Error estimates for approximate approximations on compact intervals

Frank Müller , Werner Varnhorn *

Universität Kassel, Fachbereich Mathematik/Informatik, Heinrich-Plett-Str. 40, 34109 Kassel, Germany

Abstract

The aim of this paper is the investigation of the error which results from the method of approximate approximations applied to functions defined on compact intervals, only. This method, which is based on an approximate partition of unity, was introduced by V. Maz'ya in 1991 and has mainly been used for functions defined on the whole space up to now. For the treatment of differential equations and boundary integral equations, however, an efficient approximation procedure on compact intervals is needed.

In the present paper we apply the method of approximate approximations to functions which are defined on compact intervals. In contrast to the whole space case here a truncation error has to be controlled in addition. For the resulting total error pointwise estimates and L_1-estimates are given, where all the constants are determined explicitly.

Key words: Approximate approximations, error estimates, Gaussian kernels
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1 Introduction

In 1991 V. Maz'ya proposed a new approximation method called the method of approximate approximations [2], which is based on generating functions representing an approximate partition of unity, only. As a consequence, this approximation method does not converge if the mesh size tends to zero.

* Corresponding author.
Email addresses: fmueller@mathematik.uni-kassel.de (Frank Müller), varnhorn@mathematik.uni-kassel.de (Werner Varnhorn).

For numerical purposes, however, this lack of convergence does not play an important role since the resulting error can be chosen less than machine precision. On the other hand, this method has great advantages due to nice properties of the generating functions, i.e. simplicity, smoothness and exponential decay behavior [6].

In the first part of this paper we present the method developed by V. Maz'ya for functions defined on the whole space. In the second part we consider functions defined on compact intervals. In contrast to the whole space case here the summation has to be truncated since the function is not defined outside the interval. This leads to an additional truncation error to be controlled. For the total error pointwise estimates and L_1-estimates are given, where all the constants are determined explicitly.

2 Approximate partition of unity on $\mathbb{R}$

We now construct an approximate partition of unity on $\mathbb{R}$ using Gaussian kernels.

Let $d > 0$. For $m \in \mathbb{Z}$ we define the Gaussian kernel

$$g_m : \mathbb{R} \to \mathbb{R}\,, \quad g_m(t) := \frac{1}{\sqrt{\pi d}}\, e^{-\frac{1}{d}(t-m)^2}$$

and consider the function

$$\vartheta_d : \mathbb{R} \to \mathbb{R}\,, \quad \vartheta_d(t) := \sum_{m=-\infty}^{\infty} g_m(t).$$

The function ϑ_d is smooth, periodic with the period $p = 1$ and coincides with its Fourier series. Calculating the Fourier coeffients one gets on $\mathbb{R}$ the representations [1]

$$\begin{aligned}\vartheta_d(t) &= \frac{1}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} e^{-\frac{(t-m)^2}{d}} \\ &= 1 + 2\sum_{k=1}^{\infty} e^{-\pi^2 k^2 d} \cos\left(2\pi k t\right).\end{aligned}$$

Using the second representation leads to

$$|\vartheta_d(t) - 1| \leq 2\sum_{k=1}^{\infty} e^{-\pi^2 k^2 d} \leq 2\sum_{k=1}^{\infty} e^{-\pi^2 k d} = 2\sum_{k=1}^{\infty} \left(e^{-\pi^2 d}\right)^k = \frac{2e^{-\pi^2 d}}{1 - e^{-\pi^2 d}}\,.$$

Hence we obtain the estimate

$$\|\vartheta_d - 1\|_\infty \le \frac{2e^{-\pi^2 d}}{1 - e^{-\pi^2 d}} =: \mathrm{e}_0(d)\,.$$

The mapping $d \mapsto \mathrm{e}_0(d)$ is strictly decreasing to zero. This means: If the value d increases, the error $|\vartheta_d - 1|$ decreases. In other words: The system of Gaussian kernels $\{g_m\}_{m\in\mathbb{Z}}$ represents an approximate partition of unity on $\mathbb{R}$ if the value d is chosen large enough.

In the following table the values $\vartheta_d(0) - 1$ and $\mathrm{e}_0(d)$ are given for several values of d:

d	$\vartheta_d(0) - 1$	$\mathrm{e}_0(d)$
0.01	4.6419	19.2807
0.1	0.784286	1.18831
1	$1.03446 \cdot 10^{-4}$	$1.03452 \cdot 10^{-4}$
2	$5.35058 \cdot 10^{-9}$	$5.35058 \cdot 10^{-9}$
3	$2.76668 \cdot 10^{-13}$	$2.76749 \cdot 10^{-13}$

The table shows, that the estimate of $\|\vartheta_d - 1\|_\infty$ by $\mathrm{e}_0(d)$ is quite accurate for all $d \ge 1$.

From the above representations of ϑ_d we get its derivatives of first and second order in the form

$$\begin{aligned}\vartheta_d'(t) &= -\frac{2}{d}\frac{1}{\sqrt{\pi d}}\sum_{m=-\infty}^{\infty}(t-m)e^{-\frac{(t-m)^2}{d}}\\ &= -4\pi\sum_{k=1}^{\infty} k e^{-\pi^2 k^2 d}\sin(2\pi k t)\end{aligned}$$

and

$$\begin{aligned}\vartheta_d''(t) &= -\frac{2}{d}\vartheta_d(t) + \left(\frac{2}{d}\right)^2\frac{1}{\sqrt{\pi d}}\sum_{m=-\infty}^{\infty}(t-m)^2 e^{-\frac{(t-m)^2}{d}}\\ &= -8\pi^2\sum_{k=1}^{\infty} k^2 e^{-\pi^2 k^2 d}\cos(2\pi k t).\end{aligned}$$

Using the derivatives of the geometric series in the circle of convergence we obtain for $|q| < 1$ the identities

$$\sum_{k=1}^{\infty} k\, q^k = \frac{q}{(1-q)^2} \qquad \text{and} \qquad \sum_{k=1}^{\infty} k^2\, q^k = \frac{q^2+q}{(1-q)^3}\,.$$

The Fourier series of the derivatives now lead to the estimates

$$\|\vartheta_d'\|_\infty \le \frac{4\pi\, e^{-\pi^2 d}}{(1-e^{-\pi^2 d})^2} =: \mathrm{e}_1(d)$$

and

$$\|\vartheta_d''\|_\infty \le \frac{8\pi^2\, e^{-\pi^2 d}(1+e^{-\pi^2 d})}{(1-e^{-\pi^2 d})^3} =: \mathrm{e}_2(d).$$

Here the mappings $d \mapsto \mathrm{e}_1(d)$ and $d \mapsto e_2(d)$ are also strictly decreasing to zero. The following table shows $\mathrm{e}_1(d)$ and $\mathrm{e}_2(d)$ for various values of d:

d	$\mathrm{e}_1(d)$	$\mathrm{e}_2(d)$
0.01	1289.01	164256
0.1	11.9025	163.654
1	$6.5004 \cdot 10^{-4}$	$4.08474 \cdot 10^{-3}$
2	$3.36187 \cdot 10^{-8}$	$2.11232 \cdot 10^{-7}$
3	$1.73886 \cdot 10^{-12}$	$1.09256 \cdot 10^{-11}$

3 Approximate approximations on $\mathbb{R}$

For $u \in C_0^2(\mathbb{R})$ and $h > 0$ we define

$$u_{d,h} : \mathbb{R} \to \mathbb{R}\,, \quad u_{d,h}(t) := \frac{1}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} u(mh) e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2}.$$

The function $u_{d,h}$ is called the approximate approximation of u (or the quasi-interpolant of u), since for sufficiently large d the values $u_{d,h}(mh)$ and $u(mh)$ are approximately the same.

Using Taylor's formula we find for $t \in \mathbb{R}$ and $m \in \mathbb{Z}$ the representation

$$u(mh) = u(t) + u'(t)(mh-t) + \frac{u''(t_m)}{2}(mh-t)^2,$$

where t_m lies between t and mh. Substituting this expression in $u_{d,h}$, for the difference $u_{d,h}(t) - u(t)$ we obtain

$$
\begin{aligned}
u_{d,h}(t) - u(t) = {} & \frac{u(t)}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2} - \frac{h\,u'(t)}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} \left(\frac{t}{h}-m\right) e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2} \\
& + \frac{h^2}{2\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} u''(t_m) \left(\frac{t}{h}-m\right)^2 e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2} - u(t).
\end{aligned}
$$

With

$$
\frac{1}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2} = \vartheta_d\left(\frac{t}{h}\right)
$$

and

$$
-\frac{1}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} \left(\frac{t}{h}-m\right) e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2} = \frac{d}{2}\,\vartheta_d'\left(\frac{t}{h}\right)
$$

it follows

$$
\begin{aligned}
u_{d,h}(t) - u(t) = {} & \left(\vartheta_d\left(\frac{t}{h}\right) - 1\right) u(t) + \frac{dh}{2}\,\vartheta_d'\left(\frac{t}{h}\right) u'(t) \\
& + \frac{h^2}{2\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} u''(t_m) \left(\frac{t}{h}-m\right)^2 e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2}.
\end{aligned}
$$

Since $u \in C_0^2(\mathbb{R})$, the functions u, u' and u'' are bounded, hence

$$
\begin{aligned}
|u_{d,h}(t) - u(t)| \le {} & \|\vartheta_d - 1\|_\infty \, \|u\|_\infty + \frac{dh}{2} \|\vartheta_d'\|_\infty \, \|u'\|_\infty \\
& + \frac{h^2 \|u''\|_\infty}{2\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} \left(\frac{t}{h}-m\right)^2 e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2}.
\end{aligned}
$$

Using

$$
\frac{1}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} \left(\frac{t}{h}-m\right)^2 e^{-\frac{1}{d}\left(\frac{t}{h}-m\right)^2} = \frac{d}{2}\left(\vartheta_d\left(\frac{t}{h}\right) + \frac{d}{2}\vartheta_d''\left(\frac{t}{h}\right)\right)
$$

we get

$$
\begin{aligned}
|u_{d,h}(t) - u(t)| \le {} & \|\vartheta_d - 1\|_\infty \, \|u\|_\infty + \frac{dh}{2} \|\vartheta_d'\|_\infty \, \|u'\|_\infty \\
& + \frac{dh^2}{4} \left(\vartheta_d\left(\frac{t}{h}\right) + \frac{d}{2}\vartheta_d''\left(\frac{t}{h}\right)\right) \|u''\|_\infty.
\end{aligned}
$$

Setting

$$
\|u\|_{2,\infty} := \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}
$$

and

$$
c(d) := \frac{d}{4}\left(1 + \mathrm{e}_0(d) + \frac{d}{2}\,\mathrm{e}_2(d)\right)
$$

we finally obtain the error estimate

$$\|u_{d,h} - u\|_\infty \le \|u\|_{2,\infty} \left(\mathrm{e}_0(d) + \frac{d}{2}\, \mathrm{e}_1(d)\, h + c(d)\, h^2 \right).$$

Let us summarize the above results in the following theorem.

1. Theorem. For $u \in C_0^2(\mathbb{R})$ and $d, h > 0$ we define the approximate approximation

$$u_{d,h} : \mathbb{R} \to \mathbb{R}\,, \quad u_{d,h}(t) := \frac{1}{\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} u(mh) e^{-\frac{1}{d}\left(\frac{t}{h} - m\right)^2}$$

and get the error estimate

$$\|u_{d,h} - u\|_\infty \le \|u\|_{2,\infty} \left(\mathrm{e}_0(d) + \frac{d}{2}\, \mathrm{e}_1(d)\, h + c(d)\, h^2 \right)$$

with

$$\mathrm{e}_0(d) = \frac{2e^{-\pi^2 d}}{1 - e^{-\pi^2 d}}\,, \quad \mathrm{e}_1(d) = \frac{4\pi\, e^{-\pi^2 d}}{(1 - e^{-\pi^2 d})^2}\,, \quad \mathrm{e}_2(d) = \frac{8\pi^2\, e^{-\pi^2 d}(1 + e^{-\pi^2 d})}{(1 - e^{-\pi^2 d})^3}$$

and

$$c(d) = \frac{d}{4} \left(1 + \mathrm{e}_0(d) + \frac{d}{2}\, \mathrm{e}_2(d) \right).$$

For sufficiently large values of d the first two terms in the error estimate are small and $c(d)$ is of the same order as d. So $u_{d,h}$ represents an approximation of u which is pseudo convergent of second order.

Here the notation pseudo convergence is used since on the one hand there is no convergence $u_{d,h} \to u$ for $h \to 0$, but on the other hand the first two terms in the error estimate can be chosen less than machine precision, and therefore they can be neglected for numerical purposes.

If the parameter d is not too small the term of second order is dominating, hence we call this approximation pseudo convergent of second order.

We use the compact support of u only for the boundedness of u and its derivatives. The error estimate also holds true if we require that u and its derivatives up to second order are bounded, only.

The essential advantage of functions with compact support is that $u(mh) \neq 0$ only for a finite number of m, so that the summation can be realized numerically. For functions with unbounded support the summation has to be truncated somewhere, and the resulting truncation error has to be investigated.

4 Approximate approximations on $[-1,1]$

We now consider the approximate approximation of functions which are defined on a nontrivial compact interval. Since such intervals can be mapped bijectively on the interval $[-1,1]$, in the following we restrict ourselves to this interval.

To approximate functions $u \in C^2([-1,1])$ with the method presented above, we have to take into account that $u(mh)$ is defined only for $mh \in [-1,1]$.

One way to overcome this difficulty is the prolongation of u to a $C_0^2(\mathbb{R})$-function. This leads to a function with larger support and hence to additional numerical costs. Here we propose another way: Summing up over all m with $mh \in [-1,1]$, only. For the resulting truncation error we will give pointwise estimates and L_1-estimates.

In the following, let $u \in C^2([-1,1])$ be given. Furthermore, let $N \in \mathbb{N}$ and $h = 1/N$. We define the approximate approximation of u by

$$u_{d,h} : [-1,1] \to \mathbb{R}\,, \qquad u_{d,h}(t) := \frac{1}{\sqrt{\pi d}} \sum_{m=-N}^{N} u(mh) e^{-\frac{1}{d}(tN-m)^2}.$$

To obtain estimates for the error $|u - u_{d,h}|$ we need the following two functions:

2. Definition. For $t \in [-1,1]$ we define

$$\vartheta_{d,N}(t) := \frac{1}{\sqrt{\pi d}} \sum_{m=-N}^{N} e^{-\frac{1}{d}(tN-m)^2}$$

and

$$r_{d,N}(t) := \frac{1}{\sqrt{\pi d}} \left(\frac{e^{-\frac{1}{d}((1+t)N+1)^2}}{1 - e^{-\frac{2}{d}((1+t)N+1)}} + \frac{e^{-\frac{1}{d}((1-t)N+1)^2}}{1 - e^{-\frac{2}{d}((1-t)N+1)}} \right).$$

3. Lemma. Let $\mathrm{e}_0(d)$ as in Theorem 1. Then for all $t \in [-1,1]$ we have

$$|\vartheta_{d,N}(t) - 1| \le r_{d,N}(t) + \mathrm{e}_0(d).$$

Proof. Using the formula

$$\sum_{m=0}^{\infty} q^m = \frac{1}{1-q}$$

for $|q| < 1$ we find

$$
\begin{aligned}
\sum_{m=0}^{\infty} e^{-\frac{1}{d}((1\pm t)N+1+m)^2} &= \sum_{m=0}^{\infty} e^{-\frac{1}{d}((1\pm t)N+1)^2}\, e^{-\frac{2}{d}m(1\pm t)N+1)}\, e^{-\frac{1}{d}m^2} \\
&\le e^{-\frac{1}{d}((1\pm t)N+1)^2} \sum_{m=0}^{\infty} e^{-\frac{2}{d}((1\pm t)N+1)m} \\
&= e^{-\frac{1}{d}((1\pm t)N+1)^2} \sum_{m=0}^{\infty} \left(e^{-\frac{2}{d}((1\pm t)N+1)}\right)^m \\
&= \frac{e^{-\frac{1}{d}((1\pm t)N+1)^2}}{1-e^{-\frac{2}{d}((1\pm t)N+1)}}
\end{aligned}
$$

and hence

$$
\begin{aligned}
|\vartheta_{d,N}(t)-\vartheta_d(tN)| &= \frac{1}{\sqrt{\pi d}}\left|\sum_{m=-N}^{N} e^{-\frac{1}{d}(tN-m)^2} - \sum_{m=-\infty}^{\infty} e^{-\frac{1}{d}(tN-m)^2}\right| \\
&= \frac{1}{\sqrt{\pi d}}\left|\sum_{m=-\infty}^{-N-1} e^{-\frac{1}{d}(tN-m)^2} + \sum_{m=N+1}^{\infty} e^{-\frac{1}{d}(tN-m)^2}\right| \\
&= \frac{1}{\sqrt{\pi d}} \sum_{m=N+1}^{\infty} \left(e^{-\frac{1}{d}(tN+m)^2} + e^{-\frac{1}{d}(tN-m)^2}\right) \\
&= \frac{1}{\sqrt{\pi d}} \sum_{m=0}^{\infty} \left(e^{-\frac{1}{d}((1+t)N+1+m)^2} + e^{-\frac{1}{d}((1-t)N+1+m)^2}\right) \\
&\le \frac{1}{\sqrt{\pi d}} \left(\frac{e^{-\frac{1}{d}((1+t)N+1)^2}}{1-e^{-\frac{2}{d}((1+t)N+1)}} + \frac{e^{-\frac{1}{d}((1-t)N+1)^2}}{1-e^{-\frac{2}{d}((1-t)N+1)}}\right) \\
&= r_{d,N}(t).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
|\vartheta_{d,N}(t)-1| &\le |\vartheta_{d,N}(t)-\vartheta_d(tN)| + |\vartheta_d(tN)-1| \\
&\le r_{d,N}(t) + \mathrm{e}_0(d),
\end{aligned}
$$

as asserted. $\square$

4. Lemma. Let $\mathrm{e}_1(d)$ as in Theorem 1. Then for $d \le 2$ and all $t \in [-1,1]$ we have

$$
\left|\frac{1}{\sqrt{\pi d}} \sum_{m=-N}^{N} (tN-m)\, e^{-\frac{1}{d}(tN-m)^2}\right| \le \frac{d}{2}\, \mathrm{e}_1(d) + \frac{2\, e^{-\frac{1}{d}}}{\sqrt{\pi d}} \left(1+\frac{d}{2}\right).
$$

Proof. First we use

$$
\begin{aligned}
\frac{1}{\sqrt{\pi d}} \sum_{m=-N}^{N} (tN-m)\, e^{-\frac{1}{d}(tN-m)^2} = &- \frac{d}{2}\, \vartheta_d'(tN) \\
&- \frac{1}{\sqrt{\pi d}} \sum_{m=N+1}^{\infty} (tN-m)\, e^{-\frac{1}{d}(tN-m)^2} \\
&- \frac{1}{\sqrt{\pi d}} \sum_{m=N+1}^{\infty} (tN+m)\, e^{-\frac{1}{d}(tN+m)^2}.
\end{aligned}
$$

Since $m \geq N+1$ implies

$$
m \pm tN \geq m - N \geq 1,
$$

and since the function

$$
s \mapsto s e^{-\frac{1}{d}s^2}
$$

is strictly decreasing on $[1, \infty)$ for $d \leq 2$, it follows

$$
\begin{aligned}
\sum_{m=N+1}^{\infty} (m \pm tN)\, e^{-\frac{1}{d}(m \pm tN)^2} &\leq \sum_{m=N+1}^{\infty} (m-N)\, e^{-\frac{1}{d}(m-N)^2} \\
&= \sum_{m=1}^{\infty} m\, e^{-\frac{1}{d}m^2}.
\end{aligned}
$$

With

$$
\begin{aligned}
\sum_{m=1}^{\infty} m\, e^{-\frac{1}{d}m^2} &= e^{-\frac{1}{d}} + \sum_{m=2}^{\infty} \int_{m-1}^{m} m\, e^{-\frac{1}{d}m^2}\, dy \\
&\leq e^{-\frac{1}{d}} + \sum_{m=2}^{\infty} \int_{m-1}^{m} y\, e^{-\frac{1}{d}y^2}\, dy \\
&= e^{-\frac{1}{d}} + \int_{1}^{\infty} y\, e^{-\frac{1}{d}y^2}\, dy \\
&= e^{-\frac{1}{d}} \left(1 + \frac{d}{2}\right)
\end{aligned}
$$

we get

$$
\left| \frac{1}{\sqrt{\pi d}} \sum_{m=-N}^{N} (tN-m)\, e^{-\frac{1}{d}(tN-m)^2} \right| \leq \frac{d}{2}\, \mathrm{e}_1(d) + \frac{2\, e^{-\frac{1}{d}}}{\sqrt{\pi d}} \left(1 + \frac{d}{2}\right),
$$

as asserted. $\square$

These estimates now lead to the following pointwise estimate of the total error:

5. Theorem. For $u \in C^2([-1,1])$ we define the approximate approximation

$$u_{d,h} : [-1,1] \to \mathbb{R}\,, \qquad u_{d,h}(t) := \frac{1}{\sqrt{\pi d}} \sum_{m=-N}^{N} u(mh) e^{-\frac{1}{d}(tN-m)^2}.$$

Then for $d \leq 2$ and all $t \in [-1,1]$ we have

$$\begin{aligned} |u(t) - u_{d,h}(t)| \leq {} & \|u\|_\infty \, (r_{d,N}(t) + \mathrm{e}_0(d)) \\ & + \|u'\|_\infty \, h \left(\frac{d}{2}\, \mathrm{e}_1(d) + \frac{2\, e^{-\frac{1}{d}}}{\sqrt{\pi d}} \left(1 + \frac{d}{2}\right)\right) \\ & + \|u''\|_\infty \, c(d)\, h^2, \end{aligned}$$

where the constants $\mathrm{e}_i(d)$ and $c(d)$ are defined in Theorem 1 and $r_{d,N}$ is defined in Definition 2.

Proof. For $m \in \{-N, \ldots, N\}$ and $t \in [-1,1]$ we use Taylor's formula to get the representation

$$u(mh) = u(t) + u'(t)(mh - t) + \frac{u''(t_m)}{2}(mh - t)^2,$$

where t_m lies between t and mh. Inserting this term for $u(mh)$ in $u_{d,h}$, we find for $u_{d,h}(t) - u(t)$ the representation

$$\begin{aligned} u_{d,h}(t) - u(t) = {} & (\vartheta_{d,N}(t) - 1)\, u(t) \\ & - \frac{h\, u'(t)}{\sqrt{\pi d}} \sum_{m=-N}^{N} (tN - m)\; e^{-\frac{1}{d}(tN-m)^2} \\ & + \frac{h^2}{2\sqrt{\pi d}} \sum_{m=-N}^{N} u''(t_m)\; (tN - m)^2\; e^{-\frac{1}{d}(tN-m)^2}. \end{aligned}$$

Using Lemma 3 and Lemma 4 together with

$$\begin{aligned} \frac{1}{2\sqrt{\pi d}} \sum_{m=-N}^{N} (tN - m)^2\, e^{-\frac{1}{d}(tN-m)^2} & \leq \frac{1}{2\sqrt{\pi d}} \sum_{m=-\infty}^{\infty} (tN - m)^2\, e^{-\frac{1}{d}(tN-m)^2} \\ & = \frac{d}{4} \left(\vartheta_d(tN) + \frac{d}{2}\, \vartheta_d''(tN) \right) \\ & \leq \frac{d}{4} \left(1 + \mathrm{e}_0(d) + \frac{d}{2}\, \mathrm{e}_2(d) \right) \\ & = c(d), \end{aligned}$$

the proof is done. $\square$

Comparing this error estimate with the error estimate in Theorem 1 we notice that here additional terms do appear. Furthermore we see that it is not convenient to use the supremum norm here since there remains an error in the boundary points which does not vanish for increasing N unless u itself vanishes in the boundary points.

In the following we show that the L_1-norm

$$\int_{-1}^{1} |u(t) - u_{d,h}(t)| \, dt$$

of the total error can be made small for sufficiently large d and N. To do so we need the error function:

6. Definition. For $a, b \in \mathbb{R}$ with $a \leq b$ the error function erf is defined by

$$\operatorname{erf}(a,b) := \frac{2}{\sqrt{\pi}} \int_{a}^{b} e^{-t^2} \, dt.$$

The next lemma is essential for the L_1-estimate of the total error.

7. Lemma. Let $\vartheta_{d,N}$ be given as in Definition 2 and $\mathrm{e}_0(d)$ as in Theorem 1. Then we have

$$\int_{-1}^{1} |1 - \vartheta_{d,N}(t)| \, dt \leq \frac{\operatorname{erf}\left(\frac{1}{\sqrt{d}}, \frac{2N+1}{\sqrt{d}}\right)}{N\left(1 - e^{-\frac{2}{d}}\right)} + 2 \, \mathrm{e}_0(d).$$

Proof. Since

$$\frac{1}{1 - e^{-\frac{2}{d}((1 \pm t)N+1)}} \leq \frac{1}{1 - e^{-\frac{2}{d}}}$$

for $t \in [-1, 1]$, for the function $r_{d,N}$ defined in Definition 2 we get the estimate

$$\begin{aligned} r_{d,N}(t) &= \frac{1}{\sqrt{\pi d}} \left(\frac{e^{-\frac{1}{d}((1+t)N+1)^2}}{1 - e^{-\frac{2}{d}((1+t)N+1)}} + \frac{e^{-\frac{1}{d}((1-t)N+1)^2}}{1 - e^{-\frac{2}{d}((1-t)N+1)}} \right) \\ &\leq \frac{1}{\sqrt{\pi d}} \, \frac{e^{-\frac{1}{d}((1+t)N+1)^2} + e^{-\frac{1}{d}((1-t)N+1)^2}}{1 - e^{-\frac{2}{d}}} \, . \end{aligned}$$

Using

$$\int_{-1}^{1} e^{-\frac{1}{d}((1 \pm t)N+1)^2} dt = \frac{\sqrt{d}}{N} \int_{\frac{1}{\sqrt{d}}}^{\frac{2N+1}{\sqrt{d}}} e^{-t^2} dt = \frac{\sqrt{\pi d}}{2N} \operatorname{erf}\left(\frac{1}{\sqrt{d}}, \frac{2N+1}{\sqrt{d}} \right)$$

we find

$$\int_{-1}^{1} r_{d,N}(t)\,dt \le \frac{\operatorname{erf}\left(\frac{1}{\sqrt{d}}\,,\,\frac{2N+1}{\sqrt{d}}\right)}{N\left(1-e^{-\frac{2}{d}}\right)}\,.$$

From Lemma 3 we already know that

$$|1-\vartheta_{d,N}(t)| \le r_{d,N}(t) + \mathrm{e}_0(d),$$

and this yields

$$\int_{-1}^{1} |1-\vartheta_{d,N}(t)|\,dt \le \int_{-1}^{1} (r_{d,N}(t) + \mathrm{e}_0(d))\,dt \le \frac{\operatorname{erf}\left(\frac{1}{\sqrt{d}}\,,\,\frac{2N+1}{\sqrt{d}}\right)}{N\left(1-e^{-\frac{2}{d}}\right)} + 2\;\mathrm{e}_0(d)\,,$$

as asserted. □

Now we can estimate the L_1-norm of the total error as follows:

8. Theorem. Let $u \in C^2([-1,1])$. We define the approximate approximation $u_{d,h}$ as in Theorem 5. Then for $d \le 2$ we have

$$\begin{aligned}\int_{-1}^{1} |u(t)-u_{d,h}(t)|\,dt \le \|u\|_\infty &\left(\frac{\operatorname{erf}\left(\frac{1}{\sqrt{d}}\,,\,\frac{2N+1}{\sqrt{d}}\right)}{1-e^{-\frac{2}{d}}}\,h + 2\;\mathrm{e}_0(d)\right) \\ &+ 2\,\|u'\|_\infty\, h \left(\frac{d}{2}\;\mathrm{e}_1(d) + \frac{2\,e^{-\frac{1}{d}}}{\sqrt{\pi d}}\left(1+\frac{d}{2}\right)\right) \\ &+ 2\,\|u''\|_\infty\, c(d)\, h^2,\end{aligned}$$

where the constants $\mathrm{e}_i(d)$ and $c(d)$ are defined as in Theorem 1.

Proof. The proof follows immediately from Theorem 5 and Lemma 7. □

The above estimate shows that the L_1-norm of the total error is pseudo convergent of first order even though the error in the boundary points does not vanish.

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