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On a hybrid approximation concept for self-excited periodic oscillations of large-scale dynamical systems

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When approximating periodic solutions in the context of large-scale dynamical systems involving strong local nonlinearities, efficiency is of special interest. Hence, the literature suggests a combination of two approximation methods for increasing the ratio of computational cost to accuracy. Within this contribution, a combination of *Finite Difference* and *Harmonic Balance* method is proposed. Due to the usage of *Harmonic Balance* it is shown, that the resulting equations only depend on the degrees of freedom that are affected by nonlinear forces. As an application, a self-excited limit cycle of a chain of oscillators is approximated and results are compared against numerical time integration to highlight qualitative accuracy.

1 Introduction

In the numerical analysis of dynamical systems, the reduction of computational effort is an essential component. For nonlinear systems tending to soar into a periodic oscillation, there are various methods for approximating these stationary solutions. However, for systems with a large number of degrees of freedom, a combination of two approximation methods can exploit their individual properties and, thus, increase efficiency. This approach was also discussed in [1], where a combination of HB with Shooting was suggested.

The present contribution proposes a hybrid approximation method, that combines the *Harmonic Balance Method* (HB) and *Finite Difference* (FD) method. This *Hybrid FD-HB* (HFH) method takes advantage of a sophisticated resolution of the motion within the nonlinear domains by the discretization of the FD method. On the other hand, approximating the linear domains via HB with a few harmonics allows a sufficient accuracy. For systems with locally acting nonlinearities and many degrees of freedom, this method promises to enhance the ratio of computational cost to accuracy.

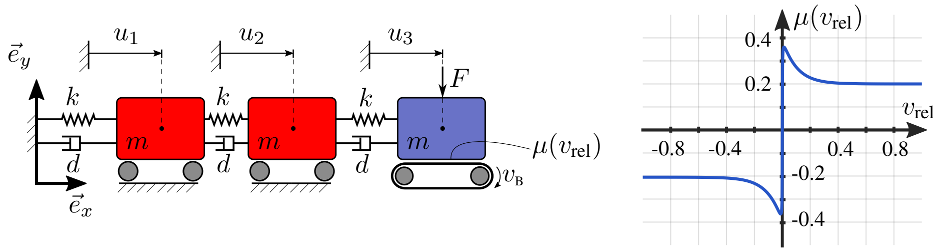

Fig. 1: Minimal model with 3 degrees of freedom and (regularized) friction curve with negative gradient. Here, the system is graphically divided into a linear (red) domain $\Gamma_{\rm lin}$ and a nonlinear (blue) domain $\Gamma_{\rm nl}$.

In the following, the system dynamics are reduced to a subset of the state space by defining the nonlinear degrees of freedom as master coordinates. This reduction scheme is explained for autonomous systems showing self-excited periodic vibrations, but is also applicable to externally excited systems. The solution of these *reduced equations of periodic motion* is approximated by the *Finite Difference* method. After that, results are presented for a self-excited dynamical system shown in fig. 1.

2 Method

The proposed method for approximating periodic oscillations focusses on large-scale systems with locally acting nonlinearities, where only a few degrees of freedom (DoF) are affected by nonlinear forces. Separating the degrees of freedom in a nonlinear and a linear domain, the autonomous governing equations of a mechanical system can be transformed to

$$\boldsymbol{M}_{NN}\ddot{\boldsymbol{u}}_{\rm N} + \boldsymbol{P}_{NN}\dot{\boldsymbol{u}}_{\rm N} + \boldsymbol{C}_{NN}\boldsymbol{u}_{\rm N} + \boldsymbol{f}_{\rm N}(\boldsymbol{u}_{\rm N}, \dot{\boldsymbol{u}}_{\rm N}) + \boldsymbol{M}_{NL}\ddot{\boldsymbol{u}}_{\rm L} + \boldsymbol{P}_{NL}\dot{\boldsymbol{u}}_{\rm L} + \boldsymbol{C}_{NL}\boldsymbol{u}_{\rm L} = \boldsymbol{0}, \tag{1a}$$

$$\boldsymbol{M}_{LL}\ddot{\boldsymbol{u}}_{\rm L} + \boldsymbol{P}_{LL}\dot{\boldsymbol{u}}_{\rm L} + \boldsymbol{C}_{LL}\boldsymbol{u}_{\rm L} + \boldsymbol{M}_{LN}\ddot{\boldsymbol{u}}_{\rm N} + \boldsymbol{P}_{LN}\dot{\boldsymbol{u}}_{\rm N} + \boldsymbol{C}_{LN}\boldsymbol{u}_{\rm N} = \boldsymbol{0}, \tag{1b}$$

where $\boldsymbol{u}_{\rm N}$ denotes the nonlinear and $\boldsymbol{u}_{\rm L}$ the linear DoF, in correspondence to [1]. Assuming eq. (1) has a periodic solution, both nonlinear and linear DoF can be expressed as a complex FOURIER series. Truncating this series gives an approximation

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$\boldsymbol{u}_\mathrm{N} \approx \Re\{\sum_{k=0}^{H} \boldsymbol{U}_{k,\mathrm{N}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\}$ resp. $\boldsymbol{u}_\mathrm{L} = \Re\{\sum_{k=0}^{H} \boldsymbol{U}_{k,\mathrm{L}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\}$ with base frequency ω and complex FOURIER coefficients $\boldsymbol{U}_{k,\mathrm{N}}$ and $\boldsymbol{U}_{k,\mathrm{L}}$ and H is a sufficient number of harmonics. Inserting this approximation of $\boldsymbol{u}_\mathrm{N}$ and $\boldsymbol{u}_\mathrm{L}$ into the linear eq. (1b) enables an analytical evaluation of the complex FOURIER coefficients

$$\boldsymbol{U}_{k,\mathrm{L}} = -\boldsymbol{G}_{LL}^{-1}(k\omega)\boldsymbol{G}_{LN}(k\omega)\boldsymbol{U}_{k,\mathrm{N}}, \quad \text{with} \quad \boldsymbol{G}_{ij}(k\omega) = -(k\omega)^2\boldsymbol{M}_{ij} + \mathrm{j}k\omega\boldsymbol{P}_{ij} + \boldsymbol{C}_{ij}, \quad i,j \in \{N, L\} \tag{2}$$

of the linear DoF for $k = 0, 1, \ldots, H$, that corresponds to the HB scheme. Hence, for prescribed periodic motion for $\boldsymbol{u}_\mathrm{N}$, the deflection $\boldsymbol{u}_\mathrm{L}$, velocity $\dot{\boldsymbol{u}}_\mathrm{L}$ and acceleration $\ddot{\boldsymbol{u}}_\mathrm{L}$ of the linear DoF are directly known. Inserting them into the nonlinear eq. (1a) gives an analytical expression for the feedback forces of the linear structure

$$\boldsymbol{f}_\mathrm{lin}(\omega, \boldsymbol{u}_\mathrm{N}) = \boldsymbol{M}_{NL}\ddot{\boldsymbol{u}}_\mathrm{L} + \boldsymbol{P}_{NL}\dot{\boldsymbol{u}}_\mathrm{L} + \boldsymbol{C}_{NL}\boldsymbol{u}_\mathrm{L} = -\Re\left\{\sum_{k=0}^{H} \boldsymbol{G}_{NL}(k\omega)\boldsymbol{G}_{LL}^{-1}(k\omega)\boldsymbol{G}_{LN}(k\omega)\boldsymbol{U}_{k,\mathrm{N}}\, \mathrm{e}^{\,\mathrm{j}k\omega t}\right\} \tag{3}$$

only depending on the base frequency ω and the FOURIER coefficients of the nonlinear DoF. Hence, the periodic dynamics of the system are described by the nonlinear DoF and the *reduced equations of periodic motion* given by

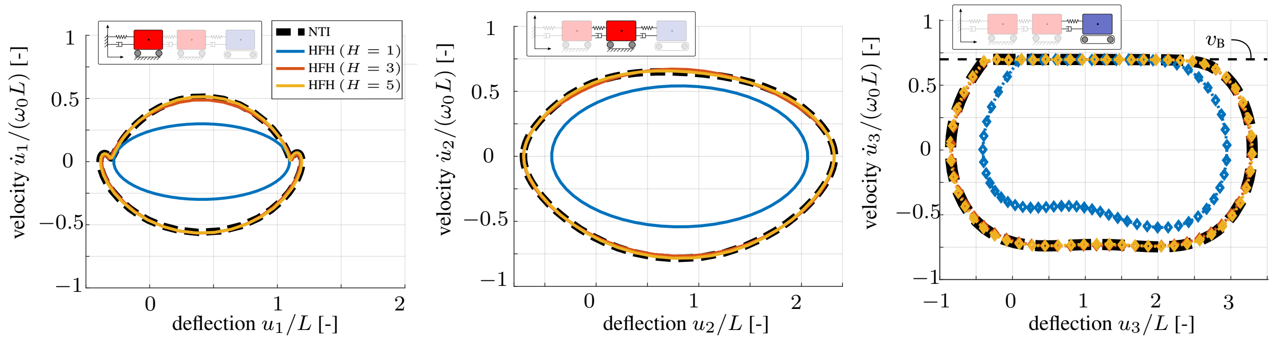
$$\boldsymbol{M}_{NN}\ddot{\boldsymbol{u}}_\mathrm{N} + \boldsymbol{P}_{NN}\dot{\boldsymbol{u}}_\mathrm{N} + \boldsymbol{C}_{NN}\boldsymbol{u}_\mathrm{N} + \boldsymbol{f}_\mathrm{N}(\boldsymbol{u}_\mathrm{N}, \dot{\boldsymbol{u}}_\mathrm{N}) + \boldsymbol{f}_\mathrm{lin}(\omega, \boldsymbol{u}_\mathrm{N}) = \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{f}_\mathrm{lin}(\omega, \boldsymbol{u}_\mathrm{N})$ includes the interaction of the nonlinear and linear subsystem. The linear DoF are directly related to $\boldsymbol{u}_\mathrm{N}$ and the base frequency ω as indicated by eq. (2). Up to this point, the approximation error depends solely on the number H of harmonics being considered.

As a last step, the approximation method for the nonlinear DoF must be selected. Approximating both the first and second time derivative within eq. (4) by finite differences and considering periodic boundary values lead to an algebraic equation system (AES) related to the deflection of the nonlinear DoF at N_FD discrete time samples [2]. In the present case of self-excited vibrations, a phase condition fixes the initial point on the limit cycle. This AES is solved numerically by a NEWTON-RAPHSON scheme.

3 Application & Outlook

As an application, periodic limit cycles of the system, shown in fig. 1 are calculated. Therefore, $N_\mathrm{FD} = 60$ time samples for the FD approximation were taken and the derivatives $\dot{\boldsymbol{u}}_N$ resp. $\ddot{\boldsymbol{u}}_N$ are approximated using a third order upwind resp. fourth order central difference scheme. Compared to numerical time integration, fig. 2 shows that only a few harmonics are essential to achieve sufficient accuracy for the present case. Here, both solutions for $H = 3$ and $H = 5$ show qualitatively sufficient



(a) Phase plot of the first DoF **(b)** Phase plot of the second DoF **(c)** Phase plot of the third DoF

Fig. 2: Periodic limit cycle of the system shown in fig. 1 at $\overline{v}_\mathrm{B} = 0.7$: comparison of *Hybrid FD-HB* (HFH) versus the *numerical time integration* (NTI).

results compared to NTI, as an approximation with $H = 1$ for $\boldsymbol{u}_L$ leads to poor agreement. Since the resulting algebraic equation system depends only on the deflection of the nonlinear DoF, this reduction will speed up the calculation process compared to classical methods that solve for both nonlinear and linear DoF. For the presented method, this hypothesis will be proven in future research by expanding the chain of oscillators and comparing the results to a sole usage of FD.

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