsymbol{\mu}\nabla + \lambda)^{-1}\|_{L\left(H^{-1}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega}), H^1_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\right)} < \infty.$$

Together with (4.7) we obtain that

$$\max_{\lambda\in[0,\mathfrak{q}]} \big(1+\lambda\big)\|(-\nabla \cdot \boldsymbol{\mu}\nabla + \lambda)^{-1}\|_{L\left(W^{-1,q}_{\boldsymbol{\Gamma}}(\boldsymbol{\Omega})\right)} < \infty.$$

This implies the claim. □

For a positive operator A, one can define its fractional power operator A^m for appropriate complex numbers m, see for instance [63, Chap. 1.15.1]. In particular, we have the following useful interpolation result involving the fractional power A^m, which will be used several times in the rest of this chapter:

Lemma 4.10 ([63, Chap. 1.15.2])**.** *Let X be a Banach space and $A : \mathrm{dom}(A) \to X$ be a positive operator in the sense of Definition 4.6. Then*

$$\big(X, \mathrm{dom}(A)\big)_{\frac{1}{2},1} \subset \mathrm{dom}(A^{\frac{1}{2}}),$$

where $\mathrm{dom}(A^{\frac{1}{2}})$ *is the definition domain of the fractional power operator* $A^{\frac{1}{2}}$ *of* A.

Here, $\big(X, \mathrm{dom}(A)\big)_{\frac{1}{2},1}$ denotes the real interpolation space of X and $\mathrm{dom}(A)$ with index $(\frac{1}{2}, 1)$. For a precise definition, we refer to Appendix C.

4.1.4 Some more embedding and regularity results

In what follows, we show that the underlying space for $\boldsymbol{P}$ is continuously embedded to some Hölder space. The Hölder continuity of $\boldsymbol{P}$ will ensure us to apply similar difference estimation arguments as the ones given in Section 3.5 to verify the condition (S) in Theorem 4.3.

Lemma 4.11. *Let $d \in \{2,3\}$ and $\mathbf{\Omega} \subset \mathbb{R}^d$ be a bounded domain. Let $\partial\mathbf{\Omega}_{\boldsymbol{P}} \subset \partial\mathbf{\Omega}$ be a $(d-1)$-set and $\mathbf{\Omega} \cup \partial\mathbf{\Omega}_{\boldsymbol{P}}$ be G2-regular. Then for all*

$$\begin{cases} (p,d) \in (2,\infty) \times \{2\} \text{ or} \\ (p,d) \in (3,6] \times \{3\} \end{cases} \tag{4.8}$$

and $r \in (\frac{2p}{p-d}, \infty)$, the space $\big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d\big)_{\frac{1}{r},r}$ is continuously embedded into the space $\big(W^{1-2\mathfrak{p},p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d$ for some $\mathfrak{p} \in (\frac{1}{r}, \frac{p-d}{2p})$. Consequently, we have the embedding

$$\big(W^{1-2\mathfrak{p},p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d \hookrightarrow \big(C^\delta(\overline{\mathbf{\Omega}})\big)^d \tag{4.9}$$

with $\delta = 1 - 2\mathfrak{p} - \frac{d}{p} \in (0,1)$.

Remark 4.12. From Lemma 4.11 we infer in particular that every $\boldsymbol{P}$ from the space $\big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d\big)_{\frac{1}{r},r}$ is also an element of $\big(W^{1-2\mathfrak{p},p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d$, which implies particularly that $\boldsymbol{P}|_{\partial\mathbf{\Omega}_{\boldsymbol{P}}} = \mathbf{0}$. △

Proof. From the condition that $\mathfrak{p} \in (\frac{1}{r}, \frac{p-d}{2p})$ we obtain immediately that

$$\delta = 1 - 2\mathfrak{p} - \frac{d}{p} \in (0,1).$$

Thus (4.9) follows immediately from Sobolev's embedding theorem. It is left to show that there exists some $\mathfrak{p} \in (\frac{1}{r}, \frac{p-d}{2p})$ such that

$$\big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d\big)_{\frac{1}{r},r} \hookrightarrow \big(W^{1-2\mathfrak{p},p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d.$$

We utilize the same idea given in [46, Lem. A.1] to prove the claim. Let $\tau \in (0, \frac{p-d}{2p})$. We obtain that

$$\begin{aligned}
&\big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d\big)_{\tau,1} \\
=&\Big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \Big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d\Big)_{\frac{1}{2},1}\Big)_{2\tau,1} \\
=&\Big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \Big(\big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d\Big)_{\frac{1}{2},1}\Big)_{2\tau,1} \\
=&\Big(\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \Big(\big(W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d, \operatorname{dom}(-\Delta)\Big)_{\frac{1}{2},1}\Big)_{2\tau,1},
\end{aligned} \tag{4.10}$$

where the first equality comes from the reiteration theorem [63, Chap. 1.10.2] (by setting

$$\begin{aligned}
&E_0 = W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega}), E_1 = \big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega}), W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)_{\frac{1}{2},1}, \\
&A_0 = W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega}), A_1 = W^{-1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega}), \\
&\theta_0 = 0, \theta_1 = \frac{1}{2}, p = 1, \lambda = 2\tau
\end{aligned}$$

therein), the second from the property that

$$(X,Y)_{\theta,q} = (Y,X)_{1-\theta,q} \tag{4.11}$$

for Banach spaces X, Y, see Theorem C.5, and the last equality from the fact that

$$\big(W^{1,p}_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}(\mathbf{\Omega})\big)^d = \operatorname{dom}(-\Delta),$$

where dom$(-\Delta)$ is defined as the domain of the operator $-\Delta : \big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d \to \big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d$. If we define

$$A = -\nabla \cdot \boldsymbol{\mu}\nabla := -\Delta$$

(or equivalently $\boldsymbol{\mu} = \mathrm{Id}$), then the coefficient function $\boldsymbol{\mu}$ is symmetric, essentially bounded and uniformly elliptic on $\boldsymbol{\Omega}$. Thus from Corollary 4.9 and Lemma 4.10 we obtain that

$$\Big(\big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d, \mathrm{dom}(-\Delta)\Big)_{\frac{1}{2},1} \hookrightarrow \mathrm{dom}\big((-\Delta)^{\frac{1}{2}}\big). \tag{4.12}$$

From [63, 1.15.2, (b)] we infer that $(-\Delta)^{-\frac{1}{2}}$ is a linear and bounded operator on $\big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d$. Thus from [52, Thm. 2.6.8, (a)] we obtain that

$$\mathrm{dom}\big((-\Delta)^{\frac{1}{2}}\big) = \mathrm{Ran}\big((-\Delta)^{-\frac{1}{2}}\big) = (-\Delta)^{-\frac{1}{2}}\Big(\big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d\Big), \tag{4.13}$$

where Ran$\big((-\Delta)^{-\frac{1}{2}}\big)$ denotes the range of $(-\Delta)^{-\frac{1}{2}}$ on $\big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d$. From [29, Thm. 4.3] it follows that

$$(-\Delta)^{-\frac{1}{2}}\Big(\big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d\Big) \hookrightarrow \big(L^p(\boldsymbol{\Omega})\big)^d. \tag{4.14}$$

Finally, from (4.12) to (4.14) we conclude that

$$\Big(\big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d, \mathrm{dom}(-\Delta)\Big)_{\frac{1}{2},1} \hookrightarrow \big(L^p(\boldsymbol{\Omega})\big)^d. \tag{4.15}$$

It follows that

$$\begin{aligned}
&(\big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d, \big(W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d)_{\tau,1}\\
\hookrightarrow&(\big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d, \big(L^p(\boldsymbol{\Omega})\big)^d)_{2\tau,1}\\
=&(\big(L^p(\boldsymbol{\Omega})\big)^d, \big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d)_{1-2\tau,1}\\
\hookrightarrow&[\big(L^p(\boldsymbol{\Omega})\big)^d, \big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d]_{1-2\tau}\\
=&\big(W^{1-2\tau,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^d,
\end{aligned} \tag{4.16}$$

where the first embedding comes from (4.10) and (4.15), the first equality is obtained by using (4.11), the second embedding comes from the fact that

$$(X, Y)_{\theta,1} \hookrightarrow [X, Y]_\theta,$$

see Theorem C.10, and the second equality is deduced from [23, Thm. 3.1] (notice that τ is in $(0, \frac{p-d}{2p})$, which implies in particular that $1 - 2\tau > \frac{d}{p} > \frac{1}{p}$ and consequently that $1 - 2\tau \neq \frac{1}{p}$, being the condition of [23, Thm. 3.1]). Now from the second and third properties of Theorem C.5 we infer that

$$\begin{aligned}
\big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega}), W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)_{\frac{1}{r},r} &\hookrightarrow \big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega}), W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)_{\frac{1}{r},1}\\
&\hookrightarrow \big(W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega}), W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)_{\tau,1}
\end{aligned}$$

for $\tau \in (\frac{1}{r}, 1)$, since $W^{1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega}) \subset W^{-1,p}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})$. The condition on r implies that the interval $(\frac{1}{r}, \frac{p-d}{2p})$ is not empty, thus one can choose some $\mathfrak{p} \in (\frac{1}{r}, \frac{p-d}{2p}) \subset (0, \frac{p-d}{2p})$ such that (4.16) is valid. This completes the proof. □

At the end, we would like to close this section by showing that the local solutions $\boldsymbol{P}$ deduced from the local existence results (Theorem 4.22, Theorem 4.39 etc. given below) are also Hölder continuous w.r.t. both time and space.

Proposition 4.13. *Let $d \in \{2, 3\}$ and $\boldsymbol{\Omega} \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let $\partial\boldsymbol{\Omega}_{\boldsymbol{P}} \subset \partial\boldsymbol{\Omega}$ be a $(d-1)$-set and $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ be G2-regular. Then for (p, d) satisfying (4.8) and $r \in (\frac{2p}{p-d}, \infty)$, the embedding*

$$W^{1,r}(J; \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \cap L^r(J; \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \hookrightarrow C^\delta(\bar{J}; \big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^d)$$

is valid for each interval $J \subset \mathbb{R}$ with some $\delta \in (0, 1)$.

Proof. From [3, Thm. 3] we have the embedding

$$W^{1,r}(J; \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \cap L^r(J; \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \hookrightarrow C^{s-\frac{1}{r}}(\bar{J}; \Big(\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega}), W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)_{\theta,1}\Big)^d)$$

for $s \in (\frac{1}{r}, 1)$ and $\theta \in [0, 1-s)$. From Theorem C.5 we have

$$\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega}), W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)_{\theta,1} = \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega}), W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)_{1-\theta,1}.$$

Notice that $1-\theta \in (s, 1]$ and $s \in (\frac{1}{r}, 1)$ and $1-\theta, s$ can be arbitrary chosen in these intervals. The condition on r implies that $\frac{1}{r} < \frac{p-d}{2p}$, thus one can choose s sufficiently small and θ sufficiently large such that $s \in (\frac{1}{r}, \frac{p-d}{2p})$ and $1-\theta \in (s, \frac{p-d}{2p}] \subset (0, \frac{p-d}{2p})$. Then (4.16) is satisfied and we obtain that

$$\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega}), W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)_{1-\theta,1} \hookrightarrow C^{1-2(1-\theta)-\frac{d}{p}}(\overline{\boldsymbol{\Omega}}).$$

Choosing $\delta := \min\{s - \frac{1}{r}, 1-2(1-\theta)-\frac{d}{p}\} \in (0, 1)$ we obtain the desired result. □

4.2 2D-local existence result for Gröger-regular domains

In this section we consider the case $d = 2$. First, we give the Assumptions B1 to B4 in the following, which give us sufficient conditions to formulate a weak form corresponding to the coupled elliptic-parabolic differential system (4.1). It turns out that the Assumptions B1 to B4 are also sufficient for proving the existence of a local solution of the differential system (4.1). We make this precise in the following.

4.2.1 Assumptions and weak formulation

B1 $\boldsymbol{\Omega} \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary, $\partial\boldsymbol{\Omega}_{\boldsymbol{u}} \dot\cup \partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}} = \partial\boldsymbol{\Omega}_\phi \dot\cup \partial\boldsymbol{\Omega}_{\boldsymbol{D}} = \partial\boldsymbol{\Omega}_{\boldsymbol{P}} \dot\cup \partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}} = \partial\boldsymbol{\Omega}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \partial\boldsymbol{\Omega}_\phi, \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ are 1-sets, $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_\phi$ are G1-regular, $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ is G2-regular (c.f. Section 2.1).

B2 $\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ (c.f. Section 2.4) are differentiable on $\mathbb{R}^2$ and their derivatives are locally Lipschitzian on $\mathbb{R}^2$; $\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ are uniformly bounded on $\mathbb{R}^2$, i.e.,

$$\Xi := \sup_{\boldsymbol{P} \in \mathbb{R}^2} \Big\{ |\mathbb{C}(\boldsymbol{P})|, |\boldsymbol{e}(\boldsymbol{P})|, |\boldsymbol{\varepsilon}^0(\boldsymbol{P})|, |\boldsymbol{\epsilon}(\boldsymbol{P})| \Big\} < \infty;$$

$\omega : \mathbb{R}^2 \to \mathbb{R}$ (c.f. Section 2.4) is a polynomial of sixth order with constant coefficients.

B3 There exists some $\alpha > 0$ such that for all $\boldsymbol{P} \in \mathbb{R}^2$, $\boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^2, \mathbb{R}^2)$, $\boldsymbol{D} \in \mathbb{R}^2$

$$\begin{aligned} \mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} &\geq \alpha|\boldsymbol{\varepsilon}|^2, \\ \boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{D} \cdot \boldsymbol{D} &\geq \alpha|\boldsymbol{D}|^2. \end{aligned}$$

B4 There exist $p^* \in (2, \infty)$ and $r^* \in [1, \infty)$, such that

$$\begin{aligned} &\boldsymbol{f}_1 \in L^{2r^*}\big(0, T; \big(L^{\frac{2p^*}{p^*+2}}(\boldsymbol{\Omega})\big)^2\big), \\ &f_2 \in L^{2r^*}\big(0, T; L^{\frac{2p^*}{p^*+2}}(\boldsymbol{\Omega})\big), \\ &\boldsymbol{f}_3 \in L^{r^*}\big(0, T; \big(L^{\frac{2p^*}{p^*+2}}(\boldsymbol{\Omega})\big)^2\big), \\ &\boldsymbol{t} \in L^{2r^*}\big(0, T; \big(L^{\frac{p^*}{2}}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^2\big), \\ &\rho \in L^{2r^*}\big(0, T; L^{\frac{p^*}{2}}(\partial\boldsymbol{\Omega}_{\boldsymbol{D}})\big), \\ &\boldsymbol{\pi} \in L^{r^*}\big(0, T; \big(L^{\frac{p^*}{2}}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^2\big), \\ &\boldsymbol{u}_D \in L^{2r^*}\big(0, T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^2\big), \\ &\phi_D \in L^{2r^*}\big(0, T; B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\phi})\big), \\ &\boldsymbol{P}_D \in W^{1,r^*}\big(0, T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^2\big). \end{aligned}$$

Remark 4.14. Using Lemma 3.8 we infer that there exist $\boldsymbol{u}_D$, ϕ_D, $\boldsymbol{P}_D$ such that

$$\begin{aligned} &\boldsymbol{u}_D \in L^{2r^*}\big(0, T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^2\big), && \boldsymbol{u}_D|_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}} = \boldsymbol{u}_D, \\ &\phi_D \in L^{2r^*}\big(0, T; W^{1,p^*}(\boldsymbol{\Omega})\big), && \phi_D|_{\partial\boldsymbol{\Omega}_{\phi}} = \phi_D, \\ &\boldsymbol{P}_D \in W^{1,r^*}\big(0, T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^2\big), && \boldsymbol{P}_D|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \boldsymbol{P}_D. \end{aligned}$$

In particular, from Lemma 3.5 and the analysis given below Lemma 3.8 we obtain that

$$\begin{aligned} \boldsymbol{f}_1,\, \boldsymbol{t},\, \boldsymbol{\varepsilon}(\boldsymbol{u}_D) &\in L^{2r^*}\big(0, T; \big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2\big), \\ f_2,\, \rho,\, \nabla\phi_D &\in L^{2r^*}\big(0, T; W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})\big), \\ \boldsymbol{f}_3,\, \boldsymbol{\pi} &\in L^{r^*}\big(0, T; \big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2\big), \\ \boldsymbol{P}_D,\, \Delta\boldsymbol{P}_D &\in W^{1,r^*}\big(0, T; \big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2\big), \end{aligned}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}_D)$ is the small strain tensor generated by $\boldsymbol{u}_D$. △

Remark 4.15. We point out that unlike the case with dissipation functional of mixed type (Chapter 3), the leading component of the polynomial ω need not be positive, since we will not use any kind of variational method involving minimizers to conclude the existence of $\boldsymbol{P}$ (and thus no coercivity will be used, which requires the positivity of the leading component of ω). △

We point out that the variables $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$ given in the differential system (4.1) and (4.2) admit Dirichlet boundary conditions $\boldsymbol{u}_D$, ϕ_D, $\boldsymbol{P}_D$ respectively. Replacing $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$, $\boldsymbol{P}_0$ in

(4.1) and (4.2) by $\tilde{\boldsymbol{u}}$, $\tilde{\phi}$, $\tilde{\boldsymbol{P}}$, $\tilde{\boldsymbol{P}}_0$ and writing

$$\begin{aligned}\tilde{\boldsymbol{u}} &= \boldsymbol{u} + \boldsymbol{u}_D,\\ \tilde{\phi} &= \phi + \phi_D,\\ \tilde{\boldsymbol{P}} &= \boldsymbol{P} + \boldsymbol{P}_D,\\ \tilde{\boldsymbol{P}}_0 &= \boldsymbol{P}_0 + \boldsymbol{P}_D(0),\end{aligned}$$

we obtain from (4.1) and (4.2) the transformed differential system

$$\boldsymbol{\sigma} = \mathbb{C}(\tilde{\boldsymbol{P}})\big(\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}) - \boldsymbol{\varepsilon}^0(\tilde{\boldsymbol{P}})\big) + \boldsymbol{e}(\tilde{\boldsymbol{P}})^T\nabla\tilde{\phi} \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.18a}$$

$$\boldsymbol{D} = \boldsymbol{e}(\tilde{\boldsymbol{P}})\big(\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}) - \boldsymbol{\varepsilon}^0(\tilde{\boldsymbol{P}})\big) - \boldsymbol{\epsilon}(\tilde{\boldsymbol{P}})\nabla\tilde{\phi} + \tilde{\boldsymbol{P}} \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.18b}$$

$$\boldsymbol{\Sigma} = \kappa\nabla\tilde{\boldsymbol{P}} \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.18c}$$

$$\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}_1 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.18d}$$

$$\operatorname{div}\boldsymbol{D} = f_2 \quad \text{in } (0,T)\times\boldsymbol{\Omega}, \tag{4.18e}$$

$$\beta\tilde{\boldsymbol{P}}' = \operatorname{div}\boldsymbol{\Sigma} - D_{\boldsymbol{P}}\tilde{H} - D_{\boldsymbol{P}}\tilde{\omega} + \boldsymbol{f}_3 \quad \text{in } (0,T)\times\boldsymbol{\Omega} \tag{4.18f}$$

and

$$\boldsymbol{u}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}} = \boldsymbol{0} \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{u}}, \tag{4.19a}$$

$$\boldsymbol{\sigma}\boldsymbol{n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} = \boldsymbol{t} \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}, \tag{4.19b}$$

$$\phi|_{\partial\boldsymbol{\Omega}_\phi} = 0 \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_\phi, \tag{4.19c}$$

$$\boldsymbol{D}\cdot\boldsymbol{n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}} = \rho \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{D}}, \tag{4.19d}$$

$$\boldsymbol{P}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \boldsymbol{0} \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{P}}, \tag{4.19e}$$

$$\boldsymbol{\Sigma}\boldsymbol{n}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}} = \boldsymbol{\pi} \quad \text{on } (0,T)\times\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}, \tag{4.19f}$$

$$\boldsymbol{P}(0) = \boldsymbol{P}_0 = \tilde{\boldsymbol{P}}_0 - \boldsymbol{P}_D(0), \tag{4.19g}$$

where $\tilde{H}$ and $\tilde{\omega}$ in (4.18f) are defined by

$$\begin{aligned}\tilde{H}(t,\boldsymbol{u},\phi,\boldsymbol{P}) &= H\big(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\nabla\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big),\\ \tilde{\omega}(t,\boldsymbol{P}) &= \omega\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)\end{aligned}$$

and H and ω are defined by (3.7) and (3.8). Using (4.18) and (4.19) we give the weak formulation which is to be investigated in the following: Find $(\boldsymbol{u},\phi,\boldsymbol{P}) : (0,T) \to \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega}) \times \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$ such that

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1\big(\boldsymbol{P}(t)+\boldsymbol{P}_D(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\\ \nabla\phi(t)\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x} = l_{t,\boldsymbol{P}(t)+\boldsymbol{P}_D(t)}(\bar{\boldsymbol{u}},\bar{\phi}), \tag{4.20a}$$

$$\beta\boldsymbol{P}'(t) - \kappa\Delta\boldsymbol{P}(t) = S\big(t,\boldsymbol{u}(t),\phi(t),\boldsymbol{P}(t)\big) \quad \text{in } \big(H^{-1}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2, \tag{4.20b}$$

$$\boldsymbol{P}(0) = \boldsymbol{P}_0 \tag{4.20c}$$

for a.a $t \in (0,T)$ and all $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times H^1_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$, where

$$S(t,\boldsymbol{u},\phi,\boldsymbol{P}) = -\mathcal{Q}(t,\boldsymbol{u},\phi,\boldsymbol{P}) - \big(\beta\boldsymbol{P}'_D(t) - \kappa\Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big), \tag{4.21}$$

$\mathbb{B}_1$, $l_{t,\boldsymbol{P}}$ are defined by (3.3) and (3.4) and $\mathcal{Q}$ is defined by

$$\begin{aligned}&\mathcal{Q}(t,\boldsymbol{u},\phi,\boldsymbol{P})[\bar{\boldsymbol{P}}]\\ &= \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\tilde{H}\big(t,\boldsymbol{u},\phi,\boldsymbol{P}\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\tilde{\omega}(t,\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}\\ &= \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}H\big(t,\boldsymbol{u},\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\omega\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x}\end{aligned} \tag{4.22}$$

for $\bar{\boldsymbol{P}} \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$.

Remark 4.16. While thanks to the Assumption B2, the tensor $\mathbb{B}_1(\boldsymbol{P})$ is uniformly bounded w.r.t. $\boldsymbol{P} \in \mathbb{R}^2$ (and hence the l.h.s. of (4.20a) is well-defined for all $(\boldsymbol{u}, \phi)$, $(\bar{\boldsymbol{u}}, \bar{\phi})$ in $\big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times H^1_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})$), we point out that the coefficient tensor $D_{\boldsymbol{P}}\mathbb{B}_1$ defined by (3.3) (which appears in $D_{\boldsymbol{P}}\tilde{H}$ here) need not be uniformly bounded on $\mathbb{R}^2$. This differs from the case with dissipation functional of mixed type studied in Chapter 3, where the uniform boundedness condition on $D_{\boldsymbol{P}}\mathbb{B}_1$ is also imposed. This is due to the fact that the underlying space of the latter case is only embedded to some Lebesgue space of finite order and therefore not necessarily bounded on $\overline{\boldsymbol{\Omega}}$, while the underlying space for $\boldsymbol{P}$ in the former case is embedded to some Hölder space defined on $\overline{\boldsymbol{\Omega}}$, see Lemma 4.11 given previously. △

Remark 4.17. At this moment we have defined the functional S as a functional taking the variable $(t, \boldsymbol{u}, \phi, \boldsymbol{P})$, which still differs from the one given in Theorem 4.3, since S therein is a functional having only the variable $(t, \boldsymbol{P})$. However, we will see that under certain conditions, the variable $(\boldsymbol{u}, \phi)$ is uniquely determined by a given pair $(t, \boldsymbol{P})$, which is seen as the unique solution of (4.20a), see Lemma 4.19 below. In this case, we can define

$$S(t, \boldsymbol{P}) = S\big(t, \boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P}), \boldsymbol{P}\big)$$

and Theorem 4.3 is applicable. △

A first fundamental question is the well-definedness of the integrals appearing in (4.20a) and $\mathcal{Q}(t, \boldsymbol{u}, \phi, \boldsymbol{P})$. In our proof of local existence result, we will deal with uniformly continuous $\boldsymbol{P}$ (more precisely, $\boldsymbol{P}$ is in the underlying space $\big(\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2, \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2\big)_{\frac{1}{r},r}$ with $p > 2$ and $r > \frac{2p}{p-2}$, which is embedded to some Hölder space $\big(C^{\delta}(\overline{\boldsymbol{\Omega}})\big)^2$ with some $\delta > 0$, see Lemma 4.11). Also, one can obtain from (4.29) below that $(t, \boldsymbol{x}) \mapsto \boldsymbol{P}_D(t, \boldsymbol{x})$ is uniformly continuous on $[0, T] \times \overline{\boldsymbol{\Omega}}$. Thus the coefficient tensors appearing in Assumption B2 and their derivatives evaluated at $\tilde{\boldsymbol{P}} = \boldsymbol{P} + \boldsymbol{P}_D$ will also be uniformly continuous on $[0, T] \times \overline{\boldsymbol{\Omega}}$, and one can therefore verify that the regularity Lemma A.8 is still applicable. Hence we obtain the well-definedness of the integrals in (4.20a) and of $\mathcal{Q}(t, \boldsymbol{u}, \phi, \boldsymbol{P})$. We summarize this result in the following lemma without giving a proof, since this is only a straightforward but tedious verification of the conditions of Lemma A.8.

Lemma 4.18. *Let the Assumptions B1 to B4 be satisfied. Then the integrals*

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1(\hat{\boldsymbol{P}}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \nabla\phi \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x}$$

and

$$\int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\tilde{H}\big(t, \boldsymbol{u}, \phi, \boldsymbol{P}\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\tilde{\omega}(t, \boldsymbol{P})(\bar{\boldsymbol{P}}) d\boldsymbol{x}$$

are well-defined for a.a. $t \in (0, T)$, *all measurable* $\hat{\boldsymbol{P}} : \mathbb{R}^2 \to \mathbb{R}^2$, $(\boldsymbol{u}, \phi)$, $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times H^1_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})$ *and all* $\boldsymbol{P}$, $\bar{\boldsymbol{P}} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^2$.

4.2.2 Admissible pair of parameters

In order to formulate the main result Theorem 4.22 we still need to give the definition of an *admissible pair* of parameters. The definition of an admissible pair is based on the following regularity result, which is a direct consequence of Proposition 3.18:

Lemma 4.19. *Let the Assumptions B1 to B3 be satisfied. For a measurable function* $\boldsymbol{P} : \boldsymbol{\Omega} \to \mathbb{R}^2$ *and a number* $q \in [1, \infty]$ *with Hölder conjugate* q'*, the operator* $L_{\boldsymbol{P}}$ *is defined by*

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi)[\bar{\boldsymbol{u}}, \bar{\phi}] := \int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \nabla \phi \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \\ \nabla \bar{\phi} \end{pmatrix} d\boldsymbol{x}$$

for $(\boldsymbol{u}, \phi) \in \big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})$ *and* $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \big(W^{1,q'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{1,q'}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})$*. Then there exists some* $p_* \in (2, \infty)$ *such that* $L_{\boldsymbol{P}}$ *is linear, continuous and bijective from*

$$\big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}) \text{ to } \big(W^{-1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{-1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})$$

for all $q \in [2, p_*]$*. In particular, the norm* $C_{\boldsymbol{P}}$ *of the inverse operator* $L_{\boldsymbol{P}}^{-1}$ *is uniformly bounded by some positive constant* C_* *for all measurable* $\boldsymbol{P}$ *and* $q \in [2, p_*]$*, and* C_**,* p_* *depend only on the upper bound* Ξ *of the coefficients given in Assumption B2 and the elliptic constant* α *given in Assumption B3 but not on the particular choice of* $\boldsymbol{P}$*.*

Proof. One sees that the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})$ of $L_{\boldsymbol{P}}$ is measurable, since $\mathbb{B}_1$ is continuous on $\mathbb{R}^2$. Also, $\mathbb{B}_1(\boldsymbol{P})$ is bounded above by some positive constant C_Ξ (depending only on Ξ) for all measurable $\boldsymbol{P}$. On the other hand, $L_{\boldsymbol{P}}$ is elliptic w.r.t. the elliptic constant α given in Assumption B3 for all measurable $\boldsymbol{P}$. Thus the conditions of Proposition 3.18 are satisfied and the claim follows immediately by applying Proposition 3.18 to $L_{\boldsymbol{P}}$. □

Motivated by Lemma 4.11 and Lemma 4.19, we give the following definition of an admissible pair of parameters:

Definiton 4.20. *A pair* (p, r) *is called admissible, if the pair* (p, r) *satisfies*

$$p \in (2, p_*),$$

where p_* *is the number given by Lemma 4.19, and*

$$r \in (\frac{2p}{p-2}, \infty).$$

Remark 4.21. In Theorem 4.22 we will assume that the pair (p^*, r^*) given in Assumption B4 satisfies $p^* \geq p$ and $r^* \geq r$. We point out that such assumption is legit and makes sense from the following viewpoint: notice that in Lemma 4.19 we have only used the Assumptions B1 to B3, thus an admissible pair is independent on the Assumption B4. It is namely legit that we first choose an admissible pair (p, r) due to the Assumptions B1 to B3, and then determine the components (p^*, r^*) given in the Assumption B4. △

4.2.3 Local existence result for Gröger-regular domains

Having given all the preliminaries, we are able to state our main result as follows:

Theorem 4.22. *Let the Assumptions B1 to B3 be satisfied and* β, κ *in* (4.20) *be given positive constants. Let* (p, r) *be an admissible pair in the sense of Definition 4.20. Let the Assumption B4 be satisfied with* $p^* \in [p, \infty)$ *and* $r^* \in [r, \infty)$*. Assume also that*

$$\boldsymbol{P}_0 \in \big(\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2, \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2\big)_{\frac{1}{r},r}.$$

Then the differential system (4.20) *has a unique local solution* $(\boldsymbol{u}, \phi, \boldsymbol{P})$ *in the time interval* $(0, \hat{T})$ *for some* $0 < \hat{T} \leq T$ *such that*

$$\begin{aligned} &\boldsymbol{u} \in L^{2r^*}(0, \hat{T}; \big(W^{1,\hat{p}}_{\partial \boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2), \\ &\phi \in L^{2r^*}(0, \hat{T}; W^{1,\hat{p}}_{\partial \boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})), \\ &\boldsymbol{P} \in W^{1,r}(0, \hat{T}; \big(W^{-1,p}_{\partial \boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2) \cap L^r(0, \hat{T}; \big(W^{1,p}_{\partial \boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2), \end{aligned} \tag{4.23}$$

where p_* *is given by Lemma 4.19 and* $\hat{p} := \min\{p_*, p^*\}$.

Remark 4.23. Notice that we have different temporal exponents $2r^*$ and r and spatial exponents $\hat{p}$ and p for $(\boldsymbol{u}, \phi)$ and $\boldsymbol{P}$ respectively. The reason is that the regularity of $(\boldsymbol{u}, \phi)$ follows from Lemma 4.19, while the regularity of $\boldsymbol{P}$ is deduced from Theorem 4.3. △

Proof. For the sake of simplicity we assume that $\beta = \kappa = 1$, since the size of β and κ has no influence on the results. We formulate the following notation, which corresponds to the one given in Theorem 4.3:

$$\begin{aligned} A(t, \boldsymbol{P}) &\equiv -\Delta, \\ \tau &= r, \\ Y &= \big(W^{1,p}_{\partial \boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2, \\ X &= \big(W^{-1,p}_{\partial \boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2 \end{aligned}$$

and

$$\begin{aligned} S(t, \boldsymbol{P}) &= S\big(t, \boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P}), \boldsymbol{P}\big) \\ &= -\mathcal{Q}\big(t, \boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P}), \boldsymbol{P}\big) - \big(\boldsymbol{P}'_D(t) - \Delta \boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big), \end{aligned} \tag{4.24}$$

where $\big(\boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P})\big)$ is the unique weak solution of the differential equation

$$L_{\boldsymbol{P}+\boldsymbol{P}_D(t)}\big(\boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P})\big) = l_{t, \boldsymbol{P}+\boldsymbol{P}_D(t)} \tag{4.25}$$

with $L_{\boldsymbol{P}+\boldsymbol{P}_D(t)}$ defined by Lemma 4.19 and $l_{t,\boldsymbol{P}+\boldsymbol{P}_D(t)}$ defined by (3.4). Here we insist on the notation $S(t, \boldsymbol{P})$ but not define a new functional $\tilde{S}(t, \boldsymbol{P})$, which is for the purpose of avoiding unnecessary redundant notation, see also Remark 4.17. The existence, uniqueness and regularity of $\big(\boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P})\big)$ are deduced from Lemma 4.19 (to see that Lemma 4.19 is applicable, we refer to Step 1c below). In particular, $\big(\boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P})\big)$ is uniquely determined by a given pair $(t, \boldsymbol{P})$, thus $S(t, \boldsymbol{P})$ is well-defined. Having defined this notation, we utilize Theorem 4.3 to show that the equation

$$\boldsymbol{P}'(t) + A\big(t, \boldsymbol{P}(t)\big) = S\big(t, \boldsymbol{P}(t)\big)$$

with initial value $\boldsymbol{P}_0$ has the claimed unique local solution $\boldsymbol{P}$ given in (4.23). We first give the following statements corresponding to part of the conditions from Theorem 4.3, which is relatively easier to verify:

1. From Corollary 4.8 it follows immediately that $Y \hookrightarrow X$ densely, $A(0, \boldsymbol{P}_0) = -\Delta$ satisfies maximal parabolic regularity on X with $\operatorname{dom}\big(A(0, \boldsymbol{P}_0)\big) = Y$ and $A(t, \boldsymbol{P})$ is continuous from $[0, T] \times (Y, X)_{\frac{1}{\tau}, \tau}$ to $L(Y, X)$ (since A is constantly valued and equal to $-\Delta$).

2. Since A is constantly equal to $-\Delta$, the validity of (A) in Theorem 4.3 is evident.

3. Now we show that

$$S(t,\cdot):(Y,X)_{\frac{1}{\tau},\tau}\to X$$

is continuous for a.a. $t\in[0,T]$. Let $\boldsymbol{P}\in(Y,X)_{\frac{1}{\tau},\tau}$ be arbitrary. For $\epsilon>0$ define

$$\mathcal{O}_\epsilon(\boldsymbol{P}):=\{\bar{\boldsymbol{P}}\in(Y,X)_{\frac{1}{\tau},\tau}:\|\boldsymbol{P}-\bar{\boldsymbol{P}}\|_{(Y,X)_{\frac{1}{\tau},\tau}}<\epsilon\}. \tag{4.26}$$

Thus to show the continuity, it suffices to show that

$$\lim_{\epsilon\to 0}\sup_{\bar{\boldsymbol{P}}\in\mathcal{O}_\epsilon(\boldsymbol{P})}\|S(t,\boldsymbol{P})-S(t,\bar{\boldsymbol{P}})\|_X=0. \tag{4.27}$$

But this turns out to be a direct consequence of the Assumption (S) of Theorem 4.3, which will be shown in the rest part of the proof below and we do not repeat here.

It is left to show that

- given fixed $\boldsymbol{P}$ in $(Y,X)_{\frac{1}{\tau},\tau}$, the mapping $t\mapsto S(t,\boldsymbol{P})$ is Bochner-measurable,
- the validity of (S) in Theorem 4.3 and
- $S(\cdot,\boldsymbol{0})$ is from $L^r(0,T;X)$.

We will show these statements in the following steps:

Step 1: Bochner-measurability of $t\mapsto S(t,\boldsymbol{P})$

Let $\boldsymbol{P}\in(Y,X)_{\frac{1}{\tau},\tau}$ be given and denote by W the space

$$W:=\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2\times W^{1,p}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega}). \tag{4.28}$$

Define

$$g^1_{\boldsymbol{P}}(t,\boldsymbol{u},\phi):=S(t,\boldsymbol{u},\phi,\boldsymbol{P})$$

and

$$g^2_{\boldsymbol{P}}(t):=L^{-1}_{\boldsymbol{P}+\boldsymbol{P}_D(t)}(l_{t,\boldsymbol{P}+\boldsymbol{P}_D(t)})\in W$$

for $t\in[0,T]$ and $(\boldsymbol{u},\phi)\in W$, where $L^{-1}_{\boldsymbol{P}+\boldsymbol{P}_D(t)}$ and $l_{t,\boldsymbol{P}+\boldsymbol{P}_D(t)}$ are defined according to (4.25). Notice that

$$S(t,\boldsymbol{P})=g^1_{\boldsymbol{P}}\big(t,g^2_{\boldsymbol{P}}(t)\big).$$

We claim that if

1. $g^1_{\boldsymbol{P}}:[0,T]\times W\to X$ is Carathéodory in the sense that $t\mapsto g^1_{\boldsymbol{P}}(t,\boldsymbol{u},\phi)$ is Bochner-measurable for all $(\boldsymbol{u},\phi)\in W$ and $(\boldsymbol{u},\phi)\mapsto g^1_{\boldsymbol{P}}(t,\boldsymbol{u},\phi)$ is continuous for a.a. $t\in[0,T]$ and
2. $g^2_{\boldsymbol{P}}:[0,T]\to W$ is Bochner-measurable,

then $t\mapsto S(t,\boldsymbol{P}):[0,T]\to X$ is Bochner-measurable. Indeed, this follows immediately from Lemma D.3 by setting $f=g^1_{\boldsymbol{P}}$ and $g=g^2_{\boldsymbol{P}}$ therein. We show 1. and 2. in three steps.

Step 1a: Bochner-measurability of $t \mapsto g^1_{\boldsymbol{P}}(t, \boldsymbol{u}, \phi)$

Recall from (4.21) and (4.22) that

$$\begin{aligned} g^1_{\boldsymbol{P}}(t, \boldsymbol{u}, \phi) &= S\big(t, \boldsymbol{u}, \phi, \boldsymbol{P}\big) \\ &= -\mathcal{Q}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) - \big(\boldsymbol{P}'_D(t) - \Delta \boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big) \end{aligned}$$

and

$$\mathcal{Q}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) = D_{\boldsymbol{P}} \tilde{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P}) + D_{\boldsymbol{P}} \tilde{\omega}(t, \boldsymbol{P}).$$

The Bochner-measurability of $\boldsymbol{P}'_D$, $\Delta \boldsymbol{P}_D$, $\boldsymbol{f}_3$, $\boldsymbol{\pi}$ follow directly from Assumption B4, we thus still need to verify the Bochner-measurability of $D_{\boldsymbol{P}}\tilde{H}$ and $D_{\boldsymbol{P}}\tilde{\omega}$ from $[0,T]$ to X for a given $(\boldsymbol{u}, \phi, \boldsymbol{P}) \in W \times (Y,X)_{\frac{1}{\tau},\tau}$. First we point out that we are not able to directly work with the space $(Y,X)_{\frac{1}{\tau},\tau}$, since $\boldsymbol{P} + \boldsymbol{P}_D$ need not be an element in the space $(Y,X)_{\frac{1}{\tau},\tau}$. But from Lemma 4.11 we know on the one hand that $(Y,X)_{\frac{1}{\tau},\tau} \hookrightarrow \big(C(\overline{\boldsymbol{\Omega}})\big)^2$; on the other hand, from Remark 4.14 we obtain that for all $t \in [0,T]$

$$\boldsymbol{P}_D(t) \in \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^2 \hookrightarrow \big(C^{1-\frac{2}{p^*}}(\overline{\boldsymbol{\Omega}})\big)^2 \hookrightarrow \big(C(\overline{\boldsymbol{\Omega}})\big)^2,$$

since $1 - \frac{2}{p^*} \in (0,1)$, which is deduced from $p^* > 2$. Together with the fact that

$$W^{1,w}(0,T;Z) \hookrightarrow C([0,T];Z)$$

for all Banach spaces Z and all $w \in [1,\infty]$ (see [18, Sec. 5.9.2, Thm.2]), we conclude that

$$\boldsymbol{P}_D \in C([0,T]; \big(C^{1-\frac{2}{p^*}}(\overline{\boldsymbol{\Omega}})\big)^2) \hookrightarrow C([0,T]; \big(C(\overline{\boldsymbol{\Omega}})\big)^2). \tag{4.29}$$

This particularly implies that $\boldsymbol{P} + \boldsymbol{P}_D(t) \in \big(C(\overline{\boldsymbol{\Omega}})\big)^2$ for all $t \in [0,T]$. Hence in the rest part of Step 1, we will work with $\boldsymbol{P}$ in the underlying space $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$. Once we have shown the Bochner-measurability of $D_{\boldsymbol{P}}\tilde{H}$ and $D_{\boldsymbol{P}}\tilde{\omega}$ taking variable $\boldsymbol{P}$ in the space $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$, we will obtain the Bochner-measurability of $D_{\boldsymbol{P}}\tilde{H}$ and $D_{\boldsymbol{P}}\tilde{\omega}$ taking variable $\boldsymbol{P}$ in the space $(Y,X)_{\frac{1}{\tau},\tau}$, since $(Y,X)_{\frac{1}{\tau},\tau}$ is a subspace of $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$.

We first show the Bochner-measurability of $D_{\boldsymbol{P}}\tilde{\omega}$. Recall that

$$D_{\boldsymbol{P}}\tilde{\omega}(t, \boldsymbol{P}) = D_{\boldsymbol{P}}\omega\big(\boldsymbol{P} + \boldsymbol{P}_D(t)\big).$$

In view of Lemma D.3, we only need to show that the mapping $D_{\boldsymbol{P}}\omega : \big(C(\overline{\boldsymbol{\Omega}})\big)^2 \to \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$ is continuous. Let $\varepsilon > 0$ and $\boldsymbol{P} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^2$. Let $\hat{\boldsymbol{P}} \in \mathcal{O}_\epsilon(\boldsymbol{P})$, where $\mathcal{O}_\epsilon(\boldsymbol{P})$ is given by (4.26) with the replacement that the set $(Y,X)_{\frac{1}{\tau},\tau}$ therein is replaced by $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$. Then estimating similarly as in (4.52) below by setting $\boldsymbol{P}_1 = \boldsymbol{P}$, $\boldsymbol{P}_2 = \hat{\boldsymbol{P}}$ and $\boldsymbol{P}_D = \boldsymbol{0}$ therein we obtain that

$$\begin{aligned} &\Big| \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\omega(\boldsymbol{P})(\bar{\boldsymbol{P}}) - D_{\boldsymbol{P}}\omega(\hat{\boldsymbol{P}})(\bar{\boldsymbol{P}}) d\boldsymbol{x} \Big| \\ \le& C \|\boldsymbol{P} - \hat{\boldsymbol{P}}\|_{L^\infty} \|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \\ \le& C\varepsilon \|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \end{aligned}$$

for $\bar{\boldsymbol{P}} \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, where in the last inequality we have used the definition of the set $\mathcal{O}_\epsilon(\boldsymbol{P})$. By taking ε to zero we obtain immediately the continuity of $D_{\boldsymbol{P}}\omega$ from $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$ to $\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$.

Next, we show the Bochner-measurability of $D_{\boldsymbol{P}}\tilde{H}$. If we define

$$h(t,\boldsymbol{u},\phi,\boldsymbol{P}) = D_{\boldsymbol{P}}H\big(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\nabla\phi,\boldsymbol{P}\big),$$

where H is defined by (3.7), then from Remark 3.11 we obtain that

$$\begin{aligned}
&\int_\Omega h(t,\boldsymbol{u},\phi,\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}\\
&= \int_{\boldsymbol{\Omega}} \frac{1}{2} D_{\boldsymbol{P}}\mathbb{B}_2(\boldsymbol{P})\bar{\boldsymbol{P}} \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u})+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}(t)\end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u})+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}(t)\end{pmatrix} \\
&\quad + \mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u})+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D}(t)\end{pmatrix} : \begin{pmatrix} -D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0(\boldsymbol{P})\bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{P}}\end{pmatrix} d\boldsymbol{x}
\end{aligned} \tag{4.30}$$

for $\bar{\boldsymbol{P}} \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, where $\boldsymbol{\varepsilon}(\boldsymbol{u})$, $\boldsymbol{\varepsilon}_D(t)$ are the small strain tensors generated by $\boldsymbol{u}$ and $\boldsymbol{u}_D(t)$ respectively and

$$\boldsymbol{D}(t) = -\boldsymbol{\epsilon}(\boldsymbol{P})\big(\nabla\phi+\nabla\phi_D(t)\big) + \boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}(\boldsymbol{u})+\boldsymbol{\varepsilon}_D(t)-\boldsymbol{\varepsilon}^0(\boldsymbol{P})\big).$$

For the tensors and functions $\mathbb{B}_2$, $\boldsymbol{\varepsilon}^0$, $\boldsymbol{e}$, $\boldsymbol{\epsilon}$ appeared in the above expression, we refer to Section 2.4, (3.12) and (3.19). An additional remark should be made here: in fact, the integral given in (4.30) should be given in terms using the tensors $\mathbb{B}_1(\boldsymbol{P})$ and $D_{\boldsymbol{P}}\mathbb{B}_1(\boldsymbol{P})$ but not $\mathbb{B}_2(\boldsymbol{P})$ and $D_{\boldsymbol{P}}\mathbb{B}_1(\boldsymbol{P})$, due to the explicit expression of the function H given by (3.7). However, due to Remark (3.11), the expression (4.30) gives an equivalent formulation of the function $D_{\boldsymbol{P}}H$ in terms using $\mathbb{B}_2(\boldsymbol{P})$ and $D_{\boldsymbol{P}}\mathbb{B}_2(\boldsymbol{P})$. There are two reasons of insisting on the expression (4.30): first, it is quite cumbersome to formulate a new functional using the tensors $\mathbb{B}_1(\boldsymbol{P})$ and $D_{\boldsymbol{P}}\mathbb{B}_1(\boldsymbol{P})$, due to the enormous number of parameters given in this thesis. Since the expression (4.30) has already appeared in the study given in Chapter 3, this will save us much work for reformulating several new notation; On the other hand, many estimates given in Chapter 3 can also be directly utilized in this case, thanks to the expression given in (4.30).

From (4.30) we obtain that

$$D_{\boldsymbol{P}}\tilde{H}(t,\boldsymbol{u},\phi,\boldsymbol{P}) = h\big(t,\hat{h}(t)\big),$$

where

$$\hat{h}(t) := \big(\boldsymbol{u},\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big).$$

Thus similarly as previously done, in view of Lemma D.3 it suffices to show that $h(t,\boldsymbol{u},\phi,\boldsymbol{P})$ is a Carathéodory function in the following sense:

- $h(\cdot,\boldsymbol{u},\phi,\boldsymbol{P}) : [0,T] \to X$ is Bochner-measurable for all $(\boldsymbol{u},\phi,\boldsymbol{P}) \in W \times \big(C(\overline{\boldsymbol{\Omega}})\big)^2$ and
- $h(t,\cdot) : W \times \big(C(\overline{\boldsymbol{\Omega}})\big)^2 \to X$ is continuous for a.a. $t \in [0,T]$.

To see the Bochner-measurability of $h(\cdot,\boldsymbol{u},\phi,\boldsymbol{P})$, we first point out that the functions $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$ can be seen as constants (in corresponding function spaces), since they are independent on t. From the expression of $h(t,\boldsymbol{u},\phi,\boldsymbol{P})$ given in (4.30) we see that $h(t,\boldsymbol{u},\phi,\boldsymbol{P})$ is nothing else but a sum of several products of constants and Bochner-measurable functions, which is again Bochner-measurable. One thing has still to be clarified, namely, we still need to show that these products, or more precisely, the integrands in the r.h.s. of (4.30), are realized as elements in the space $X = \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, which is to guarantee the compatibility of the products (of constants and Bochner-measurable functions) with the underlying space X. It suffices to show that for a.a. $t \in [0,T]$, the first integral in the r.h.s. of (4.30) is bounded by $C\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}$ for $\bar{\boldsymbol{P}} \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, with some positive constant C; the estimation for the second integral is analogous. First we point out that since $\boldsymbol{P} : \overline{\boldsymbol{\Omega}} \to \mathbb{R}^2$ is continuous on $\overline{\boldsymbol{\Omega}}$ and $D_{\boldsymbol{P}}\mathbb{B}_2$, $\boldsymbol{e}$, $\boldsymbol{\epsilon}$ are continuous on $\mathbb{R}^2$ due to Assumption B2, $D_{\boldsymbol{P}}\mathbb{B}_2(\boldsymbol{P})$, $\boldsymbol{e}(\boldsymbol{P})$, $\boldsymbol{\epsilon}(\boldsymbol{P})$ are as composition of continuous functions also continuous on $\overline{\boldsymbol{\Omega}}$, thus also uniformly bounded on $\overline{\boldsymbol{\Omega}}$, since $\overline{\boldsymbol{\Omega}}$ is compact. Hence the first integrand in the (4.30) is bounded by

$$C|\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}(\boldsymbol{u})| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi| + |\nabla\phi_D(t)|\big)^2.$$

From Assumption B4 we know that $\boldsymbol{\varepsilon}_D(t)$ and $\nabla\phi_D(t)$ are of class L^p for a.a. $t \in [0,T]$. On the other hand, for $p > 2$, simple calculation shows that the Sobolev relation

$$1 - \frac{2}{p'} \geq 0 - \frac{2}{p/(p-2)}$$

is equivalent to

$$(\sqrt{2} - 1)p \geq -2,$$

which is obviously true for $p > 2$. Thus we conclude the Sobolev embedding

$$W^{1,p'} \hookrightarrow L^{\frac{p}{p-2}}.$$

Finally, using the following Hölder relation

$$\frac{1}{p} + \frac{1}{p} + \frac{p-2}{p} = 1$$

we obtain that

$$\begin{aligned}
&\int_{\boldsymbol{\Omega}} |\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}(\boldsymbol{u})| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi| + |\nabla\phi_D(t)|\big)^2 d\boldsymbol{x} \\
\leq& C\int_{\boldsymbol{\Omega}} |\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}(\boldsymbol{u})|^2 + |\boldsymbol{\varepsilon}_D(t)|^2 + |\nabla\phi|^2 + |\nabla\phi_D(t)|^2\big) d\boldsymbol{x} \\
\leq& C\|\bar{\boldsymbol{P}}\|_{L^{\frac{p}{p-2}}}\big(1 + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L^p} + \|\boldsymbol{\varepsilon}_D(t)\|^2_{L^p} + \|\nabla\phi\|^2_{L^p} + \|\nabla\phi_D(t)\|^2_{L^p}\big) \\
\leq& C\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}\big(1 + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{L^p} + \|\boldsymbol{\varepsilon}_D(t)\|^2_{L^p} + \|\nabla\phi\|^2_{L^p} + \|\nabla\phi_D(t)\|^2_{L^p}\big) \\
\leq& C\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}
\end{aligned}$$

for a.a. $t \in [0,T]$, since $\boldsymbol{u}$, $\boldsymbol{u}_D$, ϕ, ϕ_D are thought to be given fixed functions. This completes the proof of the Bochner-measurability of $h(\cdot, \boldsymbol{u}, \phi, \boldsymbol{P})$. It is left to show the continuity of $h(t,\cdot)$. Let $(\boldsymbol{u}, \phi, \boldsymbol{P}) \in W \times \big(C(\overline{\boldsymbol{\Omega}})\big)^2$ be some given function and $(\hat{\boldsymbol{u}}, \hat{\phi}, \hat{\boldsymbol{P}})$ be in the set $\mathcal{O}_\varepsilon$ with an arbitrary but fixed $\varepsilon > 0$, where $\mathcal{O}_\varepsilon$ is given by (4.26) with the replacement that the set $(Y,X)_{\frac{1}{\tau},\tau}$ therein is replaced by $W \times \big(C(\overline{\boldsymbol{\Omega}})\big)^2$. Hence to show the continuity of $h(t,\cdot)$, it suffices to show that there exists some $C > 0$ such that for all $\bar{\boldsymbol{P}} \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$ and for all ε, say, in $(0,1]$, we have

$$\int_{\boldsymbol{\Omega}} \big(h(t, \boldsymbol{u}, \phi, \boldsymbol{P}) - h(t, \hat{\boldsymbol{u}}, \hat{\phi}, \hat{\boldsymbol{P}})\big)(\bar{\boldsymbol{P}}) d\boldsymbol{x} \leq C\varepsilon\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}. \tag{4.31}$$

Using telescoping principle we can rewrite $h(t, \boldsymbol{u}, \phi, \boldsymbol{P}) - h(t, \hat{\boldsymbol{u}}, \hat{\phi}, \hat{\boldsymbol{P}})$ as a sum of several summands, such that each summand is a product of the following terms:

- components of $\big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\phi, \boldsymbol{P}\big)$ or $\big(\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}}), \nabla\hat{\phi}, \hat{\boldsymbol{P}}\big)$;

- tensors evaluated at $\boldsymbol{P}$ or $\hat{\boldsymbol{P}}$, e.g. $D_{\boldsymbol{P}}\mathbb{B}_2(\hat{\boldsymbol{P}})$;
- difference of components, e.g. $\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\hat{\boldsymbol{u}})$, $D_{\boldsymbol{P}}\mathbb{B}_2(\boldsymbol{P}) - D_{\boldsymbol{P}}\mathbb{B}_2(\hat{\boldsymbol{P}})$.

In particular, the term of the third type occurs exactly once in each product. The term

$$\Big(D_{\boldsymbol{P}}\mathbb{B}_2(\boldsymbol{P}) - D_{\boldsymbol{P}}\mathbb{B}_2(\hat{\boldsymbol{P}})\Big)\bar{\boldsymbol{P}}\begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\\ \boldsymbol{D}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\\ \boldsymbol{D}\end{pmatrix} \tag{4.32}$$

is being an example. In order to avoid unnecessary duplicate and tedious calculation, we shall only estimate the term (4.32) here, the other terms appearing in the telescoping sum generated by the difference $h(t, \boldsymbol{u}, \phi, \boldsymbol{P}) - h(t, \hat{\boldsymbol{u}}, \hat{\phi}, \hat{\boldsymbol{P}})$ are being estimated analogously. First, we point out that similarly as argued in the step of showing continuity of $\boldsymbol{P} \to \boldsymbol{D_P}\omega(\boldsymbol{P})$, the tensors $\boldsymbol{\varepsilon}^0(\boldsymbol{P})$, $\boldsymbol{e}(\boldsymbol{P})$, $\boldsymbol{\epsilon}(\boldsymbol{P})$ appearing in (4.32) are uniformly bounded for all $\boldsymbol{x} \in \boldsymbol{\Omega}$. Next, we obtain that

$$\begin{aligned}\|\hat{\boldsymbol{P}}\|_{L^\infty(\boldsymbol{\Omega})} =&\|\hat{\boldsymbol{P}} - \boldsymbol{P} + \boldsymbol{P}\|_{L^\infty(\boldsymbol{\Omega})}\\ \le&\|\hat{\boldsymbol{P}} - \boldsymbol{P}\|_{L^\infty(\boldsymbol{\Omega})} + \|\boldsymbol{P}\|_{L^\infty(\boldsymbol{\Omega})}\\ \le&\varepsilon + \|\boldsymbol{P}\|_{L^\infty(\boldsymbol{\Omega})}\\ \le&C + \varepsilon,\end{aligned} \tag{4.33}$$

since $\boldsymbol{P}$ can be thought as a given fixed function. Thus $\boldsymbol{P}(\boldsymbol{x})$ and $\hat{\boldsymbol{P}}(\boldsymbol{x})$ are uniformly bounded by the positive constant $C + \varepsilon$ for all $\boldsymbol{x} \in \boldsymbol{\Omega}$ and $\hat{\boldsymbol{P}} \in \mathcal{O}_\varepsilon$. From Assumption B2 we know that $D_{\boldsymbol{P}}\mathbb{B}_2$ is locally Lipschitz continuous on $\mathbb{R}^2$, thus it is Lipschitz continuous on the $\mathbb{R}^2$-ball

$$B_{C+\varepsilon} := \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| \le C + \varepsilon\}.$$

We denote the Lipschitz constant of $D_{\boldsymbol{P}}\mathbb{B}_2$ on the ball $B_{C+\varepsilon}$ by $\mathcal{L}_\varepsilon$. Note that since $B_{C+\varepsilon_1} \subset B_{C+\varepsilon_2}$ if $\varepsilon_1 < \varepsilon_2$, we can w.l.o.g. assume that $\mathcal{L}_\varepsilon$ is uniformly bounded by some positive constant C (e.g. taking $C = \mathcal{L}_{1+C}$) for all $\varepsilon \in (0, 1]$. We thus obtain that

$$|D_{\boldsymbol{P}}\mathbb{B}_2\big(\boldsymbol{P}(\boldsymbol{x})\big) - D_{\boldsymbol{P}}\mathbb{B}_2\big(\hat{\boldsymbol{P}}(\boldsymbol{x})\big)| \le C|\boldsymbol{P}(\boldsymbol{x}) - \hat{\boldsymbol{P}}(\boldsymbol{x})|$$

for all $\boldsymbol{x} \in \boldsymbol{\Omega}$. Therefore we obtain the following upper bound

$$C\big(|\boldsymbol{P} - \hat{\boldsymbol{P}}|\big) \cdot |\bar{\boldsymbol{P}}| \cdot \big(1 + |\boldsymbol{\varepsilon}(\boldsymbol{u})| + |\nabla\phi| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi_D(t)|\big)^2 \tag{4.34}$$

for (4.32). Now we estimate the integral of (4.34) over $\boldsymbol{\Omega}$. First, using the fact that $p > 2$ we obtain from Sobolev's embedding relation

$$1 - \frac{2}{p'} = 0 - 1/\Big(\frac{p}{p-2}\Big) > 0 - 2/\Big(\frac{p}{p-2}\Big)$$

that $\big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2 \hookrightarrow \big(L^{\frac{p}{p-2}}(\boldsymbol{\Omega})\big)^2$. We also obtain that

$$\begin{aligned}\|\big(\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}}), \nabla\hat{\phi}\big)\|_{L^p(\boldsymbol{\Omega})} =&\|\big(\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}}), \nabla\hat{\phi}\big) - \big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\phi\big) + \big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\phi\big)\|_{L^p(\boldsymbol{\Omega})}\\ \le&\|\big(\boldsymbol{\varepsilon}(\hat{\boldsymbol{u}}), \nabla\hat{\phi}\big) - \big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\phi\big)\|_{L^p(\boldsymbol{\Omega})} + \|\big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\phi\big)\|_{L^p(\boldsymbol{\Omega})}\\ \le&\varepsilon + \|\big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\phi\big)\|_{L^p(\boldsymbol{\Omega})}\\ \le&C + \varepsilon.\end{aligned} \tag{4.35}$$

Finally, using the Hölder relation

$$\frac{p-2}{p} + \frac{1}{p} + \frac{1}{p} = 1$$

we obtain that

$$
\begin{aligned}
&\int_{\boldsymbol{\Omega}} (4.34)\ d\boldsymbol{x}\\
\leq& C\|\bar{\boldsymbol{P}}\|_{L^{\frac{p}{p-2}}}\|\boldsymbol{P}-\hat{\boldsymbol{P}}\|_{L^\infty}\big(1+\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^p}^2+\|\nabla\phi\|_{L^p}^2+\|\boldsymbol{\varepsilon}_D(t)\|_{L^p}^2+\|\nabla\phi_D(t)\|_{L^p}^2\big)\\
\leq& C\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}\|\boldsymbol{P}-\hat{\boldsymbol{P}}\|_{L^\infty}\big(1+\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^p}^2+\|\nabla\phi\|_{L^p}^2+\|\boldsymbol{\varepsilon}_D(t)\|_{L^p}^2+\|\nabla\phi_D(t)\|_{L^p}^2\big)\\
\leq& C(C+\varepsilon)^2\varepsilon\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}\\
\leq& C\varepsilon\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}
\end{aligned}
$$

by considering $\varepsilon\in(0,1]$. From this we finally obtain (4.31) and consequently the continuity of $h(t,\cdot)$, and the proof of Step 1a is complete.

Step 1b: Continuity of $(\boldsymbol{u},\phi)\mapsto g^1_{\boldsymbol{P}}(t,\boldsymbol{u},\phi)$

Notice that

$$g^1_{\boldsymbol{P}}(t,\boldsymbol{u},\phi)=h(t,\boldsymbol{u},\phi,\boldsymbol{P}).$$

Thus the proof of Step 1b is already contained in the proof of showing the continuity of $h(t,\cdot)$ previously and nothing has to be shown.

Step 1c: Bochner-measurability of $t\mapsto g^2_{\boldsymbol{P}}(t)$

Recall that

$$g^2_{\boldsymbol{P}}(t):=L^{-1}_{\tilde{\boldsymbol{P}}(t)}(l_{t,\tilde{\boldsymbol{P}}(t)})=\big(\boldsymbol{u}(t),\phi(t)\big),$$

where $\tilde{\boldsymbol{P}}(t)=\boldsymbol{P}+\boldsymbol{P}_D(t)$ and $L_{\tilde{\boldsymbol{P}}(t)}$ and $l_{t,\tilde{\boldsymbol{P}}(t)}$ are defined according to Lemma 4.19 and (4.25). Due to Lemma 4.11 and (4.29), $\tilde{\boldsymbol{P}}:t\to\big(C(\overline{\boldsymbol{\Omega}})\big)^2$ is Bochner-measurable and $\tilde{\boldsymbol{P}}(t):\boldsymbol{\Omega}\to\mathbb{R}^2$ is measurable for all $t\in[0,T]$, where the latter statement is to guarantee that Lemma 4.19 is applicable. We also recall that (p,r) is an admissible pair in the sense of Definition 4.20 and the Assumption B4 is satisfied with $p^*\in[p,\infty)$ and $r^*\in[r,\infty)$. In particular, the number $\hat{p}$ is defined by

$$\hat{p}:=\min\{p_*,p^*\},$$

where p_* is given by Lemma 4.19. Now due to Assumption B4 we know that $l_{t,\tilde{\boldsymbol{P}}(t)}$ defined by (3.4) and evaluated at $\tilde{\boldsymbol{P}}(t)$ in the second component is an element of $\big(W^{-1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2\times W^{-1,q}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ for all $q\in[2,\hat{p}]$ and a.a. $t\in[0,T]$. Thus one concludes from Lemma 4.19 the existence and uniqueness of $\boldsymbol{u}(t)$ and $\phi(t)$ in the spaces $\big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2$ and $W^{1,q}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$ respectively for all $q\in\big[2,\hat{p}\big]$ and a.a. $t\in[0,T]$. From previous analysis we know that

$$\tilde{\boldsymbol{P}}:[0,T]\to\big(C(\overline{\boldsymbol{\Omega}})\big)^2$$

is Bochner-measurable. Also we know from Assumption B4 that

$$l_{\cdot,\tilde{\boldsymbol{P}}(\cdot)}:[0,T]\to\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2\times W^{-1,p}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$$

is Bochner-measurable, since p is in $(2,\hat{p}]$ due to the definition of an admissible pair, see Definition 4.20. Thus due to Definition 4.4, there exist simple functions $\{\tilde{\boldsymbol{P}}_n\}_{n\in\mathbb{N}}$ and $\{l_n\}_{n\in\mathbb{N}}$ defined on $[0,T]$ such that

$$\tilde{\boldsymbol{P}}_n(t)\to\tilde{\boldsymbol{P}}(t)\quad\text{in }\big(C(\overline{\boldsymbol{\Omega}})\big)^2\text{ for a.a. }t\in[0,T],\tag{4.36}$$

$$l_n(t)\to l_{t,\tilde{\boldsymbol{P}}(t)}\quad\text{in }\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2\times W^{-1,p}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})\text{ for a.a. }t\in[0,T].\tag{4.37}$$

Let $t \in [0,T]$ be chosen such that (4.36) and (4.37) are satisfied. In this case we assume that the sequence $\{\tilde{\boldsymbol{P}}_n(t,\boldsymbol{x})\}_{n\in\mathbb{N}}$ and $\tilde{\boldsymbol{P}}(t,\boldsymbol{x})$ are bounded by some positive constant M uniformly for all $\boldsymbol{x} \in \overline{\boldsymbol{\Omega}}$. Let $\big(\boldsymbol{u}_n^1(t), \phi_n^1(t)\big)$ be the solution of the elliptic system

$$L_{\tilde{\boldsymbol{P}}(t)}\big(\boldsymbol{u}_n^1(t), \phi_n^1(t)\big) = l_n(t),$$

whose existence is ensured by Lemma 4.19. Then

$$L_{\tilde{\boldsymbol{P}}(t)}\big(\boldsymbol{u}(t) - \boldsymbol{u}_n^1(t), \phi(t) - \phi_n^1(t)\big) = l_{t,\tilde{\boldsymbol{P}}(t)} - l_n(t).$$

From Lemma 4.19 we obtain that

$$\|\boldsymbol{u}_n^1(t) - \boldsymbol{u}(t)\|_{W^{1,p}} + \|\phi_n^1(t) - \phi(t)\|_{W^{1,p}} \leq C_* \|l_{t,\tilde{\boldsymbol{P}}(t)} - l_n(t)\|_{W^{-1,p}}, \tag{4.38}$$

where C_* is given by Lemma 4.19. Thus, for an arbitrary $\varepsilon > 0$ we infer that for all sufficiently large n it holds

$$\|\boldsymbol{u}_n^1(t) - \boldsymbol{u}(t)\|_{W^{1,p}} + \|\phi_n^1(t) - \phi(t)\|_{W^{1,p}} \leq \frac{\varepsilon}{2}.$$

Now let $\big(\boldsymbol{u}_n(t), \phi_n(t)\big)$ be the solution of

$$L_{\tilde{\boldsymbol{P}}_n(t)}\big(\boldsymbol{u}_n(t), \phi_n(t)\big) = l_n(t), \tag{4.39}$$

whose existence is again ensured by Lemma 4.19. Then

$$L_{\tilde{\boldsymbol{P}}(t)}\big(\boldsymbol{u}_n^1(t), \phi_n^1(t)\big) = l_n(t) = L_{\tilde{\boldsymbol{P}}_n(t)}\big(\boldsymbol{u}_n(t), \phi_n(t)\big),$$

which implies

$$L_{\tilde{\boldsymbol{P}}(t)}\big(\boldsymbol{u}_n^1(t) - \boldsymbol{u}_n(t), \phi_n^1(t) - \phi_n(t)\big) = (L_{\tilde{\boldsymbol{P}}_n(t)} - L_{\tilde{\boldsymbol{P}}(t)})\big(\boldsymbol{u}_n(t), \phi_n(t)\big).$$

From Lemma 4.19 we infer that

$$\begin{aligned}
&\|\boldsymbol{u}_n(t) - \boldsymbol{u}_n^1(t)\|_{W^{1,p}} + \|\phi_n(t) - \phi_n^1(t)\|_{W^{1,p}} \\
\leq& C_* \|(L_{\tilde{\boldsymbol{P}}_n(t)} - L_{\tilde{\boldsymbol{P}}(t)})\big(\boldsymbol{u}_n(t), \phi_n(t)\big)\|_{W^{-1,p}} \\
\leq& \mathcal{L}_M C_* \|\tilde{\boldsymbol{P}}(t) - \tilde{\boldsymbol{P}}_n(t)\|_{L^\infty}\big(\|\boldsymbol{u}_n(t)\|_{W^{1,p}} + \|\phi_n(t)\|_{W^{1,p}}\big) \\
\leq& \mathcal{L}_M C_* \|\tilde{\boldsymbol{P}}(t) - \tilde{\boldsymbol{P}}_n(t)\|_{L^\infty} \cdot \big(C_* \|l_n(t)\|_{W^{-1,p}}\big)
\end{aligned} \tag{4.40}$$

where $\mathcal{L}_M$ is the Lipschitz constant of $\mathbb{B}_1$ constrained on the ball in $\mathbb{R}^2$ with center $\mathbf{0}$ and radius M (which is valid due to Assumption B2) and the last inequality comes from the application of Lemma 4.19 to the equation (4.39). Due to (4.37), $\|l_n(t)\|_{W^{-1,p}}$ is bounded for all $n \in \mathbb{N}$, thus for all sufficiently large n we obtain from (4.36) and (4.40) that

$$\|\boldsymbol{u}_n(t) - \boldsymbol{u}_n^1(t)\|_{W^{1,p}} + \|\phi_n(t) - \phi_n^1(t)\|_{W^{1,p}} \leq \frac{\varepsilon}{2}.$$

Together with (4.38) we infer that

$$\|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_{W^{1,p}} + \|\phi_n(t) - \phi(t)\|_{W^{1,p}} \leq \varepsilon$$

for all sufficiently large n. But since $\tilde{\boldsymbol{P}}_n$ and l_n are simple functions, we know that $\boldsymbol{u}_n$ and ϕ_n are also simple functions. This implies the Bochner-measurability of $\big(\boldsymbol{u}_n(t), \phi_n(t)\big)$ and completes the proof of Step 1c. Consequently we have proved the desired Step 1.

Step 2: Validity of condition (S)

We recall the condition (S): Let $M > 0$ be arbitrary. Then we need to show that there exists some $h_M \in L^r(0,T)$, such that for all $\boldsymbol{P}_1, \boldsymbol{P}_2 \in (Y,X)_{\frac{1}{\tau},\tau}$ with

$$\max\{\|\boldsymbol{P}_1\|_{(Y,X)_{\frac{1}{\tau},\tau}}, \|\boldsymbol{P}_2\|_{(Y,X)_{\frac{1}{\tau},\tau}}\} \leq M$$

we have

$$\|S(t,\boldsymbol{P}_1) - S(t,\boldsymbol{P}_2)\|_X \leq h_M(t)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{(Y,X)_{\frac{1}{\tau},\tau}}. \tag{4.41}$$

We will utilize the same idea as the one for the regularity results given in Section 3.5 to show (4.41), namely, the Lipschitz estimate (4.41) on bounded subsets of $(Y,X)_{\frac{1}{\tau},\tau}$ will rely on a corresponding difference estimate of the solutions $\big((\boldsymbol{u}_i(t), \phi_i(t)\big)$ of the piezo-system

$$L_{\tilde{\boldsymbol{P}}_i(t)}\big(\boldsymbol{u}_i(t), \phi_i(t)\big) = l_{t,\tilde{\boldsymbol{P}}_i(t)} \tag{4.42}$$

for $i = 1, 2$. From Lemma 4.11 we deduce that the set

$$\mathcal{V} := \{\boldsymbol{P} \in (Y,X)_{\frac{1}{\tau},\tau} : \|\boldsymbol{P}\|_{(Y,X)_{\frac{1}{\tau},\tau}} \leq M\}$$

is a bounded subset of $\big(L^\infty(\boldsymbol{\Omega})\big)^2$. Thus w.l.o.g. up to a prefactor we may also assume that

$$\|\boldsymbol{P}\|_{L^\infty(\boldsymbol{\Omega})} \leq M \tag{4.43}$$

for all $\boldsymbol{P} \in \mathcal{V}$. From (4.29) we know that $\boldsymbol{P}_D$ is in the space $\big(L^\infty([0,T]\times\boldsymbol{\Omega})\big)^2$. Thus

$$\begin{aligned}\|\tilde{\boldsymbol{P}}\|_{L^\infty([0,T]\times\boldsymbol{\Omega})} = \|\boldsymbol{P}+\boldsymbol{P}_D\|_{L^\infty([0,T]\times\boldsymbol{\Omega})} &\leq \|\boldsymbol{P}\|_{L^\infty(\boldsymbol{\Omega})} + \|\boldsymbol{P}_D\|_{L^\infty([0,T]\times\boldsymbol{\Omega})} \\ &\leq M + \|\boldsymbol{P}_D\|_{L^\infty([0,T]\times\boldsymbol{\Omega})} \\ &\leq M'\end{aligned}$$

for some $M' > 0$ depending on M. Using Assumption B2 we can find some positive C_M such that

$$\begin{aligned}\sup_{\boldsymbol{P}\in\mathbb{R}^3, |\boldsymbol{P}|\leq M'} \Big\{&\mathcal{L}_{D_{\boldsymbol{P}}\mathbb{C}}, \mathcal{L}_{D_{\boldsymbol{P}}\boldsymbol{e}}, \mathcal{L}_{D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0}, \mathcal{L}_{D_{\boldsymbol{P}}\boldsymbol{\epsilon}}, \mathcal{L}_{D_{\boldsymbol{P}}\boldsymbol{\epsilon}^{-1}}, \\ &|D_{\boldsymbol{P}}\mathbb{C}(\boldsymbol{P})|, |D_{\boldsymbol{P}}\boldsymbol{e}(\boldsymbol{P})|, |D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0(\boldsymbol{P})|, |D_{\boldsymbol{P}}\boldsymbol{\epsilon}(\boldsymbol{P})|, |D_{\boldsymbol{P}}\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})|, \\ &|\mathbb{C}(\boldsymbol{P})|, |\boldsymbol{e}(\boldsymbol{P})|, |\boldsymbol{\varepsilon}^0(\boldsymbol{P})|, |\boldsymbol{\epsilon}(\boldsymbol{P})|, |\boldsymbol{\epsilon}^{-1}(\boldsymbol{P})|, \Big\} \leq C_M,\end{aligned}$$

where $\mathcal{L}_\circ$ are the corresponding local Lipschitz constants of the derivatives. Define

$$\begin{aligned}h(t) :=& \|\boldsymbol{f}_1(t)\|_{L^{\frac{2p}{p+2}}} + \|\boldsymbol{t}(t)\|_{L^{\frac{p}{2}}} + \|f_2(t)\|_{L^{\frac{2p}{p+2}}} \\ &+ \|\rho(t)\|_{L^{\frac{p}{2}}} + \|\boldsymbol{u}_D(t)\|_{B^{1-\frac{1}{p}}_{p,p}} + \|\phi_D(t)\|_{B^{1-\frac{1}{p}}_{p,p}}.\end{aligned} \tag{4.44}$$

For convention we will use K_M to stand for some positive constant which depends only on M for various inequalities. Let $\boldsymbol{P}_1$, $\boldsymbol{P}_2 \in \mathcal{V}$ and $(\boldsymbol{u}_i, \phi_i)$ be given by (4.42) for $i = 1, 2$.

We also define $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}(\boldsymbol{u}_i)$. Recall that $l_{t,\boldsymbol{P}}(\bar{\boldsymbol{u}}, \bar{\phi})$ is defined by

$$
\begin{aligned}
l_{t,\boldsymbol{P}}(\bar{\boldsymbol{u}}, \bar{\phi}) = &\int_{\boldsymbol{\Omega}} \boldsymbol{f}_1(t) \cdot \bar{\boldsymbol{u}} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}} \boldsymbol{t}(t) \cdot \bar{\boldsymbol{u}} d\boldsymbol{S} \\
&- \int_{\boldsymbol{\Omega}} \Big(\mathbb{C}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \boldsymbol{e}^T(\boldsymbol{P}) \nabla \phi_D(t) \Big) : \bar{\boldsymbol{\varepsilon}} d\boldsymbol{x} \\
&- \Big(\int_{\boldsymbol{\Omega}} f_2(t) \bar{\phi} d\boldsymbol{x} + \int_{\partial\boldsymbol{\Omega}_{\boldsymbol{D}}} \rho(t) \bar{\phi} d\boldsymbol{S} \Big) \\
&+ \int_{\boldsymbol{\Omega}} \Big(\boldsymbol{e}(\boldsymbol{P})\big(\boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P})\big) - \boldsymbol{\epsilon}(\boldsymbol{P}) \nabla \phi_D(t) + \boldsymbol{P} \Big) \cdot \nabla \bar{\phi} d\boldsymbol{x}
\end{aligned}
\tag{4.45}
$$

for $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{1,p'}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})$. It follows from (4.45), Hölder's inequality and standard dual estimation that

$$
\begin{aligned}
&|l_{t,\boldsymbol{P}_i+\boldsymbol{P}_D(t)}(\bar{\boldsymbol{u}}, \bar{\phi})| \\
\le&\big(\|\boldsymbol{f}_1(t)\|_{L^{\frac{2p}{p+2}}} + \|\boldsymbol{t}(t)\|_{L^{\frac{p}{2}}} + \|f_2(t)\|_{L^{\frac{2p}{p+2}}} + \|\rho(t)\|_{L^{\frac{p}{2}}}\big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big) \\
&+ \Big(C_M\big(\|\boldsymbol{\varepsilon}_D(t)\|_{L^p} + \|\nabla\phi_D(t)\|_{L^p}\big) + C \cdot C_M^2\Big)\big(\|\bar{\boldsymbol{\varepsilon}}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big) + \|\boldsymbol{P}_i + \boldsymbol{P}_D(t)\|_{L^p}\|\nabla\bar{\phi}\|_{L^{p'}} \\
\le&\Big(C\big(1 + C_M^2\big)\big(1 + h(t)\big) + C\big(1 + \|\boldsymbol{P}_i\|_{L^\infty}\big)\Big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big) \\
\le&\Big(C\big(1 + C_M^2\big)\big(1 + h(t)\big) + CM'\Big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big) \\
\le&\Big(CM'\big(1 + C_M^2\big)\big(1 + h(t)\big)\Big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big) \\
\le&K_M\big(1 + h(t)\big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big).
\end{aligned}
\tag{4.46}
$$

Thus we obtain from Lemma 4.19 that

$$
\|\boldsymbol{u}_i\|_{W^{1,p}} + \|\phi_i\|_{W^{1,p}} \le C_* K_M\big(1 + h(t)\big) \le K_M\big(1 + h(t)\big), \; i = 1, 2. \tag{4.47}
$$

We use the same idea as the one for deriving (4.47) to estimate $\boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\phi_1 - \phi_2$: analogous to (3.41) we obtain that

$$
\begin{aligned}
&\int_{\boldsymbol{\Omega}} \mathbb{B}_1\big(\tilde{\boldsymbol{P}}_1(t)\big) \begin{pmatrix} \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \\ \nabla\phi_1 - \nabla\phi_2 \end{pmatrix} : \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x} \\
=&\Big[\big(\widehat{l}_{t,\tilde{\boldsymbol{P}}_1(t)} - \widehat{l}_{t,\tilde{\boldsymbol{P}}_2(t)}\big)(\bar{\boldsymbol{u}}, \bar{\phi})\Big] + \Big[\int_{\boldsymbol{\Omega}} (\boldsymbol{P}_1 - \boldsymbol{P}_2) \cdot \nabla\bar{\phi} d\boldsymbol{x}\Big] \\
&- \Big[\int_{\boldsymbol{\Omega}} \Big(\mathbb{B}_1\big(\tilde{\boldsymbol{P}}_1(t)\big) - \mathbb{B}_1\big(\tilde{\boldsymbol{P}}_2(t)\big)\Big) \begin{pmatrix} \boldsymbol{\varepsilon}_2 \\ \nabla\phi_2 \end{pmatrix} : \begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x}\Big] \\
=&: I_1 + I_2 + I_3.
\end{aligned}
\tag{4.48}
$$

Now the estimation of $\boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\phi_1 - \phi_2$ follow from the application of Lemma 4.19 to the differential operator given in the first line of (4.48) (namely, replacing $\boldsymbol{u}_i$ and ϕ_i in (4.47) by $\boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\phi_1 - \phi_2$ respectively) with r.h.s. given by the second and third lines of (4.48). Hence we need to estimate I_1, I_2, I_3. To estimate I_1, it suffices to consider the terms

$$
\begin{aligned}
&\int_{\boldsymbol{\Omega}} \big(\mathbb{C}(\boldsymbol{P}_1) - \mathbb{C}(\boldsymbol{P}_2)\big)\boldsymbol{\varepsilon}_D(t) : \bar{\boldsymbol{\varepsilon}} d\boldsymbol{x} =: I_{11}, \\
&\int_{\boldsymbol{\Omega}} \big(\mathbb{C}(\boldsymbol{P}_1)\boldsymbol{\varepsilon}^0(\boldsymbol{P}_1) - \mathbb{C}(\boldsymbol{P}_2)\boldsymbol{\varepsilon}^0(\boldsymbol{P}_2)\big) : \bar{\boldsymbol{\varepsilon}} d\boldsymbol{x} =: I_{12},
\end{aligned}
$$

which are summands of the difference $\widehat{l}_{t,\tilde{\boldsymbol{P}}_1(t)} - \widehat{l}_{t,\tilde{\boldsymbol{P}}_2(t)}$, and estimation for the other terms in the difference $\widehat{l}_{t,\tilde{\boldsymbol{P}}_1(t)} - \widehat{l}_{t,\tilde{\boldsymbol{P}}_2(t)}$ can be deduced analogously. It follows on the one hand that

$$\begin{aligned}|I_{11}| \leq & C_M \int_{\boldsymbol{\Omega}} |\boldsymbol{P}_1 - \boldsymbol{P}_2||\boldsymbol{\varepsilon}_D(t)||\bar{\varepsilon}|d\boldsymbol{x} \\ \leq & C_M \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \|\boldsymbol{\varepsilon}_D(t)\|_{L^p} \|\bar{\varepsilon}\|_{L^{p'}} \\ \leq & C_M h(t) \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \|\bar{\varepsilon}\|_{L^{p'}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|I_{12}| \leq & C_M^2 \int_{\boldsymbol{\Omega}} |\boldsymbol{P}_1 - \boldsymbol{P}_2||\bar{\varepsilon}|d\boldsymbol{x} \\ \leq & C_M^2 \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p} \|\bar{\varepsilon}\|_{L^{p'}} \\ \leq & C C_M^2 \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \|\bar{\varepsilon}\|_{L^{p'}}.\end{aligned}$$

We also obtain that

$$|I_2| \leq \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p} \|\nabla\bar{\phi}\|_{L^{p'}} \leq C \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \|\nabla\bar{\phi}\|_{L^{p'}}$$

and

$$\begin{aligned}|I_3| \leq & C_M \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \big(\|\boldsymbol{\varepsilon}_2\|_{L^p} + \|\nabla\phi_2\|_{L^p}\big)\big(\|\bar{\varepsilon}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big) \\ \leq & K_M\big(1 + h(t)\big) \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \big(\|\bar{\varepsilon}\|_{L^{p'}} + \|\nabla\bar{\phi}\|_{L^{p'}}\big).\end{aligned} \tag{4.49}$$

Therefore from Lemma 4.19 and (4.47) we obtain that

$$\begin{aligned}& \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,p}} + \|\phi_1 - \phi_2\|_{W^{1,p}} \\ \leq & C_*\Big(C_M h(t) + C C_M^2 + C + K_M\big(1 + h(t)\big)\Big) \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \\ \leq & K_M\big(1 + h(t)\big) \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}.\end{aligned} \tag{4.50}$$

With these in hand, we are ready to estimate the difference of $S(t, \boldsymbol{P}_i)$ for $i = 1, 2$. Recall from (4.24) that

$$S(t, \boldsymbol{P}) = -\mathcal{Q}\big(t, \boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P}), \boldsymbol{P}\big) - \big(\boldsymbol{P}_D'(t) - \Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big)$$

and from (4.22) and Remark 3.11 that

$$\begin{aligned}& \mathcal{Q}(t, \boldsymbol{u}, \phi, \boldsymbol{P})[\bar{\boldsymbol{P}}] \\ = & \int_{\boldsymbol{\Omega}} \frac{1}{2} D_{\boldsymbol{P}} \mathbb{B}_2\big(\tilde{\boldsymbol{P}}(t)\big)\bar{\boldsymbol{P}} \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\tilde{\boldsymbol{P}}(t)\big) \\ \boldsymbol{D}(t) \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\tilde{\boldsymbol{P}}(t)\big) \\ \boldsymbol{D}(t) \end{pmatrix} \\ & + \mathbb{B}_2\big(\tilde{\boldsymbol{P}}(t)\big) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\tilde{\boldsymbol{P}}(t)\big) \\ \boldsymbol{D}(t) \end{pmatrix} : \begin{pmatrix} -D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0\big(\tilde{\boldsymbol{P}}(t)\big)\bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{P}} \end{pmatrix} + D_{\boldsymbol{P}}\omega\big(\tilde{\boldsymbol{P}}(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x},\end{aligned} \tag{4.51}$$

where

$$\boldsymbol{D}(t) = -\boldsymbol{\epsilon}\big(\tilde{\boldsymbol{P}}(t)\big)\big(\nabla\phi + \nabla\phi_D(t)\big) + \boldsymbol{e}\big(\tilde{\boldsymbol{P}}(t)\big)\Big(\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\tilde{\boldsymbol{P}}(t)\big)\Big).$$

Since the terms $\boldsymbol{P}_D'$, $\Delta\boldsymbol{P}_D$, $\boldsymbol{f}_3$, $\boldsymbol{\pi}$ depend only on t, we only need to consider the three integrands in (4.51). Since ω, the third term of (4.51), is a sixth order polynomial, it is smooth and its derivative is locally Lipschitzian. Denote this Lipschitz constant for $\boldsymbol{P} \in \mathbb{R}^3, |\boldsymbol{P}| \leq M'$ by $\mathcal{L}_{M'}$. Then for $\bar{\boldsymbol{P}} \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega_P}}(\boldsymbol{\Omega})\big)^2$ we have

$$\begin{aligned}
&\Big|\int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\omega(\tilde{\boldsymbol{P}}_1(t))(\bar{\boldsymbol{P}}) - D_{\boldsymbol{P}}\omega\big(\tilde{\boldsymbol{P}}_2(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x}\Big| \\
\leq&\mathcal{L}_{M'}\int_{\boldsymbol{\Omega}} |\boldsymbol{P}_1 - \boldsymbol{P}_2||\bar{\boldsymbol{P}}|d\boldsymbol{x} \\
\leq&\mathcal{L}_{M'}\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\|\bar{\boldsymbol{P}}\|_{L^{p'}} \\
\leq&C\mathcal{L}_{M'}\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \\
\leq&K_M\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}.
\end{aligned} \tag{4.52}$$

Analogously as estimating I_1 given in (4.48), to estimate the first two complicated summands in (4.51), it suffices to estimate the following terms:

$$J_1 = \int_{\boldsymbol{\Omega}} |\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}_1| + |\nabla\phi_1| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi_D(t)| + |\boldsymbol{P}_1|\big)^2|\boldsymbol{P}_1 - \boldsymbol{P}_2|d\boldsymbol{x}$$

and

$$\begin{aligned}
J_2 = \int_{\boldsymbol{\Omega}} &|\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}_1| + |\nabla\phi_1| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi_D(t)| + |\boldsymbol{P}_1|\big) \\
&\cdot\big(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\nabla\phi_1 - \nabla\phi_2| + |\boldsymbol{P}_1 - \boldsymbol{P}_2|\big)d\boldsymbol{x}.
\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}
|J_1| &\leq C\big(1 + \|\boldsymbol{\varepsilon}_1\|_{L^p} + \|\nabla\phi_1\|_{L^p} + \|\boldsymbol{\varepsilon}_D(t)\|_{L^p} + \|\nabla\phi_D(t)\|_{L^p}\big)^2\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{L^{\frac{p}{p-2}}} \\
&\leq CK_M\Big(1 + K_M\big(1 + h(t)\big) + h(t)\Big)^2\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \\
&= CK_M\Big(\big(1 + K_M\big)\big(1 + h(t)\big)\Big)^2\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \\
&\leq K_M\big(1 + h(t)\big)^2\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
|J_2| &\leq C\big(1 + \|\boldsymbol{\varepsilon}_1\|_{L^p} + \|\nabla\phi_1\|_{L^p} + \|\boldsymbol{\varepsilon}_D(t)\|_{L^p} + \|\nabla\phi_D(t)\|_{L^p}\big)\cdot\big(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{L^p} \\
&\qquad + \|\nabla\phi_1 - \nabla\phi_2\|_{L^p} + \|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^p}\big)\|\bar{\boldsymbol{P}}\|_{L^{\frac{p}{p-2}}} \\
&\leq CK_M\Big(1 + K_M\big(1 + h(t)\big) + h(t)\Big)\Big(K_M\big(1 + h(t)\big)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\Big)\|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \\
&\leq CK_M\big(1 + h(t)\big)^2\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}} \\
&\leq K_M\big(1 + h(t)\big)^2\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty}\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}.
\end{aligned} \tag{4.54}$$

Sum up all, we obtain that

$$\begin{aligned}
\|S(t, \boldsymbol{P}_1) - S(t, \boldsymbol{P}_2)\|_X &\leq h_M(t)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{L^\infty} \\
&\leq h_M(t)\|\boldsymbol{P}_1 - \boldsymbol{P}_2\|_{(Y,X)_{\frac{1}{\tau},\tau}}
\end{aligned} \tag{4.55}$$

with

$$h_M(t) = K_M\big(1 + h(t)\big)^2.$$

From Assumption B4 and Lemma D.5 we immediately infer that $h_M \in L^r(0, T)$. This shows the validity of (S).

Step 3: Verification of $S(t,\mathbf{0}) \in L^r(0,T;X)$

From the definition of $S(t,\boldsymbol{P})$ we obtain that

$$S(t,\mathbf{0}) = -\mathcal{Q}\big(t,\boldsymbol{u}(t,\mathbf{0}),\phi(t,\mathbf{0}),\mathbf{0}\big) - \big(\boldsymbol{P}'_D(t) - \Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big).$$

Due to Assumption B4 we infer that

$$\boldsymbol{P}'_D,\ \Delta\boldsymbol{P}_D,\ \boldsymbol{f}_3,\ \boldsymbol{\pi} \in L^r(0,T;X).$$

Thus we only need to consider the term $\mathcal{Q}$. We obtain from (4.51) that

$$\begin{aligned}
&\mathcal{Q}(t,\boldsymbol{u},\phi,\mathbf{0})[\bar{\boldsymbol{P}}] \\
&= \int_{\boldsymbol{\Omega}} \frac{1}{2} D_{\boldsymbol{P}}\mathbb{B}_2\big(\boldsymbol{P}_D(t)\big)\bar{\boldsymbol{P}} \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}_D(t)\big) \\ \boldsymbol{D}_0(t) \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}_D(t)\big) \\ \boldsymbol{D}_0(t) \end{pmatrix} \\
&\quad + \mathbb{B}_2\big(\boldsymbol{P}_D(t)\big) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}_D(t)\big) \\ \boldsymbol{D}_0(t) \end{pmatrix} : \begin{pmatrix} -D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0\big(\boldsymbol{P}_D(t)\big)\bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{P}} \end{pmatrix} + D_{\boldsymbol{P}}\omega\big(\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x}
\end{aligned}$$

with

$$\boldsymbol{D}_0(t) = -\boldsymbol{\epsilon}\big(\boldsymbol{P}_D(t)\big)\big(\nabla\phi + \nabla\phi_D(t)\big) + \boldsymbol{e}\big(\boldsymbol{P}_D(t)\big)\left(\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0\big(\boldsymbol{P}_D(t)\big)\right).$$

Write

$$\begin{aligned}
\big(\boldsymbol{u}(t),\phi(t)\big) &= L^{-1}_{\boldsymbol{P}_D(t)}\Big(l_{t,\boldsymbol{P}_D(t)}\Big), \\
\boldsymbol{\varepsilon}(t) &= \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big),
\end{aligned}$$

that is, $\big(\boldsymbol{u}(t),\phi(t)\big)$ is chosen as the solution of the piezo-system corresponding to the pair $(t,\boldsymbol{P}) = (t,\mathbf{0})$ and the external loading $l_{t,\boldsymbol{P}_D(t)}$. Setting $M = 0$ in (4.46) we obtain that

$$|l_{t,\boldsymbol{P}_D(t)}(\bar{\boldsymbol{u}},\bar{\phi})| \leq K_0\big(1 + h(t)\big)\big(\|\bar{\boldsymbol{u}}\|_{W^{1,p'}} + \|\bar{\phi}\|_{W^{1,p'}}\big).$$

for $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{1,p'}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$. Together with Lemma 4.19 we obtain that

$$\begin{aligned}
&\|\boldsymbol{\varepsilon}(t)\|_{L^p} + \|\nabla\phi(t)\|_{L^p} \\
\leq&\|\boldsymbol{u}(t)\|_{W^{1,p}} + \|\phi(t)\|_{W^{1,p}} \\
\leq&C_*K_0\big(1 + h(t)\big) \\
=:&C\big(1 + h(t)\big).
\end{aligned}$$

Finally, using the Hölder relation

$$\frac{p-2}{p} + \frac{1}{p} + \frac{1}{p} = 1$$

we obtain that

$$\begin{aligned}
&\mathcal{Q}(t,\boldsymbol{u}(t),\phi(t),\boldsymbol{P}_D(t))[\bar{\boldsymbol{P}}] \\
\leq&C\Big(\int_{\boldsymbol{\Omega}} |\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}(t)| + |\nabla\phi(t)| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi_D(t)| + |\boldsymbol{P}_D(t)|\big)^2 \\
&\quad + |\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}(t)| + |\nabla\phi(t)| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi_D(t)| + |\boldsymbol{P}_D(t)|\big)d\boldsymbol{x}\Big) \\
\leq&C\int_{\boldsymbol{\Omega}} |\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}(t)| + |\nabla\phi(t)| + |\boldsymbol{\varepsilon}_D(t)| + |\nabla\phi_D(t)| + |\boldsymbol{P}_D(t)|\big)^2 d\boldsymbol{x} \\
\leq&C\|\bar{\boldsymbol{P}}\|_{L^{\frac{p}{p-2}}}\big(1 + \|\boldsymbol{\varepsilon}(t)\|^2_{L^p} + \|\nabla\phi(t)\|^2_{L^p} + \|\boldsymbol{\varepsilon}_D(t)\|^2_{L^p} + \|\nabla\phi_D(t)\|^2_{L^p} + \|\boldsymbol{P}_D(t)\|^2_{L^p}\big) \\
\leq&C\big(1 + h^2(t)\big)\|\bar{\boldsymbol{P}}\|_{L^{\frac{p}{p-2}}} \\
\leq&C\big(1 + h^2(t)\big)\|\bar{\boldsymbol{P}}\|_{W^{1,p'}}
\end{aligned} \tag{4.56}$$

for $\bar{\boldsymbol{P}} \in \big(W^{1,p'}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, where for the last inequality we have used the Sobolev's embedding relation

$$1 - \frac{2}{p'} \geq 0 - 2/\Big(\frac{p}{p-2}\Big).$$

From Assumption B4 and Lemma D.5 we infer that

$$h^2(t) \in L^r(0,T).$$

Hence we obtain that

$$S(\cdot, \mathbf{0}) \in L^r(0,T;X),$$

which completes the proof of Step 3.

Finally, from Theorem 4.3 one obtains a unique local solution

$$\boldsymbol{P} \in W^{1,r}(0,\hat{T};\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2) \cap L^r(0,\hat{T};\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2)$$

of the system

$$\boldsymbol{P}'(t) - \Delta\boldsymbol{P}(t) = S\big(t, \boldsymbol{P}(t)\big) \tag{4.57}$$

on the time interval $(0,\hat{T})$ for some $\hat{T} \in (0,T]$. Inserting the pair $\big(t, \boldsymbol{P}(t)\big)$ into (4.25) we obtain the solution pair $\big(\boldsymbol{u}(t), \phi(t)\big)$. To complete the proof we still need to show that the solution $(\boldsymbol{u}, \phi)$ has the claimed regularity given by (4.23). First we point out that the inequality (4.46) is originally derived for $\boldsymbol{P} \in (Y,X)_{\frac{1}{\tau},\tau}$, but since we have only used the property that $\boldsymbol{P}$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$, which is derived from Lemma 4.11, we see that (4.46) is also valid for $\boldsymbol{P} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^2$. On the other hand, using the embedding

$$W^{1,r}(0,\hat{T};\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \cap L^r(0,\hat{T};\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^d) \hookrightarrow C^\delta([0,\hat{T}];\big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^d)$$

from Proposition 4.13, we see that the local solution $\boldsymbol{P}$ of (4.57) is also an element of $\big(C([0,T]\times\overline{\boldsymbol{\Omega}})\big)^2$. This implies that $\boldsymbol{P}(t)$ is in the space $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$ for all $t \in [0,T]$. Thus (4.46) is in this case still applicable. We still need to clarify that the constant K_M in (4.46) is uniform for a.a. $t \in [0,T]$ by inserting the local solution $\boldsymbol{P}$ into (4.46). Notice that when we consider (4.46) for $\boldsymbol{P} \in (Y,X)_{\frac{1}{\tau},\tau}$, M is some positive constant which depends only on $\|\boldsymbol{P}\|_{(Y,X)_{\frac{1}{\tau},\tau}}$. But we see that such dependence is realized via the relation (4.43), namely, we actually have used the dependence of M on $\|\boldsymbol{P}\|_{L^\infty}$. Thus if we consider $\boldsymbol{P}$ as the local solution of (4.57), we find out that since $\boldsymbol{P}$ is in the space $\big(C([0,T]\times\overline{\boldsymbol{\Omega}})\big)^2$, K_M is uniform for a.a. $t \in [0,T]$ by inserting the local solution $\boldsymbol{P}$ into (4.46). In this case, since $\boldsymbol{P}$ is given as a fixed function, we are also able to say that K_M is bounded by some positive constant C which is uniform for a.a. $t \in [0,T]$ and $\boldsymbol{x} \in \overline{\boldsymbol{\Omega}}$. Next, from Assumption B4 and Lemma D.5 we infer that the function $h(t)$ given in (4.44) is in $L^{2r^*}(0,\hat{T})$. Thus from (4.46) we obtain immediately that

$$l_{\cdot,\tilde{\boldsymbol{P}}(\cdot)} \in L^{2r^*}(0,\hat{T};\big(W^{-1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{-1,\hat{p}}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})) \tag{4.58}$$

and for a.a. $t \in [0,T]$ we have

$$\|l_{t,\tilde{\boldsymbol{P}}(t)}\|_{W^{-1,\hat{p}}} \leq C\big(1 + h(t)\big) \tag{4.59}$$

by setting $p = \hat{p}$ in (4.46). Consequently, from Lemma 4.19 we obtain that for a.a. $t \in [0,T]$ we have

$$\big(\boldsymbol{u}(t), \phi(t)\big) \in \big(W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})$$

and

$$\|\big(\boldsymbol{u}(t),\phi(t)\big)\|_{W^{1,\hat{p}}} \leq C_*\|l_{t,\tilde{\boldsymbol{P}}(t)}\|_{W^{-1,\hat{p}}} \leq C_* C\big(1+h(t)\big),$$

where the last inequality is due to (4.59) and C_* is given by Lemma 4.19 (notice particularly that C_* is even uniform for all measurable $\boldsymbol{P}:\boldsymbol{\Omega}\to\mathbb{R}^2$ due to the last statement of Lemma 4.19). Together with the fact that $h\in L^{2r^*}(0,\hat{T})$ we obtain immediately that

$$(\boldsymbol{u},\phi)\in L^{2r^*}(0,\hat{T};\big(W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2\times W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})),$$

which is the desired regularity of $(\boldsymbol{u},\phi)$ given in (4.23). This completes the proof of Theorem 4.3. □

4.3 3D-local existence results for domains with C^1-boundary and for cuboids

In this section we will present the local existence result for the case $d=3$. Here, the main difficulty is the insufficient regularity of the solution $(\boldsymbol{u},\phi)$ given by the piezo-problem (4.20). More precisely, in order to guarantee that the functional $\mathcal{Q}$ given by (4.22) is of class $W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}$ for some $p>d=3$, the solution $(\boldsymbol{u},\phi)$ of (4.20) is required to be of class $W^{1,p}$. However, this can not be obtained in general for the piezo-problem by imposing mixed boundary conditions. As we see, the result given by Proposition 3.18 shows that the value of p is expected to be close to 2.

If we restrict ourself to the Dirichlet boundary case, then the result given by [14] guarantees that the solution $(\boldsymbol{u},\phi)$ of the piezo-problem is of class $W^{1,p}_0$ for some $p>3$, by assuming that the underlying domain $\boldsymbol{\Omega}$ has C^1-boundary and the external loadings corresponding to the piezo-problem are sufficiently regular (roughly speaking, they should be of class $W^{-1,p}$). However, the norm of the inverse piezo-operator in this case will not only depend on the upper bound of the coefficient tensors, but also on their modulus of continuity, see for instance [21, Chap. 7]. Therefore, the method introduced in last section for two dimensional case to show the validity of (S) of Theorem 4.3 is not directly applicable for three dimensional case. We will use the continuity arguments given in [46] to fix this problem.

Particularly, we see that regularity result Proposition 4.33 given below, valid for domains with C^1-boundary, will play the essential role for the proof of Theorem 4.39. We point out that the results given in [1] provide us an analogue of Proposition 4.33 for a cuboid domain. We will then use similar regularity result as Proposition 4.33, namely the Proposition 4.41 below, to extend the results from Theorem 4.39 for domains with C^1-boundary to the ones for cuboid domains, see Theorem 4.44 below.

4.3.1 Assumptions and weak formulation

C1 $\boldsymbol{\Omega}\subset\mathbb{R}^3$ is a bounded domain with C^1 boundary, $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}=\partial\boldsymbol{\Omega}_\phi=\partial\boldsymbol{\Omega}_{\boldsymbol{P}}\dot{\cup}\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}=\partial\boldsymbol{\Omega}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ is a 2-set and $\boldsymbol{\Omega}\cup\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ is G2-regular (c.f. Section 2.1).

C2 $\mathbb{C},\boldsymbol{e},\boldsymbol{\varepsilon}^0,\boldsymbol{\epsilon}$ (c.f. Section 2.4) are differentiable functions on $\mathbb{R}^3$ and their derivatives are locally Lipschitzian on $\mathbb{R}^3$; $\omega:\mathbb{R}^3\to\mathbb{R}$ (c.f. Section 2.4) is a polynomial of sixth order with constant coefficients.

C3 There exists some $\alpha > 0$ such that for all $\boldsymbol{P} \in \mathbb{R}^3$, $\boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^3, \mathbb{R}^3)$, $\boldsymbol{D} \in \mathbb{R}^3$

$$\begin{aligned} \mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} &\geq \alpha|\boldsymbol{\varepsilon}|^2, \\ \boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{D} \cdot \boldsymbol{D} &\geq \alpha|\boldsymbol{D}|^2. \end{aligned}$$

C4 There exist $p^* \in (3, \infty)$ and $r^* \in [1, \infty)$ such that

$$\begin{aligned} &\boldsymbol{f}_1 \in L^{2r^*}(0,T; \big(L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)^3), \\ &f_2 \in L^{2r^*}(0,T; L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})), \\ &\boldsymbol{f}_3 \in L^{r^*}(0,T; \big(L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)^3), \\ &\boldsymbol{\pi} \in L^{r^*}(0,T; \big(L^{\frac{2p^*}{3}}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^3), \\ &\boldsymbol{u}_D \in L^{2r^*}(0,T; \big(W^{1-\frac{1}{p^*},p^*}(\partial\boldsymbol{\Omega})\big)^3), \\ &\phi_D \in L^{2r^*}(0,T; W^{1-\frac{1}{p^*},p^*}(\partial\boldsymbol{\Omega})), \\ &\boldsymbol{P}_D \in W^{1,r^*}(0,T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^3). \end{aligned}$$

Remark 4.24. Distinguished to the two dimensional case, one sees that no uniform boundedness conditions are imposed for the coefficients in the three dimensional case. The reason is that the regularity result given by [14] guarantees that the regularity of the solution $(\boldsymbol{u}, \phi)$ given by the piezo-elliptic problem (4.62a) is as good as the r.h.s. function $l_{t,\tilde{\boldsymbol{P}}(t)}$ and independent of the upper bound of the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})$, as long as $\boldsymbol{P}$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$, which will be the case in the following. △

Remark 4.25. From Lemma 3.8 and Sobolev's trace theorem we infer that there exist $\boldsymbol{u}_D$, ϕ_D, $\boldsymbol{P}_D$ such that

$$\begin{aligned} &\boldsymbol{u}_D \in L^{2r^*}(0,T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^3), && \boldsymbol{u}_D|_{\partial\boldsymbol{\Omega}} = \boldsymbol{u}_D, \\ &\phi_D \in L^{2r^*}(0,T; W^{1,p^*}(\boldsymbol{\Omega})), && \phi_D|_{\partial\boldsymbol{\Omega}} = \phi_D, \\ &\boldsymbol{P}_D \in W^{1,r^*}(0,T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^3), && \boldsymbol{P}_D|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \boldsymbol{P}_D. \end{aligned}$$

From Lemma 3.5 and the analysis given below Lemma 3.8 we obtain that

$$\begin{aligned} \boldsymbol{f}_1,\, \boldsymbol{\varepsilon}(\boldsymbol{u}_D) &\in L^{2r^*}\big(0,T; \big(W^{-1,p^*}(\boldsymbol{\Omega})\big)^3\big), \\ f_2,\, \nabla\phi_D &\in L^{2r^*}\big(0,T; W^{-1,p^*}(\boldsymbol{\Omega})\big), \\ \boldsymbol{f}_3,\, \boldsymbol{\pi} &\in L^{r^*}\big(0,T; \big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3\big), \\ \boldsymbol{P}_D,\, \Delta\boldsymbol{P}_D &\in W^{1,r^*}\big(0,T; \big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3\big), \end{aligned}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}_D)$ is the small strain tensor generated by $\boldsymbol{u}_D$. △

Remark 4.26. Analogously as done in (4.29), we obtain from the conditions $p^* \in (3, \infty)$ and

$$\boldsymbol{P}_D \in W^{1,r^*}(0,T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^3)$$

that

$$\boldsymbol{P}_D \in C([0,T]; \big(C^{1-\frac{3}{p^*}}(\overline{\boldsymbol{\Omega}})\big)^3) \hookrightarrow C([0,T]; \big(C(\overline{\boldsymbol{\Omega}})\big)^3). \tag{4.61}$$

△

Recall from Section 4.2.1 that $\tilde{H}$ and $\tilde{\omega}$ are defined by

$$\tilde{H}(t,\boldsymbol{u},\phi,\boldsymbol{P}) = H\big(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\nabla\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big)$$

and

$$\tilde{\omega}(t,\boldsymbol{P}) = \omega\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big),$$

where H and ω are given by (3.7) and (3.8). Then using the same notation as the one given in Section 4.2.1, we give the following weak formulation which is to be investigated: Find $(\boldsymbol{u},\phi,\boldsymbol{P}) : (0,T) \to \big(H_0^1(\boldsymbol{\Omega})\big)^4 \times \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3$ such that

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1\big(\boldsymbol{P}(t)+\boldsymbol{P}_D(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\\ \nabla\phi(t)\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x} = l_{t,\boldsymbol{P}(t)+\boldsymbol{P}_D(t)}(\bar{\boldsymbol{u}},\bar{\phi}), \tag{4.62a}$$

$$\beta\boldsymbol{P}'(t) - \kappa\Delta\boldsymbol{P}(t) = S\big(t,\boldsymbol{u}(t),\phi(t),\boldsymbol{P}(t)\big) \ \text{ in } \ \big(H^{-1}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3, \tag{4.62b}$$

$$\boldsymbol{P}(0) = \boldsymbol{P}_0 \tag{4.62c}$$

for a.a. $t \in (0,T)$ and all $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(H_0^1(\boldsymbol{\Omega})\big)^4$, where

$$S(t,\boldsymbol{u},\phi,\boldsymbol{P}) = -\mathcal{Q}\big(t,\boldsymbol{u},\phi,\boldsymbol{P}\big) - \big(\beta\boldsymbol{P}'_D(t) - \kappa\Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big),$$

$\mathbb{B}_1$, $l_{t,\boldsymbol{P}}$ are defined by (3.3) and (3.4) and $\mathcal{Q}$ is defined by

$$\begin{aligned}
&\mathcal{Q}(t,\boldsymbol{u},\phi,\boldsymbol{P})[\bar{\boldsymbol{P}}]\\
&= \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\tilde{H}\big(t,\boldsymbol{u},\phi,\boldsymbol{P}\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\tilde{\omega}(t,\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}\\
&= \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}H\big(t,\boldsymbol{u},\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\omega\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x}
\end{aligned}$$

for $\bar{\boldsymbol{P}} \in \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3$.

Remark 4.27. We should also point out that in this case, the terms $\boldsymbol{t}$ and ρ given in $l_{t,\boldsymbol{P}}$ from (3.4) are irrelevant, since we are dealing with the piezo-problem with overall Dirichlet boundary conditions. △

Imitating Lemma 4.18 we present the following regularity result:

Lemma 4.28. *Let the Assumptions C1 to C4 be satisfied. Then the integrals*

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u})\\ \nabla\phi\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x}$$

and

$$\int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\tilde{H}\big(t,\boldsymbol{u},\phi,\boldsymbol{P}\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\tilde{\omega}(t,\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}$$

are well-defined for a.a. $t \in (0,T)$*, all* $(\boldsymbol{u},\phi)$*,* $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(H_0^1(\boldsymbol{\Omega})\big)^4$ *and all* $\boldsymbol{P}$*,* $\bar{\boldsymbol{P}} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^3$.

4.3.2 Strongly elliptic differential system

We first set up the system introduced in [14]. Let $n, N \in \mathbb{N}$. Let $\boldsymbol{\Omega} \subset \mathbb{R}^n$ be a bounded domain. For $i, j = 1, ..., N$ and $\alpha, \iota = 1, ..., n$ let the measurable functions $f_i^\alpha, g_i : \boldsymbol{\Omega} \to \mathbb{R}$ and coefficients $\boldsymbol{A}_{ij}^{\alpha\iota} \in L^\infty(\boldsymbol{\Omega})$ be given. We look for solutions $\boldsymbol{u} : \boldsymbol{\Omega} \to \mathbb{R}^N$ of the differential system

$$(L\boldsymbol{u})_i := \sum_{j=1}^{N} \sum_{\alpha,\iota=1}^{n} -\partial_\alpha(\boldsymbol{A}_{ij}^{\alpha\iota}\partial_\iota \boldsymbol{u}^j) = \sum_{\alpha=1}^{n} -\partial_\alpha f_i^\alpha + g_i \quad \forall i = 1, ..., N. \tag{4.63}$$

In the following we use the Einstein summation convention: the Σ summation symbol will be neglected, the Greek indices α and ι are always summed from 1 to n and Latin indices i and j are summed from 1 to N. We also use the notation

$$L\boldsymbol{u} := -\mathrm{Div}(\boldsymbol{A}\nabla\boldsymbol{u}) = -\mathrm{Div}\boldsymbol{f} + \boldsymbol{g}$$

to interpret the differential system (4.63). In the following, we give the definition of a strongly elliptic differential system and additional related definitions. For more details, we also refer to [21, Chap. 3.4] and [45, Chap. 4].

Definiton 4.29 (Strong ellipticity, coerciveness and weak solution)**.** *Let L be the differential operator defined by* (4.63) *with coefficient tensor $\boldsymbol{A}$.*

- *The operator L or the coefficient tensor $\boldsymbol{A}$ is called strongly elliptic or said to satisfy the Legendre-Hadamard condition, if there exists some constant $\nu > 0$ such that*

$$\boldsymbol{A}_{ij}^{\alpha\iota}(\boldsymbol{x})\xi_\alpha\xi_\iota\zeta^i\zeta^j \geq \nu|\xi|^2|\zeta|^2 \quad \forall \xi \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^N \text{ and a.e. } \boldsymbol{x} \in \boldsymbol{\Omega}.$$

- *The operator L or the coefficient tensor $\boldsymbol{A}$ is called coercive or said to satisfy the Gårding's inequality, if there exist constants $c, C > 0$ such that*

$$\forall \boldsymbol{u} \in \left(H_0^1(\boldsymbol{\Omega})\right)^N : \int_{\boldsymbol{\Omega}} \boldsymbol{A}_{ij}^{\alpha\iota}\partial_\iota \boldsymbol{u}^j \partial_\alpha \boldsymbol{u}^i d\boldsymbol{x} \geq c\|\boldsymbol{u}\|_{H^1}^2 - C\|\boldsymbol{u}\|_{L^2}^2.$$

- *A function $\boldsymbol{u} \in \left(H^1(\boldsymbol{\Omega})\right)^N$ is called a weak solution of* (4.63)*, if*

$$\forall \boldsymbol{u} \in \left(H_0^1(\boldsymbol{\Omega})\right)^N : \int_{\boldsymbol{\Omega}} \boldsymbol{A}_{ij}^{\alpha\iota}\partial_\iota \boldsymbol{u}^j \partial_\alpha \bar{\boldsymbol{u}}^i d\boldsymbol{x} = \int_{\boldsymbol{\Omega}} f_i\partial_\alpha \bar{\boldsymbol{u}}^i + g_i\bar{\boldsymbol{u}}^i d\boldsymbol{x}.$$

From Lemma 4.30 and Lemma 4.31 given below we obtain that the piezo-operator defined by (4.62a) is strongly elliptic.

Lemma 4.30 ([45, Thm. 4.6])**.** *Let $\boldsymbol{\Omega} \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and assume that the coefficient tensor $\boldsymbol{A}$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$. Then $\boldsymbol{A}$ is strongly elliptic if and only if it is coercive.*

Lemma 4.31. *Let the Assumptions C1 to C3 be satisfied. Let $\boldsymbol{P} \in \left(C(\overline{\boldsymbol{\Omega}})\right)^3$ and let the differential system*

$$L_{\boldsymbol{P}} : \left(H_0^1(\boldsymbol{\Omega})\right)^4 \to \left(H^{-1}(\boldsymbol{\Omega})\right)^4$$

be given by

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi)\big[(\bar{\boldsymbol{u}}, \bar{\phi})\big] := \int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \nabla\phi \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x}$$

for $(\boldsymbol{u}, \phi)$, $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \left(H_0^1(\boldsymbol{\Omega})\right)^4$. Then for all $\boldsymbol{P} \in \left(C(\overline{\boldsymbol{\Omega}})\right)^3$, $L_{\boldsymbol{P}}$ is strongly elliptic on $\left(H_0^1(\boldsymbol{\Omega})\right)^4$.

Proof. Using the Poincaré's and Korn's inequalities we obtain that

$$\begin{aligned}
&\int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon} \\ \nabla\phi \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon} \\ \nabla\phi \end{pmatrix} d\boldsymbol{x} \\
=&\int_{\boldsymbol{\Omega}} \mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \boldsymbol{\epsilon}(\boldsymbol{P})\nabla\phi \cdot \nabla\phi d\boldsymbol{x} \\
\geq&\alpha\big(\|\boldsymbol{\varepsilon}\|_{L^2}^2 + \|\nabla\phi\|_{L^2}^2\big) \\
\geq&C\big(\|\boldsymbol{u}\|_{H^1}^2 + \|\phi\|_{H^1}^2\big)
\end{aligned}$$

for $(\boldsymbol{u}, \phi) \in \big(H_0^1(\boldsymbol{\Omega})\big)^4$, where α is given by the Assumption C3 and we used the Poincaré's and Korn's inequalities to infer the last inequality. Since $\boldsymbol{P}$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$ and $\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ are continuous on the whole $\mathbb{R}^3$ due to Assumption C2, $\mathbb{B}_1\big(\boldsymbol{P}(\boldsymbol{x})\big)$ is uniformly continuous in $\boldsymbol{x} \in \overline{\boldsymbol{\Omega}}$. Now Lemma 4.30 ensures that the given differential system is strongly elliptic. □

We will utilize the following regularity result to infer higher integrability of the solution $(\boldsymbol{u}, \phi)$ of the piezo-system:

Lemma 4.32 ([14, Lem. 2])**.** *Let $\boldsymbol{\Omega} \subset \mathbb{R}^n$ be a domain with C^1-boundary. Let the coefficient tensor $\boldsymbol{A}$ be uniformly continuous on $\overline{\boldsymbol{\Omega}}$ and strongly elliptic. Let $p \in (1, \infty)$, $q \in (1, n)$ and $\boldsymbol{f} \in L^p$, $\boldsymbol{g} \in L^q$. Then the differential system*

$$-\operatorname{Div}(\boldsymbol{A}\nabla\boldsymbol{u}) = -\operatorname{Div}\boldsymbol{f} + \boldsymbol{g}$$

admits a weak solution $\boldsymbol{u} \in \big(H_0^1(\boldsymbol{\Omega})\big)^N$ and $\boldsymbol{u}$ is of class $W^{1,s}$ with $s = \min(p, \frac{nq}{n-q})$ and

$$\|\boldsymbol{u}\|_{W^{1,s}} \leq C\big(\|\boldsymbol{f}\|_{L^p} + \|\boldsymbol{g}\|_{L^q}\big).$$

We now give our regularity result for the piezo-system:

Proposition 4.33. *Let the Assumptions C1 to C3 be satisfied. Let $\boldsymbol{P} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^3$ and $p \in (3, \infty)$. Let also $\boldsymbol{l} \in \big(W^{-1,p}(\boldsymbol{\Omega})\big)^4$. Let $L_{\boldsymbol{P}}$ be the differential operator defined by*

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi) := -\operatorname{Div}\left(\mathbb{B}_1(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \nabla\phi \end{pmatrix}\right)$$

for $(\boldsymbol{u}, \phi) \in \big(H_0^1(\boldsymbol{\Omega})\big)^4$ and consider the differential equation

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi) = \boldsymbol{l}. \tag{4.64}$$

Then (4.64) has a unique weak solution $(\boldsymbol{u}, \phi) \in \big(W_0^{1,p}(\boldsymbol{\Omega})\big)^4$ and

$$\|\boldsymbol{u}\|_{W^{1,p}} + \|\phi\|_{W^{1,p}} \leq C_{\boldsymbol{P}}\|\boldsymbol{l}\|_{W^{-1,p}}, \tag{4.65}$$

where $C_{\boldsymbol{P}}$ is some positive constant depending on $\boldsymbol{P}$.

Proof. The existence and uniqueness of the weak solution follow directly from Lax-Milgram. Since $\boldsymbol{P}$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$ and the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})$ is continuous in $\boldsymbol{P} \in \mathbb{R}^3$ due to Assumption C2, the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})$ is therefore uniformly continuous on $\overline{\boldsymbol{\Omega}}$. Thus the conditions of Lemma 4.32 are satisfied and consequently, the weak solution $(\boldsymbol{u}, \phi)$ will be of class $W^{1,p}$ due to Lemma 4.32 (notice that one can choose $q \in (1, n)$ as close as possible to n such that $q < p$ and $\frac{nq}{n-q}$ in Lemma 4.32 is sufficiently large). The corresponding L^p-estimate follows also immediately from Lemma 4.32. Since the proof of Lemma 4.32 is based on the so called freezing technique for coefficients, we should point out that the norm of the inverse operator $L_{\boldsymbol{P}}^{-1}$ also depends on the modulus of continuity of the coefficients (we also refer to [21, Chap. 7] for more details). Since the modulus of continuity of the coefficients $\mathbb{B}_1(\boldsymbol{P})$ depends also on $\boldsymbol{P}$, the dependence of $L_{\boldsymbol{P}}^{-1}$ on $\boldsymbol{P}$ follows immediately. □

Remark 4.34. From Lemma 4.11 we see that the underlying space for $\boldsymbol{P}$, namely the space $\left(\left(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3, \left(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3\right)_{\frac{1}{r},r}$ with some $p \in (3,6]$ and $r > \frac{2p}{p-3}$, is continuously embedded to some Hölder space. On the other hand, due to (4.61) we also know that $\boldsymbol{P}_D$ is uniformly continuous on $[0,T] \times \overline{\boldsymbol{\Omega}}$. Thus the uniform continuity of $\tilde{\boldsymbol{P}} = \boldsymbol{P} + \boldsymbol{P}_D$ on $[0,T] \times \overline{\boldsymbol{\Omega}}$ is obtained and Proposition 4.33 is applicable by evaluating $L_{\boldsymbol{P}}$ at $\boldsymbol{P} \overset{!}{=} \tilde{\boldsymbol{P}}$, which will be the case in the proof of Theorem 4.39 below. △

Remark 4.35. Since the Proposition 4.33 plays the same role for three dimensional case as Lemma 4.19 for two dimensional case, we see that given $t \in [0,T]$, $\boldsymbol{P} \in \left(C(\overline{\boldsymbol{\Omega}})\right)^3$ and assuming that $l_{t,\boldsymbol{P}+\boldsymbol{P}_D(t)}$ is of class $\left(W^{-1,p}(\boldsymbol{\Omega})\right)^4$ for some $p > 3$, the functional $S(t,\boldsymbol{P})$ defined by

$$\begin{aligned} S(t,\boldsymbol{P}) &= S\big(t,\boldsymbol{u}(t,\boldsymbol{P}),\phi(t,\boldsymbol{P}),\boldsymbol{P}\big) \\ &= -\mathcal{Q}\big(t,\boldsymbol{u}(t,\boldsymbol{P}),\phi(t,\boldsymbol{P}),\boldsymbol{P}\big) - \big(\beta\boldsymbol{P}_D'(t) - \kappa\Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big), \end{aligned} \tag{4.66}$$

where $\big(\boldsymbol{u}(t,\boldsymbol{P}),\phi(t,\boldsymbol{P})\big)$ is the unique weak solution of the differential equation

$$-\operatorname{Div}\left(\mathbb{B}_1\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t,\boldsymbol{P})\big)\\ \nabla\phi(t,\boldsymbol{P})\end{pmatrix}\right) = l_{t,\boldsymbol{P}+\boldsymbol{P}_D(t)},$$

actually belongs to $\left(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3$, meaning that the operator $S(t,\boldsymbol{P})$ is mapped to the right underlying space given in Theorem 4.3. △

4.3.3 Uniform boundedness of the piezo-operator and admissible pair of parameters

As mentioned previously, the norm of the inverse piezo-operator will also depend on the modulus of continuity of the coefficients. However, we have the following important observations which make it possible to directly utilize the results from Section 4.2 to the case here:

- The underlying space $(Y,X)_{\frac{1}{\tau},\tau}$ of $\boldsymbol{P}$ is continuously embedded into some Hölder space due to Lemma 4.11 and
- a Hölder space is compactly embedded to the space of uniformly continuous functions, see for instance [22, Lem. 6.33].

Inspired by these observations and the continuity arguments given in [46, Cor. 3.24], we are able to give the following regularity result, which enables us to make use of the arguments given in Section 4.2:

Lemma 4.36. *Let the Assumptions C1 to C3 be satisfied and let $p \in (3,\infty)$. Let $\mathcal{M}$ be a pre-compact subset of $\left(C(\overline{\boldsymbol{\Omega}})\right)^3$. Define for $\boldsymbol{P} \in \mathcal{M}$ the operator $L_{\boldsymbol{P}}$ by*

$$L_{\boldsymbol{P}}(\boldsymbol{u},\phi)[\bar{\boldsymbol{u}},\bar{\phi}] := \int_{\boldsymbol{\Omega}} \mathbb{B}_1(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u})\\ \nabla\phi\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x}$$

for $(\boldsymbol{u}, \phi) \in \big(W_0^{1,p}(\boldsymbol{\Omega})\big)^4$ *and* $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \big(W_0^{1,p'}(\boldsymbol{\Omega})\big)^4$*. Then* $L_{\boldsymbol{P}}$ *is linear, continuous and bijective from* $\big(W_0^{1,p}(\boldsymbol{\Omega})\big)^4$ *to* $\big(W_0^{-1,p}(\boldsymbol{\Omega})\big)^4$*. In particular, the norm* $C_{\boldsymbol{P}}$ *of the inverse operator* $L_{\boldsymbol{P}}^{-1}$ *is uniformly bounded by some positive constant* C^* *for all* $\boldsymbol{P} \in \mathcal{M}$ *and* C^* *depends only on the number* p *and the set* $\mathcal{M}$*.*

Proof. W.l.o.g. we may assume that $\mathcal{M}$ is compact, since a set is always contained in its own closure. First we show that the mapping

$$\big(C(\overline{\boldsymbol{\Omega}})\big)^3 \ni \boldsymbol{P} \mapsto J(\boldsymbol{P}) := \left(-\mathrm{Div}\left(\mathbb{B}_1(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}(\cdot|_{\boldsymbol{u}})\\ \nabla\cdot|_{\phi}\end{pmatrix}\right)\right)^{-1} \in LH\Big(\big(W^{-1,p}(\boldsymbol{\Omega})\big)^4, \big(W_0^{1,p}(\boldsymbol{\Omega})\big)^4\Big)$$

is well-defined and continuous, where $LH(X,Y)$ denotes the set of linear homeomorphisms between Banach spaces X and Y. On the one hand, using similar estimation as the one given by (4.49) we infer that the mapping

$$\big(C(\overline{\boldsymbol{\Omega}})\big)^3 \ni \boldsymbol{P} \mapsto -\mathrm{Div}\left(\mathbb{B}_1(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}(\cdot|_{\boldsymbol{u}})\\ \nabla\cdot|_{\phi}\end{pmatrix}\right) \in LH\Big(\big(W_0^{1,p}(\boldsymbol{\Omega})\big)^4, \big(W^{-1,p}(\boldsymbol{\Omega})\big)^4\Big)$$

is continuous (that the image is in fact a linear homeomorphism follows from Proposition 4.33). On the other hand, due to [58, CH. III.8], the mapping $LH(X,Y) \ni B \mapsto B^{-1} \in LH(Y,X)$ is continuous. Then the claim follows, since the mapping J is a composition of continuous functions. Now since J is continuous and $\mathcal{M}$ is compact, we see that $J(\mathcal{M})$ is a bounded subset of $L\Big(\big(W^{-1,p}(\boldsymbol{\Omega})\big)^4, \big(W_0^{1,p}(\boldsymbol{\Omega})\big)^4\Big)$. In particular we obtain that

$$C^* = \sup_{\boldsymbol{P}\in\mathcal{M}} \|J(\boldsymbol{P})\|_{L(W^{-1,p}, W_0^{1,p})} < \infty.$$

This completes the proof. □

In order to formulate the main theorem, we still need to give here the definition of an admissible pair:

Definiton 4.37. *We call a pair* (p, r) *admissible, if* $p \in (3, 6]$ *and* $r \in (\frac{2p}{p-3}, \infty)$*.*

Remark 4.38. Compared to the admissible pairs given by Definition 4.20 for two dimensional case with overall mixed boundary conditions, we point out that thanks to Proposition 4.33, we do not have an upper bound p_* (which does exist in the two dimensional case due to Lemma 4.19) for p here. However, in order to utilize Lemma 4.11, we still need to assume that p is not greater than 6. △

4.3.4 Local existence result for domains with C^1-boundary

In the following, we state the local existence result for three dimensional domains with C^1-boundary. As discussed at the beginning of Chapter 4, the indirect continuity arguments given in [46] can not be directly applied to our case here, since the external loadings of our model will also depend on the variable $\boldsymbol{P}$. Instead of modifying the arguments given in [46] to fit our setting (which is still possible but tedious), we will use the continuity results given by Lemma 4.36 to show that certain uniform boundedness arguments of the inverse piezo-operator utilized in the proof of Theorem 4.22 still remain valid for the case here.

Theorem 4.39. *Let the Assumptions C1 to C3 be satisfied and β, κ in (4.62) be given positive constants. Let (p, r) be an admissible pair in the sense of Definition 4.37 and let the Assumption C4 be satisfied for some $p^* \in [p, \infty)$ and $r^* \in [r, \infty)$. Assume also that*

$$\boldsymbol{P}_0 \in \left(\left(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3, \left(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3\right)_{\frac{1}{r},r}.$$

Then the differential system (4.62) has a unique local solution $(\boldsymbol{u}, \phi, \boldsymbol{P})$ in the time interval $(0, \hat{T})$ for some $0 < \hat{T} \leq T$ such that

$$\begin{aligned}
&\boldsymbol{u} \in L^{2r^*}(0, \hat{T}; \left(W_0^{1,p^*}(\boldsymbol{\Omega})\right)^3),\\
&\phi \in L^{2r^*}(0, \hat{T}; W_0^{1,p^*}(\boldsymbol{\Omega})),\\
&\boldsymbol{P} \in W^{1,r}(0, \hat{T}; \left(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3) \cap L^r(0, \hat{T}; \left(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3).
\end{aligned} \tag{4.67}$$

Remark 4.40. Compared to the two dimensional case, we see that $(\boldsymbol{u}, \phi)$ has an improved spatial regularity in the three dimensional case (in the sense that $(\boldsymbol{u}, \phi)$ has higher integrability). The reason is that the Proposition 4.33 guarantees that the integrability of the three dimensional solution $(\boldsymbol{u}, \phi)$ is as large as the the r.h.s. functional of the piezo-system, while the one in two dimensional case will also depend on the upper bound of the coefficients of the piezo-system due to Lemma 4.19. △

Proof. Writing $\tilde{\boldsymbol{P}}(t) = \boldsymbol{P} + \boldsymbol{P}_D(t)$, we give the following notation according to Theorem 4.3:

$$\begin{aligned}
A(t, \boldsymbol{P}) &\equiv -\kappa\Delta,\\
\tau &= r,\\
Y &= \left(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3,\\
X &= \left(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\right)^3
\end{aligned}$$

and

$$\begin{aligned}
S(t, \boldsymbol{P}) &= S\big(t, \boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P}), \boldsymbol{P}\big)\\
&= -\mathcal{Q}\big(t, \boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P}), \boldsymbol{P}\big) - \big(\beta\boldsymbol{P}'_D(t) - \kappa\Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big),
\end{aligned}$$

where $\big(\boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P})\big)$ is the unique $W_0^{1,2}$-weak solution of the differential equation

$$L_{\tilde{\boldsymbol{P}}(t)}\big(\boldsymbol{u}(t, \boldsymbol{P}), \phi(t, \boldsymbol{P})\big) = l_{t,\tilde{\boldsymbol{P}}(t)}, \tag{4.68}$$

where $L_{\tilde{\boldsymbol{P}}(t)}$ is defined by Lemma 4.36 and $l_{t,\tilde{\boldsymbol{P}}(t)}$ is defined by (3.4). Having defined this notation, we utilize Theorem 4.3 to show that the equation

$$\boldsymbol{P}'(t) + A\big(t, \boldsymbol{P}(t)\big) = S\big(t, \boldsymbol{P}(t)\big)$$

with initial value $\boldsymbol{P}_0$ has the claimed unique local solution $\boldsymbol{P}$ given by (4.67). The proof is essentially the same as the proof for Theorem 4.22, all we need to do is to let the statements given in the proof of Theorem 4.22, which make use of the constant C_* given by Lemma 4.19, make use of the constant C^* from Lemma 4.36 instead. More precisely, we should replace the constant C^* appearing in Step 1c, Step 2 and Step 3 in the proof of Theorem 4.22 by the constant C_* from Lemma 4.36 and show that such replacement is applicable. We make this precise in the following.

Reverification of Step 1c

Recall from (4.36) that

$$\tilde{\boldsymbol{P}}_n(t) \to \tilde{\boldsymbol{P}}(t) \quad \text{in } \big(C(\overline{\boldsymbol{\Omega}})\big)^2 \text{ for a.a. } t \in [0,T].$$

Here we have used the fact that the constant C_* from Lemma 4.19 is a uniform upper bound for all inverse piezo-operators $L^{-1}_{\tilde{\boldsymbol{P}}_n(t)}$ and $L^{-1}_{\tilde{\boldsymbol{P}}(t)}$, where $t \in [0,T]$ here is chosen such that (4.36) is valid for this chosen t. Notice we have used the set $\big(C(\overline{\boldsymbol{\Omega}})\big)^2$ in (4.36). However, this is due to the fact that the constant C_* from Lemma 4.19 is uniform for all inverse piezo-operators $L^{-1}_{\boldsymbol{P}}$ with arbitrary measurable functions $\boldsymbol{P} : \boldsymbol{\Omega} \to \mathbb{R}^2$ and for the purpose of calculation convenience. In fact, due to Lemma 4.11 and (4.61) we know that

$$\tilde{\boldsymbol{P}} \in C([0,T]; \big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^3)$$

for some $\delta \in (0,1)$. Thus due to the definition of Bochner-measurability given in Definition 4.4 we actually have the relation

$$\tilde{\boldsymbol{P}}_n(t) \to \tilde{\boldsymbol{P}}(t) \quad \text{in } \big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^3 \text{ for a.a. } t \in [0,T]. \tag{4.69}$$

Let $t \in [0,T]$ be chosen such that (4.69) is valid for such t. Then define

$$\mathcal{M}_1 = \mathcal{M}_{1,t} := \{\tilde{\boldsymbol{P}}_n(t) : n \in \mathbb{N}\} \cup \{\tilde{\boldsymbol{P}}(t)\}.$$

From the convergence due to (4.69) we know that $\mathcal{M}_1$ is a bounded subset of $\big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^3$. Since $\boldsymbol{\Omega}$ is a bounded domain in $\mathbb{R}^3$, we know that $\mathcal{M}_1$ is pre-compact in $\big(C(\overline{\boldsymbol{\Omega}})\big)^3$ due to [22, Lem. 6.33]. Thus the conditions of Lemma 4.36 are satisfied and we obtain from Lemma 4.36 a positive constant $C^{1,t,*} \in (0,\infty)$ corresponding to the set $\mathcal{M}_1$ and the number p such that

$$\max\{\sup_{n\in\mathbb{N}} \|L^{-1}_{\tilde{\boldsymbol{P}}_n(t)}\|_{L(W^{-1,p},W_0^{1,p})}, \|L^{-1}_{\tilde{\boldsymbol{P}}(t)}\|_{L(W^{-1,p},W_0^{1,p})}\} \le C^{1,t,*} < \infty. \tag{4.70}$$

We should also point out that $C^{1,t,*}$ depends also on t. However, the Bochner-measurability is given as a local definition, that is, we only need to concentrate on the value of $C^{1,t,*}$ for a given $t \in [0,T]$, for which (4.69) is valid; the overall behavior of $C^{1,t,*}$ running over admissible $t \in [0,T]$ is of no interest here (at least within Step 1c for showing the Bochner-measurability). Thus, we are able to replace the number C_* by $C^{1,t,*}$ in this case, and the proof of Step 1c will remain true due to (4.70).

Reverification of Step 2

In Step 2, we have first defined the set

$$\mathcal{V} = \{\boldsymbol{P} \in (Y,X)_{\frac{1}{\tau},\tau} : \|\boldsymbol{P}\|_{(Y,X)_{\frac{1}{\tau},\tau}} \le M\}$$

with some given positive constant M. Consequently, the underlying set $\mathcal{M}_2$ was defined by

$$\mathcal{M}_2 := \{\boldsymbol{P} + \boldsymbol{P}_D(t) : \boldsymbol{P} \in \mathcal{V},\ t \in [0,T]\}.$$

In Step 2 we have namely used the fact that C_* is a uniform upper bound of the inverse piezo-operators $L^{-1}_{\boldsymbol{P}}$ for all $\boldsymbol{P} \in \mathcal{M}_2$. From Lemma 4.11 and (4.61) we see that $\mathcal{M}_2$ is a bounded subset of $\big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^3$ for some $\delta \in (0,1)$, and from [22, Lem. 6.33] we infer immediately that $\mathcal{M}_2$ is pre-compact in $\big(C(\overline{\boldsymbol{\Omega}})\big)^3$. The same argument as the one for the reverification of Step 1c given in Theorem 4.22 will hence still remain valid for the case here, when we replace the constant C_* therein by the positive constant $C^{2,*}$ derived from Lemma 4.36 corresponding to $\mathcal{M}_2$ and p.

Reverification of Step 3

In Step 3, the underlying set $\mathcal{M}_3$ was defined by

$$\mathcal{M}_3 = \{\boldsymbol{P}_D(t) : t \in [0,T]\}.$$

Due to (4.61), the set $\mathcal{M}_3$ is a bounded subset of $\big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^3$, and we obtain some positive constant $C^{3,*}$ derived from Lemma 4.36 corresponding to $\mathcal{M}_3$ and p, which should replace the constant C_* in the original proof of Step 3. In this case, the proof of Step 3 will still remain valid.

Sum up all the reverification results, we see that all the steps from Theorem 4.22 can be adopted to the case here, and consequently we obtain from Theorem 4.3 a unique local solution

$$\boldsymbol{P} \in W^{1,r}(0,\hat{T}; \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3) \cap L^r(0,\hat{T}; \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3)$$

of the system

$$\boldsymbol{P}'(t) - \Delta\boldsymbol{P}(t) = S\big(t, \boldsymbol{P}(t)\big) \tag{4.71}$$

on the time interval $(0,\hat{T})$ for some $\hat{T} \in (0,T]$. Inserting the pair $\big(t, \boldsymbol{P}(t)\big)$ into (4.68) we obtain the solution pair $\big(\boldsymbol{u}(t), \phi(t)\big)$. To complete the proof we still need to show that the solution $(\boldsymbol{u}, \phi)$ has the claimed regularity given by (4.67). We make use of Lemma 4.36 as follows: we define the set

$$\mathcal{M} := \{\boldsymbol{P}(t) + \boldsymbol{P}_D(t) : t \in [0,T]\},$$

where $\boldsymbol{P}$ is the local solution given by (4.71). Then similarly argued as at the end of the proof of Theorem 4.22, using the regularity result from Proposition 4.13 and (4.61) we immediately infer that $\mathcal{M}$ is a bounded subset of $\big(C^\delta(\overline{\boldsymbol{\Omega}})\big)^3$ with some $\delta \in (0,1)$. Thus $\mathcal{M}$ is pre-compact in $\big(C(\overline{\boldsymbol{\Omega}})\big)^3$ due to [22, Lem. 6.33] and Lemma 4.36 is applicable. We hence obtain the positive constant C^* from Lemma 4.36 corresponding to $\mathcal{M}$ and p^* (and note that not p here!). In particular, the definition of C^* implies that

$$\|\big(\boldsymbol{u}(t), \phi(t)\big)\|_{W_0^{1,p^*}} \le C^* \|l_{t,\tilde{\boldsymbol{P}}(t)}\|_{W^{-1,p^*}} \tag{4.72}$$

for a.a. $t \in (0,\hat{T})$. Arguing as in (4.58), we obtain that

$$l_{\cdot,\tilde{\boldsymbol{P}}(\cdot)} \in L^{2r^*}(0,\hat{T}; \big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2 \times W^{-1,p^*}_{\partial\boldsymbol{\Omega}_\phi}(\boldsymbol{\Omega})).$$

Note that C^* is independent on $t \in [0,\hat{T}]$. Thus from (4.72) we obtain immediately that

$$\|(\boldsymbol{u}, \phi)\|_{L^{2r^*}(0,\hat{T};W^{1,p^*})} \le C^* \|l_{\cdot,\tilde{\boldsymbol{P}}(\cdot)}\|_{L^{2r*}(0,\hat{T};W^{-1,p^*})} < \infty, \tag{4.73}$$

which completes the desired proof. □

4.3.5 Local existence result for cuboids

We point out that the regularity result Proposition 4.33 is essential for the proof of Theorem 4.39. Using Proposition 4.41 given below, which is an analogue of Proposition 4.33 but for the case that $\boldsymbol{\Omega}$ is an open cuboid, we are able to obtain similar existence results (Theorem 4.44 below) for a cuboid domain.

To state the regularity result Proposition 4.41, we first recall from Section 4.3.2 that a linear differential system L with in $\mathbf{\Omega}$ essentially bounded coefficients $\boldsymbol{A}_{ij}^{\alpha\iota}$ and defined by

$$L\boldsymbol{u} = -\partial_\alpha(\boldsymbol{A}_{ij}^{\alpha\iota}\partial_\iota \boldsymbol{u}^j) \tag{4.74}$$

is called strongly elliptic, if there exists some constant $\nu > 0$ such that

$$\boldsymbol{A}_{ij}^{\alpha\iota}(\boldsymbol{x})\xi_\alpha\xi_\iota\zeta^i\zeta^j \geq \nu|\xi|^2|\zeta|^2 \quad \forall\xi \in \mathbb{R}^n,\ \zeta \in \mathbb{R}^N \text{ and a.e. } \boldsymbol{x} \in \mathbf{\Omega}.$$

Proposition 4.41. *Let $\mathbf{\Omega} \subset \mathbb{R}^n$ be an open cuboid. Assume that the coefficients $\boldsymbol{A}_{ij}^{\alpha\iota}$ given in (4.74) are Hölder continuous on $\overline{\mathbf{\Omega}}$ for some Hölder exponent $\alpha \in (0,1)$. Let also $\boldsymbol{u}$ be a $W_0^{1,2}$-weak solution (see Definition 4.29) of the differential system*

$$L\boldsymbol{u} = \boldsymbol{l},$$

where L is the strongly elliptic differential operator defined by (4.74) and $\boldsymbol{l}$ is some linear functional of class $W^{-1,p}$ for some $p \in [2,\infty)$. Then the weak solution $\boldsymbol{u}$ is of class $W^{1,p}$. In particular, there exists some positive constant C, depending on the number p and the coefficient tensor related to L, such that for all $\boldsymbol{l}$ of class $W^{-1,p}$ we have

$$\|\boldsymbol{u}\|_{W^{1,p}} \leq C\|\boldsymbol{l}\|_{W^{-1,p}}.$$

Proof. The claim for a cube is proved by [1, Thm. 3.1]. Then the claim for a cuboid follows by using the transformation arguments given in the proof of [1, Thm. 2.2, Step 1B], which is applicable, since for a cuboid and a cube we can always find a C^∞-diffeomorphism mapping one to another (translation, rotation, rescaling). □

Next, we formulate a version of Lemma 4.36 for a cuboid domain:

Lemma 4.42. *Let the Assumptions C1 to C3 be satisfied, except that the domain $\mathbf{\Omega}$ is assumed to be an open cuboid. Let $p \in (3,\infty)$. Let also $\mathcal{M}$ be a bounded subset of $\big(C^\delta(\overline{\mathbf{\Omega}})\big)^3$ with some $\delta \in (0,1)$. Define for $\boldsymbol{P} \in \mathcal{M}$ the operator $L_{\boldsymbol{P}}$ by*

$$L_{\boldsymbol{P}}(\boldsymbol{u},\phi)[\bar{\boldsymbol{u}},\bar{\phi}] := \int_{\mathbf{\Omega}} \mathbb{B}_1(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u})\\ \nabla\phi\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x}$$

for $(\boldsymbol{u},\phi) \in \big(W_0^{1,p}(\mathbf{\Omega})\big)^4$ and $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(W_0^{1,p'}(\mathbf{\Omega})\big)^4$. Then $L_{\boldsymbol{P}}$ is linear, continuous and bijective from $\big(W_0^{1,p}(\mathbf{\Omega})\big)^4$ to $\big(W_0^{-1,p}(\mathbf{\Omega})\big)^4$. In particular, the norm $C_{\boldsymbol{P}}$ of the inverse operator $L_{\boldsymbol{P}}^{-1}$ is uniformly bounded by some positive constant C^* for all $\boldsymbol{P} \in \mathcal{M}$ and C^* depends only on the number p and the set $\mathcal{M}$.

Remark 4.43. Notice in Lemma 4.42 we have assumed that $\mathcal{M}$ is a bounded subset of some Hölder space, which is slightly different than the condition that $\mathcal{M}$ is pre-compact in $\big(C(\overline{\mathbf{\Omega}})\big)^3$ given in Lemma 4.36. This is due to the fact that the Hölder continuity of the coefficient tensor is a necessary condition of Proposition 4.41. △

Proof. Notice that since $\mathcal{M}$ is bounded in some Hölder space, it is still pre-compact in $\big(C(\overline{\mathbf{\Omega}})\big)^3$ due to [22, Lem. 6.33]. Thus the proof follows immediately when one replaces Proposition 4.33 by Proposition 4.41 in the proof of Lemma 4.36. □

With this lemma in hand, we can state our main result as follows:

Theorem 4.44. *Let the assumptions of Theorem 4.39 be satisfied, except that the domain* $\mathbf{\Omega}$ *is assumed to be an open cuboid. Then the differential system* (4.62) *has a unique local solution* $(\boldsymbol{u}, \phi, \boldsymbol{P})$ *in the time interval* $(0, \hat{T})$ *for some* $0 < \hat{T} \leq T$ *such that*

$$\begin{aligned} &\boldsymbol{u} \in L^{2r^*}(0, \hat{T}; \left(W_0^{1,p^*}(\mathbf{\Omega})\right)^3), \\ &\phi \in L^{2r^*}(0, \hat{T}; W_0^{1,p^*}(\mathbf{\Omega})), \\ &\boldsymbol{P} \in W^{1,r}(0, \hat{T}; \left(W_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}^{-1,p}(\mathbf{\Omega})\right)^3) \cap L^r(0, \hat{T}; \left(W_{\partial\mathbf{\Omega}_{\boldsymbol{P}}}^{1,p}(\mathbf{\Omega})\right)^3). \end{aligned} \tag{4.75}$$

Proof. The proof follows immediately when one replaces Lemma 4.36 by Lemma 4.42 in the proof of Theorem 4.22. □

4.4 3D-local existence result for polyhedral domains

In this section, we will give an existence proof to the special model introduced in [55], which is one of the state-of-the-art models arising from recent years for ferroelectric study. We have the following important observations from the precise model setting given in [55]:

- The coefficient tensors $\mathbb{C}(\boldsymbol{P})$, $\boldsymbol{e}(\boldsymbol{P})$, $\boldsymbol{\epsilon}(\boldsymbol{P})$ given in [55] are polynomials in $\boldsymbol{P}$ up to order six. Since the coefficients are polynomials, we point out that the coefficients and their derivatives are particularly smooth and locally Lipschitzian on $\mathbb{R}^3$. Thus the Assumption C2 is satisfied for these coefficients.

- We obtain that these coefficients are closely related to the Lamé operator and Laplace operator when the presenting polarization $\boldsymbol{P}$ is equal to $\mathbf{0}$. This is precisely given as follows: let $\mathbf{\Omega} \subset \mathbb{R}^3$ be a bounded domain. Recall that the piezo-operator $L_{\boldsymbol{P}}$ is defined by

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi)[\bar{\boldsymbol{u}}, \bar{\phi}] := \int_{\mathbf{\Omega}} \mathbb{B}_1(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \nabla\phi \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x}. \tag{4.76}$$

  Due to [55], the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})$ at $\boldsymbol{P} = \mathbf{0}$ is given by

$$\mathbb{B}_1(\mathbf{0}) = \begin{pmatrix} \lambda \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 + 2\mu\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \gamma\boldsymbol{E}_3 \end{pmatrix}, \tag{4.77}$$

  where $\boldsymbol{E}_3$ is the identity matrix in $\mathbb{R}^{3\times3}$, $\mathbb{1}$ is the identity tensor in the space $\mathrm{Lin}(\mathbb{R}^{3\times3}, \mathbb{R}^{3\times3})$ and λ, μ, γ are some given fixed constants. A differential system with the upper left coefficient tensor of (4.77) is the so called Lamé operator, while the bottom right coefficient tensor in (4.77) corresponds to the Laplace operator. This motivates the application of the regularity results from [44], in which the Lamé operator and Laplace operator are well studied.

We also point out that the regularity results given in [44] are derived for polyhedral domains, which means that we are able to extend the results for an underlying domain with C^1-boundary or for a cuboid presented in last section to the ones for a polyhedral domain. In what follows, we will thus first give the definition of a polyhedral domain.

4.4.1 Polyhedral domains

In order to introduce the regularity results given in [44], we first need the following definitions from [44]:

Dihedron

Definiton 4.45. *Let $(r, \phi) \in (0, \infty) \times (0, 2\pi]$ be the polar coordinate of a non zero two dimensional point $\boldsymbol{x}' = (\boldsymbol{x}_1, \boldsymbol{x}_2)$, i.e. $r = |\boldsymbol{x}'|$ and*

$$\boldsymbol{x}' = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}.$$

The two dimensional wedge $\mathcal{W}$ with open angle $\theta \in (0, 2\pi]$ is defined by

$$\mathcal{W} := \{\boldsymbol{x}' = (\boldsymbol{x}_1, \boldsymbol{x}_2) : r \in (0, \infty), \phi \in (0, \theta)\}.$$

The dihedron $\mathcal{D} \subset \mathbb{R}^3$ with open angle θ is then defined by

$$\mathcal{D} := \mathcal{W} \times \mathbb{R}.$$

Cone

Definiton 4.46. *A cone $\mathcal{K} \subset \mathbb{R}^3$ is defined by*

$$\mathcal{K} := \{\boldsymbol{x} \in \mathbb{R}^3 - \{\mathbf{0}\} : \boldsymbol{x}/|\boldsymbol{x}| \in \boldsymbol{\Gamma}\},$$

where $\boldsymbol{\Gamma}$ is a subdomain on the three dimensional unit sphere S^2 such that

$$\partial\boldsymbol{\Gamma} = \bar{\boldsymbol{\Gamma}}_1 \cup ... \cup \bar{\boldsymbol{\Gamma}}_k$$

for some $k \in \mathbb{N}_{\geq 3}$, where $\boldsymbol{\Gamma}_i$ are pairwise disjoint and non curvewise collinear open arcs on the unit sphere.

An illustration of a dihedron and a cone is given in Fig. 4.1.

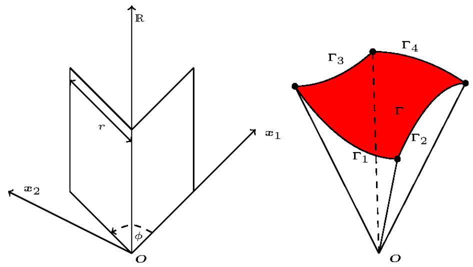

Figure 4.1: An illustration of a dihedron (left) and a cone (right).

Polyhedral domain

Definiton 4.47. *A bounded domain $\boldsymbol{\Omega} \subset \mathbb{R}^3$ is said to be a polyhedral domain, if*

1. *The boundary $\partial\boldsymbol{\Omega}$ is a disjoint union of smooth open two dimensional manifolds $\mathcal{F}_j, j = 1, ..., l$ (the faces of $\boldsymbol{\Omega}$), smooth open curves $K_j, j = 1, ..., m$ (the edges of $\boldsymbol{\Omega}$) and vertices $\boldsymbol{x}^j \in \mathbb{R}^3, j = 1, ..., k$.*

2. *For every $\boldsymbol{x} \in K_j$ there exist a neighborhood $\boldsymbol{U}_{\boldsymbol{x}} \subset \mathbb{R}^3$ of $\boldsymbol{x}$ and a C^∞-diffeomorphism $\boldsymbol{\iota}_{\boldsymbol{x}}$ such that $\boldsymbol{\iota}_{\boldsymbol{x}}$ maps $\boldsymbol{U}_{\boldsymbol{x}} \cap \boldsymbol{\Omega}$ onto $\mathcal{D}_{\boldsymbol{x}} \cap B_1$, where $\mathcal{D}_{\boldsymbol{x}}$ is a dihedron and B_1 is the unit ball in $\mathbb{R}^3$.*

3. *For every vertex $\boldsymbol{x}^j$ there exist a neighborhood $\boldsymbol{U}_j \subset \mathbb{R}^3$ of $\boldsymbol{x}^j$ and a C^∞-diffeomorphism $\boldsymbol{\iota}_j$ such that $\boldsymbol{\iota}_j$ maps $\boldsymbol{U}_j \cap \boldsymbol{\Omega}$ onto $\mathcal{K}_j \cap B_1$, where $\mathcal{K}_j$ is a cone.*

Having these definitions in hand, we are able to formulate the assumptions for the main result, which are given in the next section.

4.4.2 Assumptions and weak formulation

D1 $\boldsymbol{\Omega} \subset \mathbb{R}^3$ is a polyhedral domain with Lipschitz boundary, $\partial\boldsymbol{\Omega}_{\boldsymbol{u}} = \partial\boldsymbol{\Omega}_\phi = \partial\boldsymbol{\Omega}_{\boldsymbol{P}} \dot{\cup} \partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}} = \partial\boldsymbol{\Omega}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ is a 2-set, $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ is G2-regular (c.f. Section 2.1).

D2 $\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ (c.f. Section 2.4) are differentiable functions on $\mathbb{R}^3$ and their derivatives are locally Lipschitzian on $\mathbb{R}^3$; $\omega : \mathbb{R}^3 \to \mathbb{R}$ (c.f. Section 2.4) is a polynomial of sixth order with constant coefficients; there exist positive constants λ, μ, γ such that

$$\begin{pmatrix} \mathbb{C}(\mathbf{0}) & \boldsymbol{e}(\mathbf{0})^T \\ -\boldsymbol{e}(\mathbf{0}) & \boldsymbol{\epsilon}(\mathbf{0}) \end{pmatrix} = \begin{pmatrix} \lambda \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 + 2\mu\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \gamma \boldsymbol{E}_3 \end{pmatrix}.$$

D3 There exists some $\alpha > 0$ such that for all $\boldsymbol{P} \in \mathbb{R}^3,\ \boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^3, \mathbb{R}^3),\ \boldsymbol{D} \in \mathbb{R}^3$

$$\begin{aligned} \mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} &\geq \alpha|\boldsymbol{\varepsilon}|^2, \\ \boldsymbol{\epsilon}(\boldsymbol{P})\boldsymbol{D} \cdot \boldsymbol{D} &\geq \alpha|\boldsymbol{D}|^2. \end{aligned}$$

D4 There exist $p^* \in (3, \infty)$ and $r^* \in [1, \infty)$ such that

$$\begin{aligned} &\boldsymbol{f}_1 \in L^{2r^*}(0, T; \big(L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)^3), \\ &f_2 \in L^{2r^*}(0, T; L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})), \\ &\boldsymbol{f}_3 \in L^{r^*}(0, T; \big(L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)^3), \\ &\boldsymbol{\pi} \in L^{r^*}(0, T; \big(L^{\frac{2p^*}{3}}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^3), \\ &\boldsymbol{u}_D \in L^{2r^*}(0, T; \big(W^{1-\frac{1}{p^*}, p^*}(\partial\boldsymbol{\Omega})\big)^3), \\ &\phi_D \in L^{2r^*}(0, T; W^{1-\frac{1}{p^*}, p^*}(\partial\boldsymbol{\Omega})), \\ &\boldsymbol{P}_D \in W^{1,r^*}(0, T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^3). \end{aligned}$$

Remark 4.48. As mentioned previously, since the coefficients $\mathbb{C}(\boldsymbol{P})$, $\boldsymbol{e}(\boldsymbol{P})$, $\boldsymbol{\epsilon}(\boldsymbol{P})$ given in [55] are polynomials in $\boldsymbol{P} \in \mathbb{R}^3$ up to order 6, they are particularly smooth and locally Lipschitzian on $\mathbb{R}^3$, thus Assumption D2 is compatible with the setting given in [55]. △

Remark 4.49. Similarly as in Section 4.3, from Lemma 3.8 and Sobolev's trace theorem we infer that there exist $\boldsymbol{u}_D$, ϕ_D, $\boldsymbol{P}_D$ such that

$$\begin{aligned} &\boldsymbol{u}_D \in L^{2r^*}(0, T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^3), && \boldsymbol{u}_D|_{\partial\boldsymbol{\Omega}} = \boldsymbol{u}_D, \\ &\phi_D \in L^{2r^*}(0, T; W^{1,p^*}(\boldsymbol{\Omega})), && \phi_D|_{\partial\boldsymbol{\Omega}} = \phi_D, \\ &\boldsymbol{P}_D \in W^{1,r^*}(0, T; \big(W^{1,p^*}(\boldsymbol{\Omega})\big)^3), && \boldsymbol{P}_D|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \boldsymbol{P}_D. \end{aligned}$$

and consequently we obtain that

$$\begin{aligned} \boldsymbol{f}_1,\, \boldsymbol{\varepsilon}(\boldsymbol{u}_D) &\in L^{2r^*}\big(0,T;\big(W^{-1,p^*}(\boldsymbol{\Omega})\big)^3\big),\\ f_2,\, \nabla\phi_D &\in L^{2r^*}\big(0,T;W^{-1,p^*}(\boldsymbol{\Omega})\big),\\ \boldsymbol{f}_3,\, \boldsymbol{\pi} &\in L^{r^*}\big(0,T;\big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3\big),\\ \boldsymbol{P}_D,\, \Delta\boldsymbol{P}_D &\in W^{1,r^*}\big(0,T;\big(W^{-1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3\big), \end{aligned}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}_D)$ is the small strain tensor generated by $\boldsymbol{u}_D$. △

Analogously as in Section 4.3, writting

$$\tilde{H}(t,\boldsymbol{u},\phi,\boldsymbol{P}) = H\big(t,\boldsymbol{\varepsilon}(\boldsymbol{u}),\nabla\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big)$$

and

$$\tilde{\omega}(t,\boldsymbol{P}) = \omega\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big),$$

where H and ω are given by (3.7) and (3.8), we state the following weak formulation: Find $(\boldsymbol{u},\phi,\boldsymbol{P}) : (0,T) \to \big(H^1_0(\boldsymbol{\Omega})\big)^4 \times \big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3$ such that

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1\big(\boldsymbol{P}(t)+\boldsymbol{P}_D(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\\ \nabla\phi(t)\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x} = l_{t,\boldsymbol{P}(t)+\boldsymbol{P}_D(t)}(\bar{\boldsymbol{u}},\bar{\phi}), \tag{4.79a}$$

$$\beta\boldsymbol{P}'(t) - \kappa\Delta\boldsymbol{P}(t) = S\big(t,\boldsymbol{u}(t),\phi(t),\boldsymbol{P}(t)\big) \ \text{ in } \ \big(H^{-1}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3, \tag{4.79b}$$

$$\boldsymbol{P}(0) = \boldsymbol{P}_0 \tag{4.79c}$$

for a.a. $t \in (0,T)$ and all $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(H^1_0(\boldsymbol{\Omega})\big)^4$, where

$$S(t,\boldsymbol{u},\phi,\boldsymbol{P}) = -\mathcal{Q}(t,\boldsymbol{u},\phi,\boldsymbol{P}) - \big(\beta\boldsymbol{P}'_D(t) - \kappa\Delta\boldsymbol{P}_D(t) - \boldsymbol{f}_3(t) - \boldsymbol{\pi}(t)\big),$$

$\mathbb{B}_1$, $l_{t,\boldsymbol{P}}$ are defined by (3.3) and (3.4) and $\mathcal{Q}$ is defined by

$$\begin{aligned} &\mathcal{Q}(t,\boldsymbol{u},\phi,\boldsymbol{P})[\bar{\boldsymbol{P}}]\\ &= \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\tilde{H}\big(t,\boldsymbol{u},\phi,\boldsymbol{P}\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\tilde{\omega}(t,\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}\\ &= \int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}H\big(t,\boldsymbol{u},\phi,\boldsymbol{P}+\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\omega\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)(\bar{\boldsymbol{P}})d\boldsymbol{x}. \end{aligned}$$

The following result is an analogue of Lemma 4.28 given in Section 4.3:

Lemma 4.50. *Let the Assumptions D1 to D4 be satisfied. Then the integrals*

$$\int_{\boldsymbol{\Omega}} \mathbb{B}_1\big(\boldsymbol{P}+\boldsymbol{P}_D(t)\big)\begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u})\\ \nabla\phi\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\\ \nabla\bar{\phi}\end{pmatrix} d\boldsymbol{x}$$

and

$$\int_{\boldsymbol{\Omega}} D_{\boldsymbol{P}}\tilde{H}\big(t,\boldsymbol{u},\phi,\boldsymbol{P}\big)(\bar{\boldsymbol{P}}) + D_{\boldsymbol{P}}\tilde{\omega}(t,\boldsymbol{P})(\bar{\boldsymbol{P}})d\boldsymbol{x}$$

are well-defined for a.a. $t \in (0,T)$*, all* $(\boldsymbol{u},\phi)$*,* $(\bar{\boldsymbol{u}},\bar{\phi}) \in \big(H^1_0(\boldsymbol{\Omega})\big)^4$ *and all* $\boldsymbol{P}$*,* $\bar{\boldsymbol{P}} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^3$.

Additional assumptions on geometric singularities of the polyhedral domain

We recall that the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})$ of the piezo-system is given by

$$\mathbb{B}_1(\boldsymbol{P}) = \begin{pmatrix} \mathbb{C}(\boldsymbol{P}) & \boldsymbol{e}(\boldsymbol{P})^T \\ -\boldsymbol{e}(\boldsymbol{P}) & \boldsymbol{\epsilon}(\boldsymbol{P}) \end{pmatrix}.$$

Our goal is to derive a regularity result for a polyhedral domain, which plays the same role as Proposition 4.33 and Proposition 4.41 for domains with C^1-boundary and cuboid domains respectively. The difficulty is the lacking regularity of the weak solution $(\boldsymbol{u}, \phi)$ of the piezo-problem near the singularities of a polyhedral domain. The method applied for domains with C^1-boundary or cuboid are no longer applicable anymore, since the boundary is not smooth, the open angle of a dihedron is in general not equal to $\pi/2$ and the cone is locally not diffeomorph to a cuboid corner. Our plan is to apply the method given by [44], where certain so called operator pencils corresponding to an elliptic system are created to solve the problems, and the regularity of the weak solution of the elliptic system near the singularities will depend only on the local geometry of these singularities.

However, since the regularity results in [44] are based on some subtle spectral analysis, the self-adjointness of the piezo-operator $L_{\boldsymbol{P}}$, or more precisely, the symmetry of $\mathbb{B}_1(\boldsymbol{P})$, is essential. We see from (4.77) that $\mathbb{B}_1(\boldsymbol{P})$ is in general not symmetric unless $\boldsymbol{P} = \boldsymbol{0}$. In order to apply the results from [44], we need the following assumption:

Assumption 4.51. *Let $\boldsymbol{\Omega}$ be a polyhedral domain. Denote by $\mathrm{Sing}^1_{\boldsymbol{\Omega}}$ the set of singularities of a polyhedral domain $\boldsymbol{\Omega}$, i.e.,*

$$\mathrm{Sing}^1_{\boldsymbol{\Omega}} := \{\boldsymbol{x} \in \partial\boldsymbol{\Omega} : \boldsymbol{x} \textit{ is a vertex or lies on an edge}\}.$$

We also define

$$\mathrm{Sing}^2_{\boldsymbol{\Omega}} := \{\boldsymbol{x} \in \partial\boldsymbol{\Omega} : \boldsymbol{x} \textit{ is a point on some open edge and } \theta_{\boldsymbol{x}} = \pi/2\},$$

where $\theta_{\boldsymbol{x}}$ is the open angle of the dihedron $\mathcal{D}_{\boldsymbol{x}}$ defined in Definition 4.47, that is, the set of edge points having $\frac{\pi}{2}$-open angle;

$$\mathrm{Sing}^3_{\boldsymbol{\Omega}} := \{\boldsymbol{x} \in \partial\boldsymbol{\Omega} : \boldsymbol{x} \textit{ is a vertex and } \mathcal{K}_{\boldsymbol{x}} \cap B_1 \textit{ is } C^\infty\textit{-diffeomorph to } \mathcal{K}_c \cap B_1\},$$

where $\mathcal{K}_{\boldsymbol{x}}$ is the cone defined in Definition 4.47, B_1 is the unit ball, $\mathcal{K}_c$ is the cone formed by the positive $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$-axis, that is, the set of cube-corner-like vertices. Finally, we define

$$\mathrm{Sing}_{\boldsymbol{\Omega}} := \mathrm{Sing}^1_{\boldsymbol{\Omega}} - \mathrm{Sing}^2_{\boldsymbol{\Omega}} - \mathrm{Sing}^3_{\boldsymbol{\Omega}}.$$

Then we assume that $\mathrm{Sing}_{\boldsymbol{\Omega}} \subset \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ and

$$\forall \boldsymbol{x} \in \mathrm{Sing}_{\boldsymbol{\Omega}} \ \forall t \in (0,T) : \boldsymbol{P}_D(t, \boldsymbol{x}) = \boldsymbol{0}, \tag{4.80}$$

where $\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ and $\boldsymbol{P}_D$ are defined in (4.2).

An illustration of points in $\mathrm{Sing}^2_{\boldsymbol{\Omega}}$ and $\mathrm{Sing}^3_{\boldsymbol{\Omega}}$ is given in Fig. 4.2.

Remark 4.52. From (4.61) we see that $\boldsymbol{P}_D \in C([0,T]; \left(C^\delta(\overline{\boldsymbol{\Omega}})\right)^3)$ for some $\delta \in (0,1)$, thus (4.80) is a well-defined condition under the Assumptions D1 to D4. △

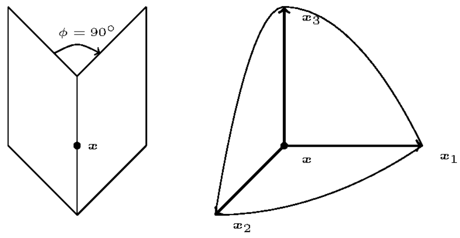

Figure 4.2: An illustration of a point $\boldsymbol{x} \in \mathrm{Sing}^2_{\boldsymbol{\Omega}}$ (left) and a point $\boldsymbol{x} \in \mathrm{Sing}^3_{\boldsymbol{\Omega}}$ (right), where the point $\boldsymbol{x}$ at the r.h.s. is given as the origin of the coordinate system $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$.

Now due to the Assumption 4.51, we see that for $\boldsymbol{P} \in \big(C(\overline{\boldsymbol{\Omega}})\big)^3$ with $\boldsymbol{P}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \mathbf{0}$ we have

$$\forall \boldsymbol{x} \in \mathrm{Sing}_{\boldsymbol{\Omega}}\ \forall t \in (0,T) : \tilde{\boldsymbol{P}}(t,\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{x}) + \boldsymbol{P}_D(t,\boldsymbol{x}) = \mathbf{0}. \tag{4.81}$$

Due to Assumption D2 we obtain that

$$\forall \boldsymbol{x} \in \mathrm{Sing}_{\boldsymbol{\Omega}}\ \forall t \in (0,T) : \mathbb{B}_1\big(\tilde{\boldsymbol{P}}(t,\boldsymbol{x})\big) = \mathbb{B}_1(\mathbf{0}) = \begin{pmatrix} \lambda \boldsymbol{E}_3 \otimes \boldsymbol{E}_3 + 2\mu\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \gamma \boldsymbol{E}_3 \end{pmatrix}. \tag{4.82}$$

Therefore the piezo-operator $L_{\boldsymbol{P}}$ defined by (4.76) will reduce to a composition of Lamé operator and Laplace operator without coupling terms, and hence the results given in [44] are applicable.

Remark 4.53. Roughly speaking, the Assumption 4.51 is to guarantee that the piezo-operator $L_{\boldsymbol{P}}$ will have "good" behavior near the geometric singularities. However, in this case, neighbored Neumann-Neumann boundary condition is in general not allowed, since in this case, the singularities will also be contained in the Neumann boundary part and the piezo-operator $L_{\boldsymbol{P}}$ can not reduce to a "good" part at such singularities. See for instance Example 4.54 below for details. △

4.4.3 Some examples of admissible and non-admissible polyhedral domains and boundary conditions

In this section we give some examples of admissible and non-admissible polyhedral domains and boundary conditions. These examples will give us a better understanding in the given assumptions, especially in the ones given by Assumption 4.51.

Example 4.54. In this example we want to show that even two polyhedrons have same shape, they can be either admissible or non-admissible polyhedral domains by imposing different boundary conditions. We make this precise in the following. In Fig. 4.3, two polyhedrons with same shape are given. The polyhedrons are composed by a pyramid and a cuboid. The neighbored (open) red faces on each polyhedron denote the Neumann boundary parts respectively, and the remaining (closed) white parts are the Dirichlet boundaries. Since $\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ in Assumption D1 is assumed to be a 2-set (see Section 2.1 for a definition of an l-set), the (open) edge corresponding to the neighbored red Neumann faces (namely the open yellow segment in Fig. 4.3) is also a part of the Neumann boundary. Due to the Assumption 4.51, the left one is an admissible candidate, while the right one is not allowed for our case, since the open angle of the red neighbored faces of the left one is $\pi/2$ and of the right one is not equal to $\pi/2$. This shows that the Neumann boundaries are generally separated in the sense that they do not share a common edge. △

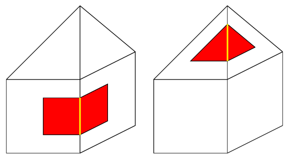

Figure 4.3: Two polyhedrons with same shape but different boundary conditions.

Example 4.55. In this example we give a precise illustration of the set $\mathrm{Sing}_{\boldsymbol{\Omega}}$. We consider the polyhedral domain given by Example 4.54. Then the set $\mathrm{Sing}_{\boldsymbol{\Omega}}$ is the collection of points on the red (open) edges and green points in Fig. 4.4. △

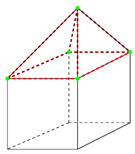

Figure 4.4: The set $\mathrm{Sing}_{\boldsymbol{\Omega}}$ of the polyhedral domain given in Example 4.54.

Example 4.56. We point out that not all polyhedral domains do have Lipschitz boundary. The domain in Fig. 4.5 is a classical counter example. It is clear that the domain is a polyhedral domain. However, it is not a domain with Lipschitz boundary in the sense that the Lipschitz boundary condition (see Section 2.1) does not hold at the point $\boldsymbol{Q}$. To see this, we first point out that the uniform cone property (see [24, Def. 1.2.2.1]) does not hold at the point $\boldsymbol{Q}$; On the other hand, it is a well-known result that the uniform cone property is equivalent to the Lipschitz boundary condition, see for instance [24, Thm. 1.2.2.2]. △

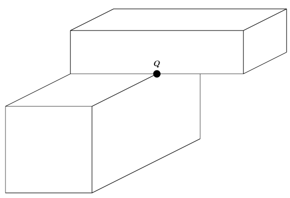

Figure 4.5: A polyhedral domain whose boundary is not Lipschitzian.

Example 4.57. We also want to show that non-convex polyhedral domains which fulfill the Assumption D1 do exist. An example is for instance the Fichera corner as shown in Fig. 4.6. As presented in the picture, a Fichera corner can be formed by removing a cuboid, which is a part of a larger cuboid and near a corner of the larger cuboid, from this larger cuboid. In Fig. 4.6, different boundary conditions are also imposed, where the red parts denote the (open) Neumann boundary part and white parts denote the (closed) Dirichlet boundary part. Particularly, the dashed line segments are contained in the Neumann boundary part. Then the Fichera corner given at the l.h.s. is an admissible candidate, since the dihedron related to the dashed segment has an open angle $\frac{\pi}{2}$, while the one at the r.h.s. is non-admissible, since the dihedron related to the dashed segment has an open angle $\frac{3\pi}{2} \neq \frac{\pi}{2}$. △

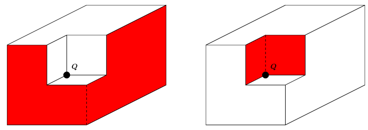

Figure 4.6: An illustration of Fichera corners with different boundary conditions.

4.4.4 Regularity result for strongly elliptic system on polyhedral domain and admissible pair of parameters

We first introduce the regularity result from [44], stated in Lemma 4.58 below. Based on this result, we will apply the perturbation arguments given in [40, Lem. A.18] to obtain Proposition 4.59 given below, which plays the similar role as Proposition 4.33 but for polyhedral domains.

Lemma 4.58. *Let λ, μ, γ be given positive constants. Let also the differential operator L be defined by*

$$L : \big(H_0^1(\mathbf{\Omega})\big)^4 \to \big(H^{-1}(\mathbf{\Omega})\big)^4,$$
$$L(\boldsymbol{u}, \phi)[\bar{\boldsymbol{u}}, \bar{\phi}] := \int_{\mathbf{\Omega}} \lambda \mathrm{tr}\big(\boldsymbol{\varepsilon}(\boldsymbol{u})\big)\mathrm{tr}\big(\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\big) + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) + \gamma\nabla\phi \cdot \nabla\bar{\phi} d\boldsymbol{x}$$

for $(\boldsymbol{u}, \phi)$, $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \big(H_0^1(\mathbf{\Omega})\big)^4$. Then there exists some $\tilde{p} \in (3, \infty)$ such that L is linear, continuous and invertible from the space $\big(W_0^{1,\mathfrak{p}}(\mathbf{\Omega})\big)^4$ to its dual $\big(W^{-1,\mathfrak{p}}(\mathbf{\Omega})\big)^4$ for all $\mathfrak{p} \in [2, \tilde{p}]$.

Proof. This follows immediately from [44, Thm. 4.3.2] and [44, Thm. 4.3.3] and the explanation text below the proofs therein. □

Finally we state our main regularity result, the Proposition 4.59:

Proposition 4.59. *Let the Assumptions D1 to D3 be satisfied. Let $\boldsymbol{P} \in \big(C(\overline{\mathbf{\Omega}})\big)^3$ with $\boldsymbol{P}|_{\mathrm{Sing}_{\mathbf{\Omega}}} = \mathbf{0}$, where the set $\mathrm{Sing}_{\mathbf{\Omega}}$ is defined by Assumption 4.51. Define the operator $L_{\boldsymbol{P}}$ by*

$$L_{\boldsymbol{P}} : \big(H_0^1(\mathbf{\Omega})\big)^4 \to \big(H^{-1}(\mathbf{\Omega})\big)^4,$$
$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi)[\bar{\boldsymbol{u}}, \bar{\phi}] := \int_{\mathbf{\Omega}} \mathbb{B}_1(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \nabla\phi(t) \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) \\ \nabla\bar{\phi} \end{pmatrix} d\boldsymbol{x} \tag{4.83}$$

for $(\boldsymbol{u}, \phi)$, $(\bar{\boldsymbol{u}}, \bar{\phi}) \in \big(H_0^1(\mathbf{\Omega})\big)^4$. Then there exists some $p_* \in (3, 6]$ such that the operator $L_{\boldsymbol{P}}$ is linear, continuous and invertible from $\big(W_0^{1,\mathfrak{p}}(\mathbf{\Omega})\big)^4$ to $\big(W^{-1,\mathfrak{p}}(\mathbf{\Omega})\big)^4$ for all $\mathfrak{p} \in [2, p_*]$.

Proof. We follow the lines of [40, Lem. A.18] to show the claim. Consider a point $\boldsymbol{x} \in \overline{\mathbf{\Omega}}$. Let $L_{\boldsymbol{P}}^0$ be the piezo-operator defined by (4.83), but with fixed constant coefficient tensor $\mathbb{B}_1\big(\boldsymbol{P}(\boldsymbol{x}_0)\big)$ fixing at $\boldsymbol{x}_0 = \boldsymbol{x}$. First we want to show that for each $\boldsymbol{x} \in \overline{\mathbf{\Omega}}$, there always exist some neighborhood $\boldsymbol{U}_{\boldsymbol{x}}$ of $\boldsymbol{x}$ and some $p_{\boldsymbol{x}} \in (3, \infty)$ such that for all $\mathfrak{p} \in [2, p_{\boldsymbol{x}}]$ we have

$$\begin{aligned} &L_{\boldsymbol{P}}^0 \text{ is linear, continuous and invertible from} \\ &\big(W_0^{1,\mathfrak{p}}(\boldsymbol{U}_{\boldsymbol{x}} \cap \mathbf{\Omega})\big)^4 \text{ to } \big(W^{-1,\mathfrak{p}}(\boldsymbol{U}_{\boldsymbol{x}} \cap \mathbf{\Omega})\big)^4. \end{aligned} \tag{4.84}$$

If $\boldsymbol{x}$ is in the interior, then the claim follows from Proposition 4.33, since $\boldsymbol{U}_{\boldsymbol{x}} \subset\subset \mathbf{\Omega}$ can be chosen as an open ball. If $\boldsymbol{x}$ is on an open face, then due to Definition 4.47, $\boldsymbol{x}$ is on a smooth open two dimensional manifold. Thus we know that there exist some neighborhood $\boldsymbol{U_x}$ of $\boldsymbol{x}$ and a C^∞-diffeomorphism $\boldsymbol{\iota}$ such that $\boldsymbol{\iota}$ maps $\boldsymbol{U_x} \cap \boldsymbol{\Omega}$ onto an open cube $\mathcal{C}$. But due to the transformation arguments given in [1, Thm. 2.2, Step 1B], (4.84) is equivalent to the $W_0^{1,\mathfrak{p}}$ to $W^{-1,\mathfrak{p}}$ isomorphism property of the differential operator with transformed coefficient tensor $\boldsymbol{B}$ (defined by [1, (2.22)], which is still strongly elliptic due to [1, (2.25)]) on $\mathcal{C}$, and the isomorphism property on a cube follows immediately from Proposition 4.41.

Finally, we consider a point $\boldsymbol{x} \in \mathrm{Sing}^1_{\boldsymbol{\Omega}}$, where $\mathrm{Sing}^1_{\boldsymbol{\Omega}}$ is defined in Assumption 4.51.

1. If $\boldsymbol{x} \in \mathrm{Sing}^2_{\boldsymbol{\Omega}} \cup \mathrm{Sing}^3_{\boldsymbol{\Omega}}$, where $\mathrm{Sing}^2_{\boldsymbol{\Omega}}, \mathrm{Sing}^3_{\boldsymbol{\Omega}}$ are defined in Assumption 4.51, then due to the definition of $\mathrm{Sing}^2_{\boldsymbol{\Omega}}$ and $\mathrm{Sing}^3_{\boldsymbol{\Omega}}$ we know that there exists a neighborhood $\boldsymbol{U_x}$ such that the intersection $\boldsymbol{U_x} \cap \boldsymbol{\Omega}$ is C^∞-diffeomorph to an open cuboid. Then the claim follows from Proposition 4.41, by using the similar transformation arguments as the ones for the points on a face given previously.

2. If $\boldsymbol{x} \in \mathrm{Sing}_{\boldsymbol{\Omega}}$, where $\mathrm{Sing}_{\boldsymbol{\Omega}}$ is defined in Assumption 4.51, then due to (4.82), the operator $L^0_{\boldsymbol{P}}$ will reduce to the operator L defined in Lemma 4.58. Due to the Definition 4.47 of a polyhedral domain, we know that there exists a neighborhood $\boldsymbol{U_x}$ of $\boldsymbol{x}$ such that $\boldsymbol{U_x} \cap \boldsymbol{\Omega}$ is still a polyhedral domain with Lipschitz boundary. Then the claim follows from Lemma 4.58. This completes the proof of (4.84).

To finish the proof we will still need the following local regularity argument: for $\boldsymbol{P} \in \big(C(\bar{\boldsymbol{\Omega}})\big)^3, \boldsymbol{l} \in \big(W^{-1,\mathfrak{p}}(\boldsymbol{\Omega})\big)^4$ with $\mathfrak{p} \in [2,6]$, let $(\boldsymbol{u}, \phi)$ be the unique $W_0^{1,2}$-solution of

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi) = \boldsymbol{l} \quad \text{in } \boldsymbol{\Omega},$$

whose existence and uniqueness are guaranteed by Lax-Milgram. Consider a point $\boldsymbol{x} \in \bar{\boldsymbol{\Omega}}$. Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a real valued smooth function such that $\mathrm{supp}(\eta) \subset \boldsymbol{U_x}$, where $\boldsymbol{U_x}$ is as defined in (4.84). We also assume that $\mathfrak{p} \le p_{\boldsymbol{x}}$, where $p_{\boldsymbol{x}}$ is defined by (4.84). Define

$$\boldsymbol{w} := (\boldsymbol{u}, \phi),$$
$$\boldsymbol{v} := (\bar{\boldsymbol{u}}, \bar{\phi})$$

and rewriting $L_{\boldsymbol{P}}(\boldsymbol{u}, \phi)[\bar{\boldsymbol{u}}, \bar{\phi}]$ as

$$L_{\boldsymbol{P}}(\boldsymbol{u}, \phi)[\bar{\boldsymbol{u}}, \bar{\phi}] = \int_{\boldsymbol{\Omega}} \boldsymbol{A}^{\alpha\iota}_{ij} \partial_\iota \boldsymbol{w}^j \partial_\alpha \boldsymbol{v}^i d\boldsymbol{x} \tag{4.85}$$

with coefficient tensor $\boldsymbol{A}$, as described in Section 4.3.2. For $\boldsymbol{v} \in \big(C_0^\infty(\boldsymbol{U_x} \cap \boldsymbol{\Omega})\big)^4$ we obtain that

$$\begin{aligned}
&\int_{\boldsymbol{U_x}\cap\boldsymbol{\Omega}} \boldsymbol{A}^{\alpha\iota}_{ij} \partial_\iota (\eta \boldsymbol{w})^j \partial_\alpha \boldsymbol{v}^i d\boldsymbol{x} \\
=&\int_{\boldsymbol{U_x}\cap\boldsymbol{\Omega}} \boldsymbol{A}^{\alpha\iota}_{ij} (\partial_\iota \eta) \boldsymbol{w}^j \partial_\alpha \boldsymbol{v}^i + \boldsymbol{A}^{\alpha\iota}_{ij} \eta \partial_\iota \boldsymbol{w}^j \partial_\alpha \boldsymbol{v}^i d\boldsymbol{x} \\
=&\int_{\boldsymbol{U_x}\cap\boldsymbol{\Omega}} \underbrace{\boldsymbol{A}^{\alpha\iota}_{ij} (\partial_\iota \eta) \boldsymbol{w}^j \partial_\alpha \boldsymbol{v}^i}_{=:\boldsymbol{l}_1(\boldsymbol{v})} + \underbrace{\boldsymbol{A}^{\alpha\iota}_{ij} \partial_\iota \boldsymbol{w}^j \partial_\alpha (\eta \boldsymbol{v})^i}_{=:\boldsymbol{l}(\eta\boldsymbol{v})} + \underbrace{\Big(- \boldsymbol{A}^{\alpha\iota}_{ij} \partial_\iota \boldsymbol{w}^j (\partial_\alpha \eta) \boldsymbol{v}^i \Big)}_{=:\boldsymbol{l}_2(\boldsymbol{v})} d\boldsymbol{x} \\
=&\boldsymbol{l}(\eta \boldsymbol{v}) + \boldsymbol{l}_1(\boldsymbol{v}) + \boldsymbol{l}_2(\boldsymbol{v}) =: \tilde{\boldsymbol{l}}(\boldsymbol{v}),
\end{aligned} \tag{4.86}$$

where $\boldsymbol{l}(\eta\boldsymbol{v})$ is understood as $\boldsymbol{l}$ evaluated at the extension of $\eta\boldsymbol{v}$ on whole $\boldsymbol{\Omega}$ such that the extension is zero on $\boldsymbol{\Omega} \setminus \boldsymbol{U_x}$. From fundamental calculus one obtains the Sobolev's embedding relations

$$q \in [2,6] \Rightarrow 1 - \frac{3}{2} \geq 0 - \frac{3}{q},$$
$$q \in [2,6] \Rightarrow 0 - \frac{3}{2} \geq -1 - \frac{3}{q}.$$

From the Sobolev's embedding one then verifies that for $q \in [2,6]$, $\mathrm{Div}(\boldsymbol{u},\phi)$ and $\nabla(\boldsymbol{u},\phi)$ are of class $W^{-1,q}$ if $(\boldsymbol{u},\phi)$ is of class H^1. Thus $\boldsymbol{l}_1, \boldsymbol{l}_2$ are of class $W^{-1,q}$ for $q \in [2,6]$. Consequently we infer that $\tilde{\boldsymbol{l}}$ is of class $W^{-1,q}$ iff $\boldsymbol{l}$ is of class $W^{-1,q}$ for $q \in [2,6]$. Since $\boldsymbol{P}$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$, the coefficient tensor $\boldsymbol{A}$ is also uniformly continuous on $\overline{\boldsymbol{\Omega}}$. Letting $q = \mathfrak{p}$ (notice that $\mathfrak{p} \leq 6$), we then obtain from Hölder's inequality that

$$\|\tilde{\boldsymbol{l}}\|_{(W^{-1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4} \leq C(\|\boldsymbol{l}\|_{(W^{-1,\mathfrak{p}}(\boldsymbol{\Omega}))^4} + \|(\boldsymbol{u},\phi)\|_{(L^{\mathfrak{p}}(\boldsymbol{\Omega}))^4} + \|\nabla(\boldsymbol{u},\phi)\|_{(L^2(\boldsymbol{\Omega}))^4}). \quad (4.87)$$

Additionally we obtain that $\eta(\boldsymbol{u},\phi)$ is the $W_0^{1,2}$−solution of

$$L_{\boldsymbol{P}}\big(\eta(\boldsymbol{u},\phi)\big) = \tilde{\boldsymbol{l}} \quad \text{in } \boldsymbol{U_x} \cap \boldsymbol{\Omega}.$$

Define the tensor $\mathbb{B}_1(\boldsymbol{P})^*$ on $\boldsymbol{U_x} \cap \boldsymbol{\Omega}$ by

$$\mathbb{B}_1(\boldsymbol{P})^*(\boldsymbol{y}) := \begin{cases} \mathbb{B}_1\big(\boldsymbol{P}(\boldsymbol{x})\big), & \boldsymbol{y} \in (\boldsymbol{U_x} \cap \boldsymbol{\Omega}) \setminus B_{r_{\boldsymbol{x}}}(\boldsymbol{x}); \\ \mathbb{B}_1\big(\boldsymbol{P}(\boldsymbol{y})\big), & \boldsymbol{y} \in (\boldsymbol{U_x} \cap \boldsymbol{\Omega}) \cap B_{r_{\boldsymbol{x}}}(\boldsymbol{x}), \end{cases}$$

where $B_{r_{\boldsymbol{x}}}(\boldsymbol{x})$ with $\mathrm{cl}\big(B_{r_{\boldsymbol{x}}}(\boldsymbol{x})\big) \subset \boldsymbol{U_x}$ is the open ball with center $\boldsymbol{x}$ and radius $r_{\boldsymbol{x}}$ (to be determined). Denote by $L_{\boldsymbol{P}}^*$ the piezo-operator with the coefficient tensor $\mathbb{B}_1(\boldsymbol{P})^*$. Using Hölder's inequality, the fact that the coefficient tensor $\mathbb{B}_1$ is continuous on $\mathbb{R}^3$ from Assumption D2 and that $\boldsymbol{P}$ is uniformly continuous on the whole $\overline{\boldsymbol{\Omega}}$, we obtain that there exists sufficiently small $r_{\boldsymbol{x}} > 0$ such that

$$\begin{aligned} &\|L_{\boldsymbol{P}}^* - L_{\boldsymbol{P}}^0\|_{L((W_0^{1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4,(W^{-1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4)}\|(L_{\boldsymbol{P}}^0)^{-1}\|_{L((W^{-1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4,(W_0^{1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4)} \\ \leq &C\|\mathbb{B}_1\big(\boldsymbol{P}(\cdot)\big) - \mathbb{B}_1\big(\boldsymbol{P}(\boldsymbol{x})\big)\|_{L^\infty(B_{r_{\boldsymbol{x}}}(\boldsymbol{x})\cap\boldsymbol{\Omega})}\|(L_{\boldsymbol{P}_0}^0)^{-1}\|_{L((W^{-1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4,(W_0^{1,\mathfrak{p}}(\boldsymbol{U_x}\cap\boldsymbol{\Omega}))^4)} < 1. \end{aligned} \quad (4.88)$$

Here, we used the fact that $\mathbb{B}_1(\boldsymbol{P})$ is uniformly continuous on $\overline{\boldsymbol{\Omega}}$. Therefore using the small perturbation theorem [36, Chap. 4, Thm. 1.16] and (4.84) we infer that $L_{\boldsymbol{P}}^*$ is linear, continuous and invertible from $\big(W_0^{1,\mathfrak{p}}(\boldsymbol{U_x} \cap \boldsymbol{\Omega})\big)^4$ to $\big(W^{-1,\mathfrak{p}}(\boldsymbol{U_x} \cap \boldsymbol{\Omega})\big)^4$ for all $\mathfrak{p} \in [2, p_{\boldsymbol{x}}]$. If we additionally let $\mathrm{supp}(\eta) \subset B_{r_{\boldsymbol{x}}(\boldsymbol{x})}$, then we deduce that $\eta(\boldsymbol{u},\phi)$ is the $W_0^{1,2}$-solution of

$$L_{\boldsymbol{P}}^*\big(\eta(\boldsymbol{u},\phi)\big) = \tilde{\boldsymbol{l}} \quad \text{in } \boldsymbol{U_x} \cap \boldsymbol{\Omega}.$$

This shows that $\eta(\boldsymbol{u},\phi)$ is of class $W^{1,\mathfrak{p}}$ on $\boldsymbol{U_x} \cap \boldsymbol{\Omega}$ for all $\mathfrak{p} \in [2, \min\{p_{\boldsymbol{x}}, 6\}]$.

Now we are ready to finish the desired proof. Since $\boldsymbol{\Omega}$ is a bounded domain in $\mathbb{R}^3$, we can find some $s \in \mathbb{N}$ such that $\overline{\boldsymbol{\Omega}} \subset \cup_{i=1}^s B_{r_{\boldsymbol{x}_i}}(\boldsymbol{x}_i)$ with $\boldsymbol{x}_i \in \overline{\boldsymbol{\Omega}}$. Let $\boldsymbol{U}_{\boldsymbol{x}_i}, p_{\boldsymbol{x}_i}$ be defined as in the proof of (4.84) such that $\mathrm{cl}\big(B_{r_{\boldsymbol{x}_i}}(\boldsymbol{x}_i)\big) \subset \boldsymbol{U}_{\boldsymbol{x}_i}$. Define

$$p_* := \min\{p_{\boldsymbol{x}_1}, ..., p_{\boldsymbol{x}_s}, 6\} \in (3,6]$$

and let $\mathfrak{p} \in [2, p_*]$. Let $\{\eta_i\}_{i=1}^s$ be a partition of unity subordinated to $\{B_{r_{\boldsymbol{x}_i}}(\boldsymbol{x}_i)\}_{i=1}^s$. Then there is a corresponding $\tilde{\boldsymbol{l}}_i$ of class $W^{-1,\mathfrak{p}}$, similarly defined as in (4.86), such that $\eta_i(\boldsymbol{u},\phi)$ is the $W_0^{1,2}$-solution of

$$L_{\boldsymbol{P}}^*\big(\eta_i(\boldsymbol{u},\phi)\big) = \tilde{\boldsymbol{l}}_i \quad \text{in } \boldsymbol{U}_{\boldsymbol{x}_i} \cap \boldsymbol{\Omega}.$$

We obtain that

$$\begin{aligned}
&\|(\boldsymbol{u},\phi)\|_{(W_0^{1,\mathfrak{p}}(\boldsymbol{\Omega}))^4}\\
\leq&\sum_{i=1}^{s}\|\eta_i(\boldsymbol{u},\phi)\|_{(W_0^{1,\mathfrak{p}}(\boldsymbol{U}_{\boldsymbol{x}_i}\cap\boldsymbol{\Omega}))^4}\\
\leq&C\sum_{i=1}^{s}\|\tilde{\boldsymbol{l}}_i\|_{(W^{-1,\mathfrak{p}}(\boldsymbol{U}_{\boldsymbol{x}_i}\cap\boldsymbol{\Omega}))^4}\\
\leq&C(\|\boldsymbol{l}\|_{(W^{-1,\mathfrak{p}}(\boldsymbol{\Omega}))^4}+\|(\boldsymbol{u},\phi)\|_{(L^{\mathfrak{p}}(\boldsymbol{\Omega}))^4}+\|\nabla(\boldsymbol{u},\phi)\|_{(L^2(\boldsymbol{\Omega}))^4})<\infty.
\end{aligned}\tag{4.89}$$

This completes the proof of the main claim. □

Finally, to formulate the main result of this section given below we will need to give the definition of an admissible pair for the case here, which is similarly defined as in Definition 4.20 and 4.37:

Definiton 4.60. *A pair (p,r) is called admissible, if $p\in(3,p_*]$ and $r\in(p,\infty)$, where $p_*\in(3,6]$ is the number given by Proposition 4.59.*

4.4.5 Local existence result for polyhedral domains

Theorem 4.61. *Let the Assumptions D1 to D3 and Assumption 4.51 be satisfied and β,κ in (4.79) be given positive constants. Let (p,r) be an admissible pair in the sense of Definition 4.60. Let the Assumption D4 be satisfied with $p^*\in[p,\infty)$ and $r^*\in[r,\infty)$. Assume also that*

$$\boldsymbol{P}_0\in\big(\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3,\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3\big)_{\frac{1}{r},r}.$$

Then the differential system (4.79) has a unique local solution $(\boldsymbol{u},\phi,\boldsymbol{P})$ in the time interval $(0,\hat{T})$ for some $0<\hat{T}\leq T$ such that

$$\begin{aligned}
&\boldsymbol{u}\in L^{2r^*}(0,\hat{T};\big(W_0^{1,\hat{p}}(\boldsymbol{\Omega})\big)^3),\\
&\phi\in L^{2r^*}(0,\hat{T};W_0^{1,\hat{p}}(\boldsymbol{\Omega})),\\
&\boldsymbol{P}\in W^{1,r}(0,\hat{T};\big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3)\cap L^r(0,\hat{T};\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3),
\end{aligned}\tag{4.90}$$

where p_* is given by Proposition 4.59 and $\hat{p}:=\min\{p_*,p^*\}$.

Remark 4.62. Compared to Theorem 4.39 and Theorem 4.44, we point out that the integrability exponent of $(\boldsymbol{u},\phi)$ is $\hat{p}$ but not p^*. This is due to the lacking regularity from Proposition 4.59, while from Proposition 4.33 and Proposition 4.41 we see that the solution $(\boldsymbol{u},\phi)$ will have the same integrability exponent as the one the external forces have if the domain has C^1-boundary or the domain is a cuboid. △

Proof. In view of the proof of Theorem 4.39 and Theorem 4.44, we only need to show that Proposition 4.59 is applicable for all admissible functions $\tilde{\boldsymbol{P}}$. More precisely, since we will apply Proposition 4.59 to the differential piezo-operator $L_{\tilde{\boldsymbol{P}}(t)}$ with underlying function $\tilde{\boldsymbol{P}}(t)=\boldsymbol{P}+\boldsymbol{P}_D(t)$ with $\boldsymbol{P}\in(Y,X)_{\frac{1}{\tau},\tau}$ (where the space $(Y,X)_{\frac{1}{\tau},\tau}$ is being similarly defined as given previously, see e.g. the proof of Theorem 4.39), we need to show that for all $\boldsymbol{P}\in(Y,X)_{\frac{1}{\tau},\tau}$ and all $t\in[0,T]$ we have

$$\tilde{\boldsymbol{P}}(t)|_{\mathrm{Sing}_{\boldsymbol{\Omega}}}=\boldsymbol{0},\tag{4.91}$$

where (4.91) is being a prerequisite of Proposition 4.59. To show this, we first obtain from Assumption 4.51 that

$$\mathrm{Sing}_{\boldsymbol{\Omega}} \subset \partial\boldsymbol{\Omega}_{\boldsymbol{P}}.$$

But from Remark 4.12 we see that for $\boldsymbol{P} \in (Y, X)_{\frac{1}{\tau},\tau}$ we have

$$\boldsymbol{P}|_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}} = \mathbf{0}$$

and therefore

$$\boldsymbol{P}|_{\mathrm{Sing}_{\boldsymbol{\Omega}}} = \mathbf{0}. \tag{4.92}$$

Combining (4.92) and (4.80), we immediately obtain (4.91), which completes the desired proof. □

4.5 Global existence result based on the Rothe's method

In this section, we utilize the Rothe's method given in Chapter 3 to show global results. Throughout this section, the space dimension d is supposed to be an element of the set $\{2, 3\}$ and the dissipation potential Ψ_β is defined by

$$\Psi_\beta(\boldsymbol{P}) = \frac{\beta}{2}\|\boldsymbol{P}\|_{L^2}^2.$$

Also, the gradient energy replacement Assumption 3.1 for a given $s \in [\max\{1, \frac{d}{2}\}, 2)$ is kept for this section. We formulate our main problem: Find $\boldsymbol{P} : (0,T) \to \big(H^s(\boldsymbol{\Omega})\big)^d$ such that

$$\mathbf{0} \in D_{\boldsymbol{P}}\mathcal{I}\big(t, \boldsymbol{P}(t)\big) + \partial\Psi_\beta\big(\boldsymbol{P}'(t)\big),\ \boldsymbol{P}(0) = \boldsymbol{P}_0 \tag{4.93}$$

for a.a. $t \in (0,T)$, where $\mathcal{I}$ is as defined in (3.32).

Theorem 4.63. *Let the Assumptions A1 to A6 be satisfied. Suppose also that* $\boldsymbol{P}_0 \in \big(H^s(\boldsymbol{\Omega})\big)^d$ *and* $D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)$ *is of class* L^2*. Then for every* $\beta > 0$ *the differential system* (4.93) *admits a solution* $\boldsymbol{P} \in H^1(0,T; \big(H^s(\boldsymbol{\Omega})\big)^d)$.

Proof. We only need to clarify that the proof of Theorem 3.30 also works for the current case (namely setting $\Psi_1 = 0$ in Theorem 3.30) without using the Assumption A7. We recall the explicit statement given in the Assumption A7: $\Psi_1 : \big(H^s(\boldsymbol{\Omega})\big)^d \to [0,\infty)$ is assumed to be convex, positively 1-homogeneous, weakly lower semi-continuous in $\big(H^s(\boldsymbol{\Omega})\big)^d$ and there exist $d_1, d_2 > 0$ such that for all $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$

$$d_1\|\boldsymbol{P}\|_{L^1} \leq \Psi_1(\boldsymbol{P}) \leq d_2\|\boldsymbol{P}\|_{L^1}. \tag{4.94}$$

Notice that $\Psi_1 = 0$ is still non negative, convex, positively 1-homogeneous and weakly lower semi-continuous in $\big(H^s(\boldsymbol{\Omega})\big)^d$, but (4.94) does not hold anymore. Having a look at the proof of Theorem 3.30, the Assumption A7 is used for Lemmas 3.27, 3.28, 3.29 and B.1. In Lemmas 3.27 and 3.28, only the non negativity and weak lower semi-continuity of Ψ_1 have been used; in Lemma 3.29, only the positive 1-homogeneity of Ψ_1 has been used; Lemma B.1 is to guarantee that the equation (3.82) is satisfied, namely

$$\int_s^t \Psi_1\big(\boldsymbol{P}'(\sigma)\big)d\sigma \leq \lim_{\tau\to 0}\int_s^t \Psi_1\big((\hat{\boldsymbol{P}}_\tau^2)'(\sigma)\big)d\sigma.$$

But this trivially holds for $\Psi_1 = 0$. Thus the proof for Theorem 3.30 also works for the problem (4.93) and we obtain the desired result. □

Corresponding to the solution $\boldsymbol{P}$, we are able to obtain a solution $(\boldsymbol{u}, \phi, \boldsymbol{D})$ given from (3.2) and (3.20). Similarly as stated in Proposition 3.32, we obtain the following regularity result for $(\boldsymbol{u}, \phi, \boldsymbol{D})$:

Proposition 4.64. *Let the Assumptions A1 to A6 be satisfied and* $\beta > 0$ *be a given positive constant. Suppose also that* $\boldsymbol{P}_0 \in \left(H^s(\boldsymbol{\Omega})\right)^d$ *and* $D_{\boldsymbol{P}}\mathcal{I}(0, \boldsymbol{P}_0)$ *is of class* L^2*. Let* $\boldsymbol{P}$ *be the solution of* (4.93) *obtained from Theorem 4.63. Then the differential system* (3.2) *and* (3.20) *admit a solution* $(\boldsymbol{u}, \phi, \boldsymbol{P})$ *and* $(\boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ *(*$\boldsymbol{u}$ *being identical in former and latter) respectively such that*

$$\begin{aligned} \boldsymbol{u} &\in H^1(0,T; \left(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\right)^d), \\ \phi &\in H^1(0,T; W^{1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega})), \\ \boldsymbol{D} &\in H^1(0,T; M_{\boldsymbol{D}} \cap \left(L^q(\boldsymbol{\Omega})\right)^d), \\ \boldsymbol{P} &\in H^1(0,T; \left(H^s(\boldsymbol{\Omega})\right)^d), \end{aligned} \tag{4.95}$$

where $M_{\boldsymbol{D}}$ *is the space defined by* (3.18) *and* $q \in (2, \infty)$ *is the number given in Lemma 3.19.*

Proof. The proof is being identical as the proof of Proposition 3.32 and we thus omit the details here. □

Chapter 5

Summary and comparison of the main results

We close the thesis by giving the Table 5.1 in the following, which gives a comparison of the main results given in the thesis for different model settings. Some supplementary explanation is made for Table 5.1 as follows:

- The dimension number d is always a number in the set $\{2,3\}$. The domain $\boldsymbol{\Omega} \subset \mathbb{R}^d$ is always assumed to be a bounded domain with Lipschitz boundary.
- $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}$, $\partial\boldsymbol{\Omega}_{\phi}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ and $\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{D}}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}}$ correspond to the Dirichlet boundary and Neumann boundary of $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$ respectively. $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}$, $\partial\boldsymbol{\Omega}_{\phi}$, $\partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ are always $(d-1)$-sets (see Section 2.1). In particular, we assume that $\partial\boldsymbol{\Omega}_{\boldsymbol{u}}\dot{\cup}\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}} = \partial\boldsymbol{\Omega}_{\phi}\dot{\cup}\partial\boldsymbol{\Omega}_{\boldsymbol{D}} = \partial\boldsymbol{\Omega}_{\boldsymbol{P}}\dot{\cup}\partial\boldsymbol{\Omega}_{\boldsymbol{\Sigma}} = \partial\boldsymbol{\Omega}$
- For the definition of the G1-regular and G2-regular sets, we refer to Section 2.1.
- The replacement of gradient energy is referred to Assumption 3.1.
- The functional Ψ_1 is always assumed to satisfy the Assumption A7.
- For the definitions of the coefficient tensors and external loadings, we refer to Section 2.4.
- Due to the relation of $\boldsymbol{D}$ and $\nabla\phi$ given by (3.23) we will only deal with the variable ϕ within the Table 5.1.
- For the construction of the vanishing viscosity solution $(\tilde{t}, \tilde{\boldsymbol{u}}, \tilde{\phi}, \tilde{\boldsymbol{P}})$, we refer to Section 3.8.

Table 5.1: Comparison of the main results for different models

	Model with the dissipation functional $\Psi(\boldsymbol{P}) = \Psi_1(\boldsymbol{P}) + \beta\|\boldsymbol{P}\|^2_{L^2(\boldsymbol{\Omega})}$	**Model with the dissipation functional** $\Psi(\boldsymbol{P}) = \frac{\beta}{2}\|\boldsymbol{P}\|^2_{L^2(\boldsymbol{\Omega})}$		
Local or Global solution	Global	Local, $d=2$	Local, $d=3$	Global
Dirichlet or Neumann or Mixed b.c.	Mixed for $\boldsymbol{u}$, ϕ Neumann for $\boldsymbol{P}$	Mixed for $\boldsymbol{u}$, ϕ, $\boldsymbol{P}$	Dirichlet for $\boldsymbol{u}$, ϕ Mixed for $\boldsymbol{P}$	Mixed for $\boldsymbol{u}$, ϕ Neumann for $\boldsymbol{P}$
Regularity and boundedness of the coefficient tensors	$\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ differentiable uniformly bounded uniformly Lipschitzian $D_{\boldsymbol{P}}\mathbb{C}, D_{\boldsymbol{P}}\boldsymbol{e}, D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0, D_{\boldsymbol{P}}\boldsymbol{\epsilon}$ uniformly bounded uniformly Lipschitzian ω is a polynomial of sixth order and $\exists C>0,\ C' \in \mathbb{R} : \omega(\boldsymbol{P}) \geq C\|\boldsymbol{P}\|^2 + C'$	$\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ differentiable uniformly bounded locally Lipschitzian $D_{\boldsymbol{P}}\mathbb{C}, D_{\boldsymbol{P}}\boldsymbol{e}, D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0, D_{\boldsymbol{P}}\boldsymbol{\epsilon}$ locally Lipschitzian ω is a polynomial of sixth order	$\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ differentiable locally Lipschitzian $D_{\boldsymbol{P}}\mathbb{C}, D_{\boldsymbol{P}}\boldsymbol{e}, D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0, D_{\boldsymbol{P}}\boldsymbol{\epsilon}$ locally Lipschitzian ω is a polynomial of sixth order	$\mathbb{C}, \boldsymbol{e}, \boldsymbol{\varepsilon}^0, \boldsymbol{\epsilon}$ differentiable uniformly bounded uniformly Lipschitzian $D_{\boldsymbol{P}}\mathbb{C}, D_{\boldsymbol{P}}\boldsymbol{e}, D_{\boldsymbol{P}}\boldsymbol{\varepsilon}^0, D_{\boldsymbol{P}}\boldsymbol{\epsilon}$ uniformly bounded uniformly Lipschitzian ω is a polynomial of sixth order and $\exists C>0,\ C' \in \mathbb{R} : \omega(\boldsymbol{P}) \geq C\|\boldsymbol{P}\|^2 + C'$
Characterization of domain and boundaries	$\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{u}}$ G1-regular $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\phi}$ G1-regular	$\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{u}}$ G1-regular $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\phi}$ G1-regular $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ G2-regular	$\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{P}}$ G2-regular; $\boldsymbol{\Omega}$ has C^1-boundary or is a cuboid or $\boldsymbol{\Omega}$ is a polyhedral domain and $\boldsymbol{\Omega}$ and $\boldsymbol{P}_D$ satisfy Assumption 4.51	$\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\boldsymbol{u}}$ G1-regular $\boldsymbol{\Omega} \cup \partial\boldsymbol{\Omega}_{\phi}$ G1-regular
Replacement of gradient	Both are possible if $d=2$ Yes if $d=3$	No	No	Both are possible if $d=2$ Yes if $d=3$
Admissible pair (p,r)	Not relevant	$p \in (2, p_*]$, $r \in (\frac{2p}{p-2}, \infty)$ p_* is given by Lemma 4.19	$p \in (3, p_*]$, $r \in (\frac{2p}{p-3}, \infty)$ $p_* = 6$ if $\boldsymbol{\Omega}$ has C^1-boundary or $\boldsymbol{\Omega}$ is a cuboid; p_* is given by Proposition 4.59 if $\boldsymbol{\Omega}$ is a polyhedral domain and $\boldsymbol{\Omega}$ and $\boldsymbol{P}_D$ satisfy Assumption 4.51	Not relevant
Initial value $\boldsymbol{P}_0$	$\boldsymbol{P}_0 \in \big(H^s(\boldsymbol{\Omega})\big)^d \wedge D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0) \in \big(L^2(\boldsymbol{\Omega})\big)^d$ Here, $s \in [\max\{1, \frac{d}{2}\}, 2)$	$\boldsymbol{P}_0 \in \big(\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2, \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2\big)_{\frac{1}{r},r}$	$\boldsymbol{P}_0 \in \big(\big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3, \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3\big)_{\frac{1}{r},r}$	$\boldsymbol{P}_0 \in \big(H^s(\boldsymbol{\Omega})\big)^d \wedge D_{\boldsymbol{P}}\mathcal{I}(0,\boldsymbol{P}_0) \in \big(L^2(\boldsymbol{\Omega})\big)^d$ Here, $s \in [\max\{1, \frac{d}{2}\}, 2)$
Regularity of external loadings	$\boldsymbol{f}_1 \in C^{1,1}([0,T]; \big(L^{\frac{p^*d}{p^*+d}}(\boldsymbol{\Omega})\big)^d)$ $\boldsymbol{t} \in C^{1,1}([0,T]; \big(L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}})\big)^d)$ $f_2 \in C^{1,1}([0,T];, L^{\frac{p^*d}{p^*+d}}(\boldsymbol{\Omega}))$ $\rho \in C^{1,1}([0,T]; L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{D}}))$ $\boldsymbol{u}_D \in C^{1,1}([0,T]; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^d)$ $\phi_D \in C^{1,1}([0,T]; B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\phi}))$ $\boldsymbol{f}_3 \in C^{1,1}([0,T]; \big(L^{q^*}(\boldsymbol{\Omega})\big)^d)$ $\boldsymbol{\pi} \in C^{1,1}([0,T]; \big(L^{q^*}(\partial\boldsymbol{\Omega})\big)^d)$ Here, $p^* \in (2,\infty)$ and $q^* \in (1,\infty)$	$\boldsymbol{f}_1 \in L^{2r^*}\big(0,T; \big(L^{\frac{2p^*}{p^*+2}}(\boldsymbol{\Omega})\big)^2\big)$ $f_2 \in L^{2r^*}\big(0,T; L^{\frac{2p^*}{p^*+2}}(\boldsymbol{\Omega})\big)$ $\boldsymbol{f}_3 \in L^{r^*}\big(0,T; \big(L^{\frac{2p^*}{p^*+2}}(\boldsymbol{\Omega})\big)^2\big)$ $\boldsymbol{t} \in L^{2r^*}\big(0,T; \big(L^{\frac{p^*}{2}}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^2\big)$ $\rho \in L^{2r^*}\big(0,T; L^{\frac{p^*}{2}}(\partial\boldsymbol{\Omega}_{\boldsymbol{D}})\big)$ $\boldsymbol{\pi} \in L^{r^*}\big(0,T; \big(L^{\frac{p^*}{2}}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^2\big)$ $\boldsymbol{u}_D \in L^{2r^*}\big(0,T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^2\big)$ $\phi_D \in L^{2r^*}\big(0,T; B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\phi})\big)$ $\boldsymbol{P}_D \in W^{1,r^*}\big(0,T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^2\big)$ Here, $p^* \in [p,\infty)$ and $r^* \in [r,\infty)$	$\boldsymbol{f}_1 \in L^{2r^*}\big(0,T; \big(L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)^3\big)$ $f_2 \in L^{2r^*}\big(0,T; L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)$ $\boldsymbol{f}_3 \in L^{r^*}\big(0,T; \big(L^{\frac{3p^*}{p^*+3}}(\boldsymbol{\Omega})\big)^3\big)$ $\boldsymbol{\pi} \in L^{r^*}\big(0,T; \big(L^{\frac{2p^*}{3}}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^3\big)$ $\boldsymbol{u}_D \in L^{2r^*}\big(0,T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^3\big)$ $\phi_D \in L^{2r^*}\big(0,T; B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\phi})\big)$ $\boldsymbol{P}_D \in W^{1,r^*}\big(0,T; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{P}})\big)^3\big)$ Here, $p^* \in [p,\infty)$ and $r^* \in [r,\infty)$	$\boldsymbol{f}_1 \in C^{1,1}([0,T]; \big(L^{\frac{p^*d}{p^*+d}}(\boldsymbol{\Omega})\big)^d)$ $\boldsymbol{t} \in C^{1,1}([0,T]; \big(L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{\sigma}})\big)^d)$ $f_2 \in C^{1,1}([0,T];, L^{\frac{p^*d}{p^*+d}}(\boldsymbol{\Omega}))$ $\rho \in C^{1,1}([0,T]; L^{\frac{p^*(d-1)}{d}}(\partial\boldsymbol{\Omega}_{\boldsymbol{D}}))$ $\boldsymbol{u}_D \in C^{1,1}([0,T]; \big(B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\boldsymbol{u}})\big)^d)$ $\phi_D \in C^{1,1}([0,T]; B^{1-\frac{1}{p^*}}_{p^*,p^*}(\partial\boldsymbol{\Omega}_{\phi}))$ $\boldsymbol{f}_3 \in C^{1,1}([0,T]; \big(L^{q^*}(\boldsymbol{\Omega})\big)^d)$ $\boldsymbol{\pi} \in C^{1,1}([0,T]; \big(L^{q^*}(\partial\boldsymbol{\Omega})\big)^d)$ Here, $p^* \in (2,\infty)$ and $q^* \in (1,\infty)$
Regularity of solution $(\boldsymbol{u}, \phi, \boldsymbol{P})$	From Proposition 3.32: $\boldsymbol{u} \in H^1(0,T; \big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d)$ $\phi \in H^1(0,T; W^{1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}))$ $\boldsymbol{P} \in H^1(0,T; \big(H^s(\boldsymbol{\Omega})\big)^d)$ Here, $q \in (2,\infty)$ is given by Lemma 3.19	From Theorem 4.22: $\boldsymbol{u} \in L^{2r^*}(0,\hat{T}; \big(W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2)$ $\phi \in L^{2r^*}(0,\hat{T}; W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}))$ $\boldsymbol{P} \in W^{1,r}(0,\hat{T}; X) \cap L^r(0,\hat{T}; Y)$ Here, $\hat{T} \in (0,T]$, $X = \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, $Y = \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^2$, $\hat{p} = \{p^*, p_*\}$	From see Theorem 4.39, 4.44 and 4.61: If $\boldsymbol{\Omega}$ has C^1-boundary or $\boldsymbol{\Omega}$ is a cuboid, then $\boldsymbol{u} \in L^{2r^*}(0,\hat{T}; \big(W^{1,p^*}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2)$ $\phi \in L^{2r^*}(0,\hat{T}; W^{1,p^*}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}))$ $\boldsymbol{P} \in W^{1,r}(0,\hat{T}; X) \cap L^r(0,\hat{T}; Y)$; If $\boldsymbol{\Omega}$ is a polyhedral domain and $\boldsymbol{\Omega}$ and $\boldsymbol{P}_D$ satisfy Assumption 4.51, then $\boldsymbol{u} \in L^{2r^*}(0,\hat{T}; \big(W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^2)$ $\phi \in L^{2r^*}(0,\hat{T}; W^{1,\hat{p}}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}))$ $\boldsymbol{P} \in W^{1,r}(0,\hat{T}; X) \cap L^r(0,\hat{T}; Y)$ Here, $\hat{T} \in (0,T]$, $X = \big(W^{-1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3$, $Y = \big(W^{1,p}_{\partial\boldsymbol{\Omega}_{\boldsymbol{P}}}(\boldsymbol{\Omega})\big)^3$, $\hat{p} = \{p^*, p_*\}$	From Theorem 4.63: $\boldsymbol{u} \in H^1(0,T; \big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d)$ $\phi \in H^1(0,T; W^{1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}))$ $\boldsymbol{P} \in H^1(0,T; \big(H^s(\boldsymbol{\Omega})\big)^d)$ Here, $q \in (2,\infty)$ is given by Lemma 3.19
Uniqueness of solution $(\boldsymbol{u}, \phi, \boldsymbol{P})$	Unknown	Yes	Yes	Unknown
Behavior as $\beta \to 0$	From Proposition 3.39: Vanishing viscosity solution $(\tilde{t}, \tilde{\boldsymbol{u}}, \tilde{\phi}, \tilde{\boldsymbol{P}})$ exists, with $\tilde{t} \in W^{1,\infty}(0,S;[0,T])$ $\tilde{\boldsymbol{u}} \in W^{1,\infty}(0,S; \big(W^{1,q}_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d)$ $\tilde{\phi} \in W^{1,\infty}(0,S; W^{1,q}_{\partial\boldsymbol{\Omega}_{\phi}}(\boldsymbol{\Omega}))$ $\tilde{\boldsymbol{P}} \in W^{1,\infty}(0,S; \big(H^s(\boldsymbol{\Omega})\big)^d))$ Here, S is some positive number with $S \in [T,\infty)$ and q is given by Lemma 3.19	Not relevant	Not relevant	Not relevant

Appendix A

Basic calculus

A.1 Inequalities

Lemma A.1 (Poincaré's inequality). *Let* $\mathbf{\Omega} \subset \mathbb{R}^d$*,* $d \in \{2,3\}$ *be a bounded domain with Lipschitz boundary and let* $\mathbf{\Gamma} \subset \partial\mathbf{\Omega}$ *be a* $(d-1)$*-set with positive surface measure. Then there exists a constant* $c_P > 0$ *such that for all* $\phi \in H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})$

$$\|\phi\|_{H^1} \leq c_P \|\nabla\phi\|_{L^2}.$$

Lemma A.2 (Korn's inequality). *Let* $\mathbf{\Omega} \subset \mathbb{R}^d$*,* $d \in \{2,3\}$ *be a bounded domain with Lipschitz boundary and let* $\mathbf{\Gamma} \subset \partial\mathbf{\Omega}$ *be a* $(d-1)$*-set with positive surface measure. Then there exists a constant* $c_K > 0$ *such that for all* $\boldsymbol{u} \in (H^1_{\mathbf{\Gamma}}(\mathbf{\Omega}))^d$

$$\|\boldsymbol{u}\|_{H^1} \leq c_K \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2}.$$

To show these, we need the following lemmas:

Lemma A.3. *Let* $\mathbf{\Omega} \subset \mathbb{R}^d$*,* $d \in \{2,3\}$ *be a bounded domain with Lipschitz boundary.*

- *Let* $V \subset H^1(\mathbf{\Omega})$ *be a closed subset such that*

$$\phi \in V, \nabla\phi = \mathbf{0} \Rightarrow \phi = 0. \tag{A.1}$$

Then there exists some $c_1 > 0$ *such that for all* $\phi \in V$

$$\|\nabla\phi\|_{L^2} \geq c_1 \|\phi\|_{L^2}. \tag{A.2}$$

- *Let* $V \subset \left(H^1(\mathbf{\Omega})\right)^d$ *be a closed subset such that*

$$\boldsymbol{u} \in V, \boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{0} \Rightarrow \boldsymbol{u} = \mathbf{0}.$$

Then there exists some $c_2 > 0$ *such that for all* $\boldsymbol{u} \in V$

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2} \geq c_2 \|\boldsymbol{u}\|_{L^2}.$$

Proof. The proof is based on a contradiction proof and embedding theorems. We refer to [18, Chap. 5.8, Thm. 1] and [20, Prop. 3] for details. We also refer to [37, Lem 1.11] for a complete proof. □

Lemma A.4. *Let $\mathbf{\Omega} \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded domain with Lipschitz boundary and let $\mathbf{\Gamma} \subset \partial\mathbf{\Omega}$ be a $(d-1)$-set with positive surface measure. Then the spaces $V_1 = H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})$ and $V_2 = (H^1_{\mathbf{\Gamma}}(\mathbf{\Omega}))^d$ fulfill the conditions* (A.1) *and* (A.2) *respectively.*

Proof. That the space $H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})$ is a closed subspace of $H^1(\mathbf{\Omega})$ is a direct consequence of the trace theorem, see for instance the proof of [9, Thm. 6.3-4]. Now let $\phi \in H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})$ with $\nabla\phi = \mathbf{0}$. For $\varepsilon > 0$, let η_ε be the standard mollifier having support in the closed ball

$$\bar{B}_\varepsilon(\mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \leq \varepsilon\}.$$

Let $\boldsymbol{x} \in \mathbf{\Omega}$ be an arbitrary point in $\mathbf{\Omega}$. Choose $\xi \in (0,\infty)$ sufficiently small such that the closed ball $\bar{B}_\xi(\boldsymbol{x})$ centered at $\boldsymbol{x}$ with radius ξ lies in $\mathbf{\Omega}$, i.e. $\bar{B}_\xi(\boldsymbol{x}) \subset \mathbf{\Omega}$. Then for all sufficiently small ε we have $\phi * \eta_\varepsilon \in C^\infty\big(B_\xi(\boldsymbol{x})\big)$ and

$$\nabla(\phi * \eta_\varepsilon) = (\nabla\phi) * \eta_\varepsilon = \mathbf{0},$$

whence $\phi * \eta_\varepsilon$ is constant in $B_\xi(\boldsymbol{x})$. Now let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a (sufficiently small) vanishing sequence, i.e. $\varepsilon_n \to 0$ as $n \to \infty$. Then $\phi * \eta_{\varepsilon_n}$ converges to ϕ in $L^2\big(B_\xi(\boldsymbol{x})\big)$ as $n \to \infty$. Thus up to a subsequence, $\phi * \eta_{\varepsilon_n}$ converges to ϕ a.e. in $B_\xi(\boldsymbol{x})$ as $n \to \infty$. But $\phi * \eta_{\varepsilon_n}$ is constant in $B_\xi(\boldsymbol{x})$, therefore ϕ is a.e. constant in $B_\xi(\boldsymbol{x})$ and we conclude that ϕ is locally a.e. constant in $\mathbf{\Omega}$. Since $\mathbf{\Omega}$ is connected, this implies that ϕ is a.e. constant in $\mathbf{\Omega}$, see for instance [7, Lem. 6.3]. Since $\mathbf{\Omega}$ has Lipschitz boundary, the trace of ϕ is well-defined, and since $\mathbf{\Gamma}$ has positive measure, if ϕ is not constantly equal to zero a.e. in $\mathbf{\Omega}$, then ϕ is constantly equal to some non zero number a.e. on $\mathbf{\Gamma}$ (w.r.t. the corresponding Hausdorff measure on $\mathbf{\Gamma}$), which is a contradiction to the fact that $\phi \in H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})$. Thus $\phi \equiv 0$ a.e. in $\mathbf{\Omega}$ and $H^1_{\mathbf{\Gamma}}(\mathbf{\Omega})$ satisfies the property (A.1). That the condition (A.2) fulfills follows directly from the proof of [9, Thm. 6.3-4]. ☐

Lemma A.5 ([45, Thm. 10.2])**.** *Let $\mathbf{\Omega}$ be a bounded domain with Lipschitz boundary. Then there exists some $c > 0$ such that for all $\boldsymbol{u} \in \big(H^1(\mathbf{\Omega})\big)^d$*

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2} + \|\boldsymbol{u}\|_{L^2} \geq c\|\nabla\boldsymbol{u}\|_{L^2}.$$

Proof of Lemma A.1 and A.2. The Poincaré's inequality follows directly from Lemma A.3 and A.4. Now we obtain that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2} \geq c\big(\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^2} + \|\boldsymbol{u}\|_{L^2}\big) \geq c\|\nabla\boldsymbol{u}\|_{L^2} \geq c\|\boldsymbol{u}\|_{H^1},$$

where the first inequality follows from Lemma A.3, the second from Lemma A.5 and the last one is the Poincaré's inequality. ☐

Lemma A.6. *Let $d \in \mathbb{N}$ and let the Assumptions A2 and A3 be satisfied. Then there exists a constant $\mu > 0$ such that*

$$\mathbb{B}_2(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{D}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{D}\end{pmatrix} \geq \mu\big(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{D}|^2\big)$$

for all $\boldsymbol{P} \in \mathbb{R}^d, \boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d,\mathbb{R}^d), \boldsymbol{D} \in \mathbb{R}^d$.

Proof. It follows from Assumption A3 that

$$\begin{aligned}&\mathbb{B}_2(\boldsymbol{P})\begin{pmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{D}\end{pmatrix} : \begin{pmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{D}\end{pmatrix}\\ =&\mathbb{C}(\boldsymbol{P})\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \boldsymbol{\epsilon}^{-1}(\boldsymbol{P})\big(\boldsymbol{D} - \boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}\big) : \big(\boldsymbol{D} - \boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}\big)\\ \geq&\alpha\big(|\varepsilon|^2 + |\boldsymbol{D} - \boldsymbol{e}(\boldsymbol{P})\varepsilon|^2\big)\end{aligned}$$

for $\boldsymbol{\varepsilon} \in \mathrm{Lin}_{\mathrm{sym}}(\mathbb{R}^d, \mathbb{R}^d)$ and $\boldsymbol{D} \in \mathbb{R}^d$. If $\boldsymbol{\varepsilon} = 0$, the claim follows; otherwise we write $|\boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}|^2 = \nu|\boldsymbol{\varepsilon}|^2$, then

$$\nu := \frac{|\boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}|^2}{|\boldsymbol{\varepsilon}|^2} \leq C_1$$

for some positive constant C_1, since $\boldsymbol{e}$ is uniformly bounded on $\mathbb{R}^3$ due to Assumption A3. It follows

$$\begin{aligned}
&\mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} \\
\geq&\alpha\big(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{D} - \boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}|^2\big) \\
\geq&\alpha\big(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{D}|^2 + |\boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}|^2 - 2|\boldsymbol{D}||\boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}|\big) \\
\geq&\alpha\Big(\big(1 + \nu(1 - \frac{1}{\nu'})\big)|\boldsymbol{\varepsilon}|^2 + (1 - \nu')|\boldsymbol{D}|^2\Big)
\end{aligned}$$

for all $\nu' > 0$, where the last inequality follows from Young's inequality, applied to the term $2|\boldsymbol{D}||\boldsymbol{e}(\boldsymbol{P})\boldsymbol{\varepsilon}|$. Let

$$1 + \nu(1 - \frac{1}{\nu'}) > 0 \wedge 1 - \nu' > 0,$$

that is

$$1 > \nu' > \frac{\nu}{\nu + 1}.$$

Since $\frac{\nu}{\nu+1}$ is monotone increasing in ν and $\nu \leq C_1$, we can take ν' as

$$\nu' := \frac{1}{2}(1 + \frac{C_1}{C_1 + 1}) = \frac{2C_1 + 1}{2C_1 + 2}.$$

Now it follows

$$\begin{aligned}
&\mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} \\
\geq&\alpha\Big(\big(1 + \nu(1 - \frac{1}{\nu'})\big)|\boldsymbol{\varepsilon}|^2 + (1 - \nu')|\boldsymbol{D}|^2\Big) \\
\geq&\alpha\Big(\big(1 - \frac{\nu}{2C_1 + 1}\big)|\boldsymbol{\varepsilon}|^2 + \frac{|\boldsymbol{D}|^2}{2C_1 + 2}\Big) \\
\geq&\alpha\Big(\big(1 - \frac{C_1}{2C_1 + 1}\big)|\boldsymbol{\varepsilon}|^2 + \frac{|\boldsymbol{D}|^2}{2C_1 + 2}\Big) \\
=&\alpha\Big(\frac{C_1 + 1}{2C_1 + 1}|\boldsymbol{\varepsilon}|^2 + \frac{|\boldsymbol{D}|^2}{2C_1 + 2}\Big).
\end{aligned}$$

Take

$$\mu := \min\big\{\alpha, \alpha\min\{\frac{C_1 + 1}{2C_1 + 1}, \frac{1}{2C_1 + 2}\}\big\} > 0,$$

we obtain that

$$\mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{D} \end{pmatrix} \geq \mu\big(|\boldsymbol{\varepsilon}|^2 + |\boldsymbol{D}|^2\big)$$

and the claim follows. □

A.2 Differentiability and integrability of functionals

Lemma A.7. *Let $d \in \{2,3\}$ and let the Assumptions A1 to A6 be satisfied. Let $s \in [\max\{1, \frac{d}{2}\}, 2)$ be some given number. Let $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ be given by (3.14). Then for each $t \in [0,T]$ and $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, $\mathcal{E}\big(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}\big)$ is Gâteaux-differentiable w.r.t. $(\boldsymbol{u}, \boldsymbol{D})$ on $\big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$.*

Proof. Recall that

$$\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) = \mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) + \mathcal{E}_2(\boldsymbol{P}) - l_3(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}),$$

where $\mathcal{E}_1$, $\mathcal{E}_2$, l_3 are defined by (3.15) to (3.17). The differentiability of l_3 and $\mathcal{E}_2$ w.r.t. $(\boldsymbol{u}, \boldsymbol{D})$ on $\big(H^1_{\partial\boldsymbol{\Omega}_{\boldsymbol{u}}}(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$ are being trivial, thus we only need to consider the term $\mathcal{E}_1$. Now recall that

$$\begin{aligned}\mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}) &= \int_{\boldsymbol{\Omega}} \frac{1}{2}\mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - \boldsymbol{P} \end{pmatrix} d\boldsymbol{x} \\ &= \int_{\boldsymbol{\Omega}} U_1\big(t, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{D}, \boldsymbol{P}\big) d\boldsymbol{x}.\end{aligned}$$

Define

$$\mathfrak{B}_{\boldsymbol{P}}(\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{D} \end{pmatrix}, \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{E} \end{pmatrix}) := \int_{\boldsymbol{\Omega}} \mathbb{B}_2(\boldsymbol{P}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) \\ \boldsymbol{D} \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{v}) \\ \boldsymbol{E} \end{pmatrix} d\boldsymbol{x} \tag{A.3}$$

for $(\boldsymbol{u}, \boldsymbol{D}) \in \big(H^1(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$, where $M_{\boldsymbol{D}}$ is the space defined by (3.18). Then $\mathfrak{B}_{\boldsymbol{P}}$ defines a continuous bilinear form on the space $\big(H^1(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$ due to the Assumptions A1 to A3. We then obtain the Gâteaux-differentiability of the quadratic functional

$$\mathfrak{B}_{\boldsymbol{P}}(\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{D} \end{pmatrix}, \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{D} \end{pmatrix})$$

on $\big(H^1(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$. Since $U_1\big(t, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{D}, \boldsymbol{P}\big)$ is a multiple of the integrand given in (A.3) with $\boldsymbol{u} = \boldsymbol{v}$ and $\boldsymbol{D} = \boldsymbol{E}$ (and up to constant translation for fixed $(t, \boldsymbol{P})$), the Gâteaux-differentiability of $\mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ w.r.t. $(\boldsymbol{u}, \boldsymbol{D})$ follows immediately from the Gâteaux-differentiability of the quadratic functional $\mathfrak{B}_{\boldsymbol{P}}(\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{D} \end{pmatrix}, \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{D} \end{pmatrix})$ w.r.t. $(\boldsymbol{u}, \boldsymbol{D})$ on $\big(H^1(\boldsymbol{\Omega})\big)^d \times M_{\boldsymbol{D}}$. □

Lemma A.8. *Let $d \in \{2,3\}$ and let the Assumptions A1 to A6 be satisfied. Let $s \in [\max\{1, \frac{d}{2}\}, 2)$ be some given number. Let $\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P})$ and $\mathcal{E}(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P})$ be defined by (3.5) and (3.14) respectively. Then*

(1) For each $p > 2$ and each $(t, \boldsymbol{u}, \phi) \in [0,T] \times \big(W^{1,p}(\boldsymbol{\Omega})\big)^d \times W^{1,p}(\boldsymbol{\Omega})$, $\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P})$ is Gâteaux-differentiable w.r.t. $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and

$$\begin{aligned}&D_{\boldsymbol{P}}\mathcal{H}(t, \boldsymbol{u}, \phi, \boldsymbol{P})[\bar{\boldsymbol{P}}] \\ &= \int_{\boldsymbol{\Omega}} \Big(D_{\boldsymbol{P}}H(t, \boldsymbol{u}, \phi, \boldsymbol{P}) + D_{\boldsymbol{P}}\omega(\boldsymbol{P})\Big)(\bar{\boldsymbol{P}}) d\boldsymbol{x} + \kappa\langle \boldsymbol{P}, \bar{\boldsymbol{P}}\rangle_s - l_3^2(t, \bar{\boldsymbol{P}})\end{aligned}$$

for all $\bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$.

(2) For each $p > 2$ and each $(t, \boldsymbol{u}, \boldsymbol{D}) \in [0,T] \times \big(W^{1,p}(\boldsymbol{\Omega})\big)^d \times (L^p(\boldsymbol{\Omega}))^d$, $\mathcal{E}\big(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}\big)$ is Gâteaux-differentiable w.r.t. $\boldsymbol{P} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and

$$\begin{aligned}
&D_{\boldsymbol{P}}\mathcal{E}\big(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}\big)[\bar{\boldsymbol{P}}] \\
&= \int_{\boldsymbol{\Omega}} \Big(D_{\boldsymbol{P}} U_1\big(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}\big) + D_{\boldsymbol{P}}\omega(\boldsymbol{P})\Big)(\bar{\boldsymbol{P}}) d\boldsymbol{x} + \kappa \langle \boldsymbol{P}, \bar{\boldsymbol{P}} \rangle_s - l_3^2(t, \bar{\boldsymbol{P}})
\end{aligned}$$

for all $\bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$, where U_1 is defined by (3.15).

Proof. It suffices to show the second statement, the proof for the first statement is being identical. The differentiability of l_3^2 and of the fractional gradient term $\langle \boldsymbol{P}, \boldsymbol{P} \rangle_s$ are being trivial, we thus only need to consider the energy terms $\mathcal{E}_1$ and $\mathcal{E}_2$, namely the energy integrals with integrands U_1 and ω respectively. For the polynomial ω, it follows

$$\begin{aligned}
&|\omega(\boldsymbol{P})| \le C(1 + |\boldsymbol{P}|^6), \\
&|D_{\boldsymbol{P}}\omega(\boldsymbol{P})| \le C(1 + |\boldsymbol{P}|^5), \\
&|D_{\nabla \boldsymbol{P}}\omega(\boldsymbol{P})| = 0.
\end{aligned}$$

Thus the conditions 3.32 and 3.35 in [12] are fulfilled and due to the proof of [12, Thm. 3.37], we obtain the Gâteaux-differentiability of $\mathcal{E}_2(\boldsymbol{P})$ w.r.t. $\boldsymbol{P}$ in space $\big(H^1(\boldsymbol{\Omega})\big)^d$. Since $\big(H^s(\boldsymbol{\Omega})\big)^d \subset \big(H^1(\boldsymbol{\Omega})\big)^d$, we also obtain the differentiability of $\mathcal{E}_2(\boldsymbol{P})$ w.r.t. $\boldsymbol{P}$ on $\big(H^s(\boldsymbol{\Omega})\big)^d$. For $\mathcal{E}_1$, it follows from the chain rule that for $\bar{\boldsymbol{P}} \in \big(H^s(\boldsymbol{\Omega})\big)^d$ and $h \in \mathbb{R}_{\neq 0}$

$$\begin{aligned}
&\frac{1}{h}\Big(\mathcal{E}_1(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P} + h\bar{\boldsymbol{P}}) - \mathcal{E}_1\big(t, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{P}\big)\Big) \\
&= \int_{\boldsymbol{\Omega}} \int_0^1 \frac{1}{2} D_{\boldsymbol{P}} \mathbb{B}_2(\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \bar{\boldsymbol{P}} \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - (\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \end{pmatrix} : \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - (\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \end{pmatrix} \\
&\quad + \mathbb{B}_2(\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \begin{pmatrix} \boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}_D(t) - \boldsymbol{\varepsilon}^0(\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \\ \boldsymbol{D} + \boldsymbol{D}_\nu(t) - (\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \end{pmatrix} : \begin{pmatrix} -D_{\boldsymbol{P}} \boldsymbol{\varepsilon}^0(\boldsymbol{P} + \sigma h \bar{\boldsymbol{P}}) \bar{\boldsymbol{P}} \\ -\bar{\boldsymbol{P}} \end{pmatrix} d\sigma d\boldsymbol{x} \\
&=: \int_{\boldsymbol{\Omega}} \int_0^1 I_1 + I_2 \, d\sigma d\boldsymbol{x}.
\end{aligned} \tag{A.4}$$

For the notation of the coefficient tensors and the function $\boldsymbol{D}_\nu$ appearing in (A.4) we refer to Section 2.4 and (3.12). Since I_1, I_2 converge pointwise to the corresponding integrands as $h \to 0$, it suffices to show that $|I_1|, |I_2| \le I$ for some $I \in L^1([0,1] \times \boldsymbol{\Omega})$ and then to apply the Lebesgue dominated convergence theorem for $h \to 0$. It follows from the triangular inequality and Cauchy-Schwarz that

$$\begin{aligned}
|I_1| &\le C|\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}_D(t)| + |\boldsymbol{D}_\nu(t)| + |\boldsymbol{\varepsilon}(\boldsymbol{u})| + |\boldsymbol{D}| + |\boldsymbol{P}| + |\bar{\boldsymbol{P}}|\big)^2 =: J_1, \\
|I_2| &\le C|\bar{\boldsymbol{P}}|\big(1 + |\boldsymbol{\varepsilon}_D(t)| + |\boldsymbol{D}_\nu(t)| + |\boldsymbol{\varepsilon}(\boldsymbol{u})| + |\boldsymbol{D}| + |\boldsymbol{P}| + |\bar{\boldsymbol{P}}|\big) =: J_2.
\end{aligned}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\boldsymbol{D}$ are of class L^p for some $p > 2$, the Hölder's inequality implies that

$$\begin{aligned}
\|J_1\|_{L^1([0,1]\times\boldsymbol{\Omega})} &\le C\|\bar{\boldsymbol{P}}\|_{L^\alpha}\big(1 + \Lambda + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^p} + \|\boldsymbol{D}\|_{L^p} + \|\boldsymbol{P}\|_{L^p} + \|\bar{\boldsymbol{P}}\|_{L^p}\big)^2, \\
\|J_2\|_{L^1([0,1]\times\boldsymbol{\Omega})} &\le C\|\bar{\boldsymbol{P}}\|_{L^\alpha}\big(1 + \Lambda + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{L^p} + \|\boldsymbol{D}\|_{L^p} + \|\boldsymbol{P}\|_{L^p} + \|\bar{\boldsymbol{P}}\|_{L^p}\big),
\end{aligned}$$

where $\alpha := \frac{p}{p-2}$ and Λ is the constant defined by (3.36) (also notice that $\boldsymbol{P}$ belongs to the class L^p for all $p \in [1, \infty)$ due to the Sobolev's embedding $H^s \hookrightarrow L^p$ for $s \ge \frac{d}{2}$). Then we obtain the desired result. □

A.3 Properties of convex functional and subdifferential

Lemma A.9 (Fenchel-Moreau). *Let X be a reflexive Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semi-continuous function. Then f^*, the Legendre-transform of f, is proper, convex, lower semi-continuous and $f = f^{**}$*

Proof. We refer to [68, Thm. 2.3.3]. □

Lemma A.10. *Let X be a reflexive Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semi-continuous function. Then the following statements are equivalent:*

- $x^* \in \partial f(x)$;
- $x \in \partial f^*(x^*)$;
- $f(x) + f^*(x^*) = \langle x^*, x\rangle_X$;
- $f(x) - \langle x^*, x\rangle_X \le f(z) - \langle x^*, z\rangle_X$ *for all* $z \in X$.

In particular, for $x \in \partial f^*(x^*), x^* \in \partial f(x)$, we always have

$$f(x) + f^*(x^*) \ge \langle x^*, x\rangle_X.$$

Proof. For the first four statements we refer to [12, Thm. 2.48]. For the last statement we refer to [68, Thm. 2.3.1]. □

Appendix B

Helly's selection theorem and Simon's embedding

In what follows, we introduce the Helly's selection theorem and Simon's embedding from [42] and [59] respectively. These regularity and embedding results will be essential for the proof of Theorem 3.30.

Lemma B.1 (Helly's selection theorem). *Let $\mathbf{\Omega} \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let $\Psi_1 : \mathbb{R}^d \to \mathbb{R}$ be some given functional such that Ψ_1 satisfies the Assumption A7. Let $\mathcal{A}$ be a weakly compact subset of $\big(H^s(\mathbf{\Omega})\big)^d$. If a sequence $\{\boldsymbol{P}_n\}_{n\in\mathbb{N}} \subset H^1(0,T;\big(H^s(\mathbf{\Omega})\big)^d)$ satisfies*

$$\sup_{n\in\mathbb{N}} \int_0^T \Psi_1\big(\boldsymbol{P}_n'(\tau)\big)d\tau \leq C$$

for some constant $C > 0$ and

$$\forall n \in \mathbb{N}\, \forall t \in [0,T] : \ \boldsymbol{P}_n(t) \in \mathcal{A},$$

then there exist a subsequence $\{\boldsymbol{P}_{n_k}\}_{k\in\mathbb{N}}$ of $\{\boldsymbol{P}_n\}_{n\in\mathbb{N}}$, a function $\varphi_\infty : [0,T] \to \mathbb{R}$ and some $\boldsymbol{P} : (0,T) \to \big(H^s(\mathbf{\Omega})\big)^d$ such that the following hold:

- $\varphi_{n_k}(t) := \int_0^t \Psi_1\big(\boldsymbol{P}_{n_k}'(\tau)\big)d\tau \to \varphi_\infty(t)$ *for all $t \in [0,T]$ as $k \to \infty$;*
- $\boldsymbol{P}_{n_k}(t) \rightharpoonup \boldsymbol{P}(t) \in \mathcal{A}$ *in $\big(H^s(\mathbf{\Omega})\big)^d$ for all $t \in [0,T]$ as $k \to \infty$.*

If in addition that $\boldsymbol{P} \in H^1(0,T;\big(H^s(\mathbf{\Omega})\big)^d)$, then

$$\int_{t_0}^{t_1} \Psi_1\big(\boldsymbol{P}'(\tau)\big)d\tau \leq \varphi_\infty(t_1) - \varphi_\infty(t_0)$$

for all $0 \leq t_0 < t_1 \leq T$.

Proof. This is a direct consequence of the Helly's selection theorem given in [42, Thm. 3.2]. We only need to check that the condition (A4) in [42, Thm. 3.2] is fulfilled, which is given as follows: let $\{\boldsymbol{P}_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ and

$$\min\{\Psi_1(\boldsymbol{P}_n - \boldsymbol{P}), \Psi_1(\boldsymbol{P} - \boldsymbol{P}_n)\} \to 0 \tag{B.1}$$

as $n \to \infty$, then it follows that $\boldsymbol{P}_n \rightharpoonup \boldsymbol{P}$ in $\big(H^s(\mathbf{\Omega})\big)^d$ as $n \to \infty$. We show that under the condition (B.1), every subsequence of $\{\boldsymbol{P}_n\}_{n\in\mathbb{N}}$ has a weak converging subsequence and all possible weak limits are equal to $\boldsymbol{P}$. Let $\{\boldsymbol{P}_{n_j}\}_{j\in\mathbb{N}}$ be a subsequence of $\{\boldsymbol{P}_n\}_{n\in\mathbb{N}}$. Since $\{\boldsymbol{P}_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, we know that there is a weak converging subsequence $\{\boldsymbol{P}_{n_{j_l}}\}_{l\in\mathbb{N}}$ of $\{\boldsymbol{P}_{n_j}\}_{j\in\mathbb{N}}$ which converges to a limit $\boldsymbol{P}^1 \in \left(H^s(\boldsymbol{\Omega})\right)^d$ as $l \to \infty$. Due to Sobolev's embedding (Lemma 3.5), $\boldsymbol{P}_{n_{j_l}} \to \boldsymbol{P}^1$ in $\left(L^1(\boldsymbol{\Omega})\right)^d$ as $l \to \infty$. But from the Assumption A7 and (B.1) we also know that $\boldsymbol{P}_{n_{j_l}} \to \boldsymbol{P}$ in $\left(L^1(\boldsymbol{\Omega})\right)^d$ as $l \to \infty$, therefore $\boldsymbol{P}^1 = \boldsymbol{P}$ and consequently the condition (A4) is fulfilled. □

Remark B.2. In the proof of Theorem 3.30, we need not only the weak convergence of a bounded sequences in certain reflexive Banach spaces, but also pointwise convergence of integrands having variable in t, which motivates the use of Lemma B.1. △

The next lemma shows that certain compactness of a set of functions follows from the uniform boundedness of its elements' derivatives.

Lemma B.3 ([59, Cor. 4], Simon's embedding)**.** *Let $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$, X, B, Y Banach spaces. Let $\mathcal{M}$ be a bounded subset in $L^\infty(0,T;X)$ and $\frac{\partial \mathcal{M}}{\partial t} = \{\frac{\partial f}{\partial t} : f \in \mathcal{M}\}$ be bounded in $L^r(0,T;Y)$ for some $r > 1$. Then $\mathcal{M}$ is relatively compact in $C(0,T;B)$.*

Appendix C

Basics of interpolation theory

In this chapter, we introduce several useful results concerning the interpolation theory between function spaces. The following definitions and results are mainly cited from [6] and [63].

C.1 Real interpolation

In order to define real interpolation spaces we need the following preliminaries: Let A_0, A_1 be compatible topological vector spaces, i.e., there exists a Hausdorff topological vector space U such that A_0, A_1 are subspaces of U. In this case, $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined.

Proposition C.1. *Let A_0, A_1 be compatible normed vector spaces. Then $A_0 \cap A_1$ and $A_0 + A_1$ are normed vector spaces with norms*

$$\|a\|_{A_0 \cap A_1} = \max\big(\|a\|_{A_0}, \|a\|_{A_1}\big),$$
$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1}\big(\|a_0\|_{A_0} + \|a_1\|_{A_1}\big)$$

respectively. Moreover, if A_0, A_1 are complete, then $A_0 \cap A_1$ and $A_0 + A_1$ are also complete.

Now we are able to give the precise definition of the real interpolation space between two spaces. There are two ways to define the real interpolation: The K-method and the J-method. Nevertheless, one can show that interpolation spaces defined by both methods are in general the same spaces with equivalent norms.

Proposition C.2 (K-method)**.** *Let A_0, A_1 be compatible Banach spaces. For $t > 0$ and $a \in A_0 + A_1$ let*

$$K(a, t; A_0, A_1) := \inf_{a = a_0 + a_1}\big\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\big\}.$$

For $0 < \theta < 1, 1 \leq q < \infty$ define

$$\|a\|_{\theta,q,K} := \Big(\int_0^\infty \frac{1}{t}\big(t^{-\theta}K(a, t; A_0, A_1)\big)^q dt\Big)^{\frac{1}{q}}$$

and for $\theta \in [0, 1]$

$$\|a\|_{\theta,\infty,K} := \sup_{t>0} t^{-\theta}K(a, t; A_0, A_1).$$

Then

$$(A_0, A_1)_{\theta,q,K} := \big\{a \in A_0 + A_1 : \|a\|_{\theta,q,K} < \infty\big\}$$

with the norm $\|\cdot\|_{\theta,q,K}$ *is a normed subspace of* $A_0 + A_1$. *The space* $(A_0, A_1)_{\theta,q,K}$ *is called the interpolation space induced by* (A_0, A_1) *of components* (θ, q) *by the K-method.*

Proposition C.3 (J-method)**.** *Let* A_0, A_1 *be compatible Banach spaces. For* $t > 0$ *and* $a \in A_0 \cap A_1$ *let*

$$J(a, t; A_0, A_1) := \max\left\{\|a_0\|_{A_0}, t\|a_1\|_{A_1}\right\}.$$

For $0 < \theta < 1, 1 \leq q < \infty$ *define*

$$\Phi_{\theta,q,J}(a) := \Big(\int_0^\infty \frac{1}{t}\big(t^{-\theta}J(a,t;A_0,A_1)\big)^q dt\Big)^{\frac{1}{q}}$$

and for $\theta \in [0, 1]$

$$\Phi_{\theta,\infty,J}(a) := \sup_{t>0} t^{-\theta}J(a, t; A_0, A_1).$$

The space $(A_0, A_1)_{\theta,q,J}$ *is defined as the space of elements* $a \in A_0+A_1$ *which can be written as*

$$a = \int_0^\infty u(t)\frac{dt}{t} \tag{C.1}$$

and

$$\Phi_{\theta,q,J}(u) < \infty, \tag{C.2}$$

where $u : (0, \infty) \to A_0 \cap A_1$ *is some Bochner-measurable function (see Definition D.1 below) with values in* $A_0 \cap A_1$. *Then* $(A_0, A_1)_{\theta,q,J}$ *is a normed subspace of* $A_0 + A_1$ *with the norm*

$$\|a\|_{\theta,q,J} := \inf\{\Phi_{\theta,q,J}(u) : u \text{ satisfies (C.1) and (C.2)}\}.$$

The space $(A_0, A_1)_{\theta,q,J}$ *is called the interpolation space induced by* (A_0, A_1) *of components* (θ, q) *by the J-method.*

The following theorem shows that both interpolation spaces are in general the same spaces with equivalent norms.

Theorem C.4. *If* $\theta \in (0, 1)$ *and* $q \in [1, \infty]$, *then* $(A_0, A_1)_{\theta,q,K} = (A_0, A_1)_{\theta,q,J}$ *with equivalent norms.*

Let $\theta \in (0, 1)$ and $q \in [1, \infty]$. Due to Theorem C.4 we are thus able to denote by $(A_0, A_1)_{\theta,q}$ the real interpolation space without referring to K-method or J-method.

C.1.1 Basic properties of real interpolation spaces

Theorem C.5. *Let* $\theta \in (0, 1)$ *and* $q \in [1, \infty]$. *Let* (A_0, A_1) *be a compatible pair. Then we have*

- $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$;
- $(A_0, A_1)_{\theta,q} \subset (A_0, A_1)_{\theta,r}$ *if* $q \leq r$;
- $A_1 \subset A_0 \Rightarrow (A_0, A_1)_{\theta_1,q} \subset (A_0, A_1)_{\theta_2,q}$ *if* $\theta_2 < \theta_1$.

Moreover, if A_0, A_1 *are complete, then* $(A_0, A_1)_{\theta,q}$ *is also complete.*

Theorem C.6 (Reiteration theorem)**.** *Let* (A_0, A_1) *be a compatible Banach spaces pair. Let* $\theta_1, \theta_2, \eta \in (0, 1)$, $q_1, q_2, q \in [1, \infty]$ *and*

$$\theta = (1 - \eta)\theta_1 + \eta\theta_2.$$

Then

$$\Big((A_0, A_1)_{\theta_1,q_1}, (A_0, A_1)_{\theta_2,q_2}\Big)_{\eta,q} = (A_0, A_1)_{\theta,q}.$$

C.2 Complex interpolation

Given compatible pair (A_0, A_1), the linear space $\mathcal{F}(A_0, A_1)$ is the space of functions $f : \mathbb{C} \to A_0 + A_1$ which are analytic in the strip $S = \{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\}$, continuous on the closure of S and satisfy the following conditions:

1. $\{f(z) : z \in S\}$ is a bounded subset of $A_0 + A_1$;
2. $\{f(it) : t \in \mathbb{R}\}$ is a bounded subset of A_0;
3. $\{f(1+it) : t \in \mathbb{R}\}$ is a bounded subset of A_1.

Then the space $\mathcal{F}(A_0, A_1)$ is a Banach space with the norm

$$\|f\|_{\mathcal{F}(A_0,A_1)} = \max\big\{\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1}\big\}.$$

Proposition C.7 (Complex interpolation)**.** *For $\theta \in (0,1)$, the complex interpolation space $[A_0, A_1]_\theta$ is the set of elements $x \in A_0 + A_1$ such that $x = f(\theta)$ for some $f \in \mathcal{F}(A_0, A_1)$. $[A_0, A_1]_\theta$ is a normed linear subspace of $A_0 + A_1$ with the norm*

$$\|x\|_{[A_0,A_1]_\theta} = \inf\big\{\|f\|_{\mathcal{F}(A_0,A_1)} : x = f(\theta), f \in \mathcal{F}(A_0, A_1)\big\}.$$

C.2.1 Basic properties of complex interpolation spaces

Theorem C.8. *Let (A_0, A_1) be a compatible pair. Then*

- $[A_0, A_1]_\theta = [A_1, A_0]_{1-\theta}$;
- $A_1 \subset A_0 \Rightarrow [A_0, A_1]_{\theta_1} \subset [A_0, A_1]_{\theta_2}$ *if* $\theta_2 < \theta_1$.

Moreover, if A_0, A_1 are complete, then $[A_0, A_1]_\theta$ is also complete.

Theorem C.9 (Reiteration theorem)**.** *Let (A_0, A_1) be a compatible Banach spaces pair. Let $\theta_j \in (0,1), j = 1, 2$. Define*

$$X_j = [A_0, A_1]_{\theta_j}.$$

Suppose that $A_0 \cap A_1$ is dense in A_0, A_1 and $X_1 \cap X_2$. Then for all $\eta \in [0,1]$ we have

$$[X_1, X_2]_\theta = [A_0, A_1]_\theta,$$

where $\theta = (1-\eta)\theta_1 + \eta\theta_2$.

C.3 Relation between real and complex interpolation

Theorem C.10. *Let (A_0, A_1) be a complete compatible pair. Let $\theta \in (0,1)$. Then*

$$(A_0, A_1)_{\theta,1} \subset [A_0, A_1]_\theta \subset (A_0, A_1)_{\theta,\infty}. \tag{C.3}$$

Theorem C.11. *Let (A_0, A_1) be a complete compatible pair. Let $0 < \theta_1 < \theta_2 < 1, \eta \in (0,1), \theta = (1-\eta)\theta_1 + \eta\theta_2$ and $p \in [1, \infty]$. Then*

$$\Big([A_0, A_1]_{\theta_1}, [A_0, A_1]_{\theta_2}\Big)_{\eta,p} = (A_0, A_1)_{\theta,p}.$$

Moreover, if there exist $p_1, p_2 \in [1, \infty]$ such that $1/p = (1-\eta)/p_1 + \eta/p_2$, then

$$\Big[(A_0, A_1)_{\theta_1,p_1}, (A_0, A_1)_{\theta_2,p_2}\Big]_\eta = (A_0, A_1)_{\theta,p}.$$

C.4 Equivalent norms on $H^s(\mathbf{\Omega})$

Lemma C.12. *Let $d \in \mathbb{N}_{\geq 2}$ and $\mathbf{\Omega} \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let $s \in (1,2)$. Then the norm $\|\cdot\|_s$ defined by*

$$\|f\|_s := \Big(\|f\|^2_{L^2(\mathbf{\Omega})} + \int_{\mathbf{\Omega}}\int_{\mathbf{\Omega}} \frac{|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s \rfloor)}} d\boldsymbol{x} d\boldsymbol{y}\Big)^{\frac{1}{2}}$$

defines an equivalent norm on $H^s(\mathbf{\Omega})$.

Proof. It suffices to show that $\|\cdot\|_{H^s(\mathbf{\Omega})} \leq C\|\cdot\|_s$. By expanding terms within norms we only need to show that

$$\|f\|_{H^1} \leq C\|f\|_s$$

for all $f \in H^s(\mathbf{\Omega})$. From [45, Thm. B.8] we obtain the real interpolation identity

$$(L^2(\mathbf{\Omega}), H^s(\mathbf{\Omega}))_{\theta,2} = H^1(\mathbf{\Omega}) \tag{C.4}$$

with $\theta = s^{-1} \in (\frac{1}{2}, 1)$. On the other hand, using [6, Chap. 3.5, (1)] we obtain from (C.4) that

$$\|f\|_{H^1} \leq C\|f\|^{1-\theta}_{L^2}\|f\|^{\theta}_{H^s}.$$

for $f \in H^s(\mathbf{\Omega})$. We also define

$$|f|_s := \Big(\int_{\mathbf{\Omega}}\int_{\mathbf{\Omega}} \frac{|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(s-\lfloor s \rfloor)}} d\boldsymbol{x} d\boldsymbol{y}\Big)^{\frac{1}{2}}.$$

Then using Young's inequality we obtain that

$$\begin{aligned}
\|f\|_{L^2} + \|\nabla f\|_{L^2} &= \|f\|_{H^1} \\
&\leq C\|f\|^{1-\theta}_{L^2}\|f\|^{\theta}_{H^s} \\
&= C\|f\|^{1-\theta}_{L^2}(\|f\|_{L^2} + \|\nabla f\|_{L^2} + |f|_s)^{\theta} \\
&\leq C_\varepsilon\|f\|_{L^2} + \varepsilon(\|f\|_{L^2} + \|\nabla f\|_{L^2} + |f|_s)
\end{aligned}$$

for $f \in H^s(\mathbf{\Omega})$ and arbitrary $\varepsilon > 0$, where $C_\varepsilon > 0$ is some positive constant depending on ε. Choosing for instance $\varepsilon = \frac{1}{2}$ and rearranging terms, we obtain the desired result. □

Appendix D

Concepts of measurability and Carathéodory functions

We have introduced the concept of Bochner-measurability in Definition 4.4. If the image set X is separable, then we are able to formulate different but equivalent measurability concepts to the Bochner-measurability. We indicate that this will be the case in our problem, since the underlying sets under consideration in our problem are all separable. The motivation of imposing different measurability concepts is that the problems can sometimes be dealt much easier when we use the equivalent measurability definitions other than the original one. We make all these concepts precise in the following. Here we will also generalize the concept of Bochner-measurability from the one given in Definition 4.4 to the one defined for a general finite measure space (Ω, Σ, μ). For simplicity we will also assume that all Banach spaces and metric spaces are over the real field $\mathbb{R}$, which is sufficient for the purpose of this thesis.

Definiton D.1 ([13]). *Let (Ω, Σ, μ) be a measure space and X be a metric space. Let $f : \Omega \to X$ be a function. Then f is said to be* ***Borel-measurable*** *if f is $(\Sigma, \mathcal{B}_X)$-measurable, where $\mathcal{B}_X$ is the Borel σ-algebra generated by the open subsets of X. Suppose in addition that (Ω, Σ, μ) is a finite measure space and X is a Banach space. Then*

- *f is said to be* ***Bochner-measurable*** *or* ***strongly measurable****, if there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions such that $f_n \to f$ in X μ-a.e. By simple function we mean a function $g : \Omega \to X$ such that $g(t) = \sum_{i=1}^{k} \mathbb{1}_{A_i}(t)x_i$ for $t \in \Omega$, where $k \in \mathbb{N}$, A_i are sets in Σ and x_i are elements in X.*

- *f is said to be* ***weakly measurable****, if for all $x^* \in X^*$ the function $t \mapsto \langle x^*, f(t)\rangle$ is $(\Sigma, \mathcal{B}_{\mathbb{R}})$-measurable, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$.*

We have the following result due to Pettis, which states that all measurability concepts are equivalent, as long as (Ω, Σ, μ) is a finite measure space and X is a separable Banach space:

Theorem D.2 ([13, Cor. 3.10.5]). *If (Ω, Σ, μ) is a finite measure space and X is a separable Banach space, then all the definitions of measurable functions given in Definition D.1 are equivalent.*

The following result states that a Carathéodory function maps measurable functions to measurable functions.

Lemma D.3 ([13, Cor. 2.5.24, Prop. 2.5.27])**.** *Let (Ω, Σ, μ) be a σ-finite measurable space, X be a separable metric space and Y be a metric space. Denote also by $\mathcal{B}_X$ and $\mathcal{B}_Y$ the Borel σ-algebras corresponding to X and Y respectively. Let $f : \Omega \times X \to Y$ be a Carathéodory function in the sense that*

- *$t \mapsto f(t, x)$ is $(\Sigma, \mathcal{B}_Y)$-measurable for all $x \in X$ and*
- *$x \mapsto f(t, x)$ is continuous for μ-a.a. $t \in \Omega$.*

Then if $g : \Omega \to X$ is $(\Sigma, \mathcal{B}_X)$-measurable, then $t \mapsto f(t, g(t))$ is $(\Sigma, \mathcal{B}_Y)$-measurable.

Remark D.4. In view of Theorem D.5, the Borel-measurability used in Lemma D.3 can be replaced by Bochner-measurability or weak measurability, if (Ω, Σ, μ) is a finite measure space and X, Y are separable Banach spaces. $\triangle$

We end up this chapter by giving the following very useful result for the special case where $\Omega = [0, T]$ with $T \in (0, \infty)$, Σ is the Lebesgue σ-algebra on $[0, T]$ and X is a Banach space:

Lemma D.5 ([17, Lem. 7.1.10])**.** *Let $f : [0, T] \to X$ be Bochner-measurable. Then $t \mapsto \|f(t)\|_X : [0, T] \to \mathbb{R}$ is Lebesgue-measurable.*

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