

Functions satisfying holonomic q -differential equations

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Abstract. In a similar manner as in the papers [7] and [8], where explicit algorithms for finding the differential equations satisfied by holonomic functions were given, in this paper we deal with the space of the q -holonomic functions which are the solutions of linear q -differential equations with polynomial coefficients. The sum, product and the composition with power functions of q -holonomic functions are also q -holonomic and the resulting q -differential equations can be computed algorithmically.

Mathematics Subject Classification: 39A13, 33D15

Key words: q -derivative, q -differential equation, algorithm, algebra of q -holonomic functions.

1 Preliminaries

The purpose of this paper is to continue the research exposed in [7] and [8]. There, the authors discussed *holonomic* functions which are the solutions of homogeneous linear differential equations with polynomial coefficients.

In the present investigation, we consider a similar problem from the point of view of q -calculus. As general references for q -calculus see [2] and [4]. We begin with a few definitions.

Let $q \neq 1$. The q -complex number $[a]_q$ is given by

$$[a]_q := \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{C}.$$

Of course

$$\lim_{q \rightarrow 1} [a]_q = a.$$

The q -factorial of a positive integer $[n]_q$ and the q -binomial coefficient are defined by

$$[0]_q! := 1, \quad [n]_q! := [n]_q [n-1]_q \cdots [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

The q -Pochhammer symbol is given as

$$\begin{aligned}(a; q)_0 &= 1, \\ (a; q)_k &= (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \quad k = 1, 2, \dots, \\ (a; q)_\infty &= \lim_{k \rightarrow \infty} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1})\end{aligned}$$

and

$$(a; q)_\lambda = \frac{(a; q)_\infty}{(aq^\lambda; q)_\infty} \quad (|q| < 1, \lambda \in \mathbb{C}).$$

The q -derivative of a function $f(x)$ is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx} \quad (x \neq 0), \quad D_q f(0) := \lim_{x \rightarrow 0} D_q f(x), \quad (1)$$

and higher order q -derivatives are defined recursively

$$D_q^0 f := f, \quad D_q^n f := D_q D_q^{n-1} f, \quad n = 1, 2, 3, \dots \quad (2)$$

Of course, if f is differentiable at x , then

$$\lim_{q \rightarrow 1} D_q f(x) = f'(x).$$

The next four lemmas are well-known in q -calculus and their proofs can be seen, for example, in [3] or [4].

Lemma 1.1. *For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathbb{C}$ and $q \neq 1$, we have linearity and product rules*

$$\begin{aligned}D_q(\alpha u(x) + \beta v(x)) &= \alpha D_q u(x) + \beta D_q v(x), \\ D_q(u(x) \cdot v(x)) &= u(qx) D_q v(x) + v(x) D_q u(x) \\ &= u(x) D_q v(x) + v(qx) D_q u(x).\end{aligned}$$

Lemma 1.2. *The Leibniz rule for the higher order q -derivatives of a product of functions is given as*

$$D_q^n(u(x) \cdot v(x)) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{n-k} u(q^k x) D_q^k v(x).$$

Lemma 1.3. *For an arbitrary function $u(x)$ and for $t(x) = cx^k$ ($c, k \in \mathbb{C}$, $q^k \neq 1$) we have for the composition with $t(x)$*

$$D_q(u \circ t)(x) = D_{q^k} u(t) \cdot D_q t(x).$$

Lemma 1.4. *The values of the function for the shifted argument and for higher q -derivatives are connected by the two relations:*

$$f(q^n x) = \sum_{k=0}^n (-1)^k (1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k D_q^k f(x), \quad (3)$$

$$D_q^n f(x) = \frac{1}{(1-q)^n x^n} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2} - (n-1)k} f(q^k x). \quad (4)$$

For our further work, it is useful to write the product rule in slightly different form.

Lemma 1.5. *The product rule for the q -derivative can be written in the form*

$$D_q(u(x) \cdot v(x)) = u(x)D_q v(x) + v(x)D_q u(x) - (1-q)x D_q u(x) D_q v(x) . \quad (5)$$

In the same manner, higher q -derivatives can be expressed by

$$D_q^n(u(x) \cdot v(x)) = \sum_{\nu=0}^n \sum_{\mu=0}^n \alpha_{\nu\mu}(x) D_q^\nu u(x) D_q^\mu v(x) ,$$

where $\alpha_{\nu\mu}(x)$ are appropriate polynomials.

Let us finally recall that the q -hypergeometric series is given by ([2], [6])

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q, x \right) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^r (a_j; q)_k}{\prod_{j=1}^s (b_j; q)_k} \frac{x^k}{(q; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} .$$

2 On q -holonomic functions

For every function $f(x)$ which is a solution of a *polynomial homogeneous linear q -differential equation*

$$\sum_{k=0}^n \tilde{p}_k(x) D_q^k f(x) = 0 \quad (\tilde{p}_n \neq 0) \quad (\tilde{p}_k \in \mathbb{K}(q)[x], n \in \mathbb{N}) \quad (6)$$

we say that $f(x)$ is a *q -holonomic function*. The smallest such n is called the *holonomic order* of $f(x)$. Here \mathbb{K} is a field, typically $\mathbb{K} = \mathbb{Q}(a_1, a_2, \dots)$ or $\mathbb{K} = \mathbb{C}(a_1, a_2, \dots)$ where a_1, a_2, \dots denote some parameters. An equation of type (6) is called a *q -holonomic equation*.

Example 2.1. Since

$$D_q x^s = [s]_q x^{s-1} \quad (x, \alpha, s \in \mathbb{R}),$$

we have

$$f(x) = x^s \Rightarrow x D_q f(x) - [s]_q f(x) = 0 ,$$

or

$$(q-1)x D_q f(x) - (q^s - 1) f(x) = 0 ,$$

i.e. the power function is (for integer s) a q -holonomic function of first order.

Example 2.2. For $0 < |q| < 1$, $\lambda \in \mathbb{R}$, $x \neq 0, 1$, we have

$$D_q((x; q)_\lambda) = -[\lambda]_q (qx; q)_{\lambda-1} = \frac{-[\lambda]_q}{1-x} (x; q)_\lambda .$$

Hence

$$f(x) = (x; q)_\lambda \Rightarrow (x-1)D_q f(x) - [\lambda]_q f(x) = 0$$

or

$$(q-1)(x-1)D_q f(x) - (q^\lambda - 1)f(x) = 0.$$

Therefore the q -Pochhammer symbol is (for integer λ) also q -holonomic of first order.

Similarly, from

$$D_q((x; q)_\infty) = -(1-q)^{-1}(qx; q)_\infty = -\frac{1}{1-q} \frac{1}{1-x} (x; q)_\infty,$$

we get

$$f(x) = (x; q)_\infty \Rightarrow (1-x)D_q f(x) + \frac{1}{1-q} f(x) = 0.$$

Example 2.3. The small q -exponential function

$$e_q(x) = {}_1\phi_0 \left(\begin{matrix} 0 \\ - \end{matrix} \middle| q, x \right) = \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} x^n, \quad |x| < 1, \quad 0 < |q| < 1,$$

has q -derivative

$$\begin{aligned} D_q e_q(x) &= \frac{e_q(x) - e_q(qx)}{x - qx} \\ &= \frac{1}{x - qx} \left(\sum_{n=0}^{\infty} \frac{1}{(q; q)_n} x^n - \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} (qx)^n \right) \\ &= \frac{1}{x - qx} \sum_{n=0}^{\infty} \frac{x^n - (qx)^n}{(q; q)_n} \\ &= \frac{1}{x - qx} \left\{ x + \sum_{n=2}^{\infty} \frac{1 - q^n}{(1-q)(1-q^2) \cdots (1-q^{n-1})(1-q^n)} x^n \right\} \\ &= \frac{x}{x - qx} \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{(1-q)(1-q^2) \cdots (1-q^k)} x^k \right\} \\ &= \frac{1}{1-q} e_q(x), \end{aligned}$$

i.e. the small q -exponential function is q -holonomic of first order:

$$f(x) = e_q(x) \Rightarrow (1-q)D_q f(x) - f(x) = 0.$$

Note that this q -differential equation as well the resulting q -differential equations of the next four examples and similar ones can be obtained completely automatically by the `qsumdiffEq` command of the Maple package `qsum` by Böing and Koepf [1]. The above equation, e.g., is obtained using the command

```
qsumdiffEq(1/qpochhammer(q, q, n) * x^n, q, n, f(x))
```

Example 2.4. The big q -exponential function

$$E_q(x) = {}_0\phi_0 \left(\begin{matrix} - \\ - \end{matrix} \middle| q, -x \right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} x^n, \quad 0 < |q| < 1$$

has q -derivative

$$D_q E_q(x) = \frac{1}{x - qx} \left(\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} x^n - \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} (qx)^n \right) = \frac{1}{1 - q} E_q(qx).$$

which can be obtained in a similar way as in Example 2.3. Since

$$f(qx) = f(x) - (1 - q)x(D_q f)(x),$$

we conclude that the big q -exponential function is also q -holonomic of first order:

$$f(x) = E_q(x) \Rightarrow (1 - q)(x + 1)D_q f(x) - f(x) = 0.$$

Example 2.5. For $0 < |q| < 1$, q -sine and q -cosine functions

$$\begin{aligned} \sin_q(x) &= \frac{e_q(ix) - e_q(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q; q)_{2n+1}} x^{2n+1}, \\ \cos_q(x) &= \frac{e_q(ix) + e_q(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q; q)_{2n}} x^{2n}, \end{aligned}$$

satisfy

$$(1 - q)^2 D_q^2 f(x) + f(x) = 0$$

and are therefore q -holonomic of second order.

Example 2.6. The q -hypergeometric series ${}_r\phi_s$ is q -holonomic. The `qsundiffseq` command computes in particular for

$$f(x) = {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| q, x \right)$$

the q -holonomic equation

$$\begin{aligned} 0 &= (xabq - c)x(q - 1)^2 D_q^2 f(x) \\ &\quad + (-xb - xa + 1 + xabq - c + xab)(q - 1)D_q f(x) \\ &\quad + (-1 + a)(-1 + b)f(x). \end{aligned}$$

Example 2.7 Many q -orthogonal polynomials are q -holonomic. The Big q -Jacobi polynomials (see e.g. [5], 3.5) are given by

$$f(x) = P_n(x; a, b, c; q) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q, q \right).$$

They satisfy the q -holonomic equation

$$\begin{aligned} 0 &= q^n a(bqx - c)(q - 1)^2(1 - qx) D_q^2 f(x) \\ &\quad + (q - 1)(abq^{n+1} + abq^{2n+1}x + x - q^n a - q^n c - abq^{n+1}x - abq^{n+2}x + q^{n+1}ac) D_q f(x) \\ &\quad + (q^n - 1)(abq^{n+1} - 1)f(x) \end{aligned}$$

which is again easily determined by the `qsundiffeq` command.

The following lemma will be the crucial tool for the investigations of the next section.

Lemma 2.1. *If $f(x)$ is a function satisfying a holonomic equation (6) of order n , then the functions $D_q^l f(x)$ ($l = n, n + 1, \dots$) can be expressed as*

$$D_q^l f(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) D_q^k f(x), \quad (7)$$

where $p_k^{(l)}(x)$ are rational functions defined by

$$p_k^{(l)}(x) = \begin{cases} \delta_{kl}, & 0 \leq l < n - 1, \\ -\frac{\tilde{p}_k(x)}{\tilde{p}_n(x)}, & l = n \\ p_{k-1}^{(l-1)}(qx) + D_q p_k^{(l-1)}(x) + p_{n-1}^{(l-1)}(qx) p_k^{(n)}(x), & l > n, \end{cases}$$

for $0 \leq k \leq n - 1$ and 0 for other k 's.

Proof. The representations (7) and the corresponding coefficients are evident by Equation (6) for $l = 0, 1, \dots, n$. By q -deriving and using Lemma 1.1, from

$$D_q^n f(x) = \sum_{k=0}^{n-1} p_k^{(n)}(x) D_q^k f(x)$$

we get

$$\begin{aligned} D_q^{n+1} f(x) &= \sum_{k=0}^{n-1} D_q(p_k^{(n)}(x) D_q^k f(x)) \\ &= \sum_{k=0}^{n-1} p_k^{(n)}(qx) D_q^{k+1} f(x) + \sum_{k=0}^{n-1} D_q(p_k^{(n)}(x)) D_q^k f(x) \\ &= \sum_{k=0}^{n-1} (p_{k-1}^{(n)}(qx) + D_q(p_k^{(n)}(x)) D_q^k f(x)) + p_{n-1}^{(n)}(x) D_q^n f(x) \\ &= \sum_{k=0}^{n-1} p_k^{(n+1)}(x) D_q^k f(x), \end{aligned}$$

with

$$p_k^{(n+1)}(x) = p_{k-1}^{(n)}(qx) + D_q p_k^{(n)}(x) + p_{n-1}^{(n)}(qx) p_k^{(n)}(x) \quad (0 \leq k \leq n-1).$$

Repeating the procedure, we get the representation and coefficients for arbitrary $l > n$.
 \diamond

We finish this section by noting that there are functions which are not q -holonomic.

Lemma 2.2. *The exponential function $f(x) = a^x$ ($a > 0, a \neq 1$) is not q -holonomic.*

Proof. Taking successive q -derivatives of $f(x) := a^x$ generates iteratively the functions of the list $L := \{a^x, a^{qx}, a^{q^2x}, \dots\}$. Since the members of L are linearly independent over $\mathbb{K}(q)[x]$, the linear space over $\mathbb{K}(q)[x]$ generated by L has infinite dimension. This is equivalent to the fact that there is no q -holonomic equation for $f(x)$. \diamond

3 Operations with q -holonomic functions

In this section, we will formulate and prove a few theorems about q -holonomic functions provided by derivation, addition or multiplication of the given q -holonomic functions.

Theorem 3.1. *If $f(x)$ is a q -holonomic function of order n , then the function $h_m(x) = D_q^m f(x)$ is a q -holonomic function of order at most n for every $m \in \mathbb{N}$. Furthermore, there is an algorithm to compute the corresponding q -differential equation.*

Proof. If we prove the statement for $m = 1$, the final conclusion follows by mathematical induction.

Let $h(x) = D_q f(x)$, where the function $f(x)$ satisfies (6). If $\tilde{p}_0(x) \equiv 0$, then immediately $h(x)$ is a q -holonomic function of order $n - 1$.

Hence, let $\tilde{p}_0(x) \not\equiv 0$. Then, by Lemma 2.1, we have

$$D_q^n f(x) = \sum_{k=0}^{n-1} p_k^{(n)}(x) D_q^k f(x),$$

wherefrom

$$\begin{aligned} f(x) &= \frac{1}{p_0^{(n)}(x)} \left(D_q^n f(x) - \sum_{k=1}^{n-1} p_k^{(n)}(x) D_q^k f(x) \right) \\ &= \frac{1}{p_0^{(n)}(x)} \left(D_q^{n-1} h(x) - \sum_{k=0}^{n-2} p_{k+1}^{(n)}(x) D_q^k h(x) \right). \end{aligned}$$

Also, by q -deriving, we get

$$\begin{aligned}
D_q^n h(x) &= D_q^{n+1} f(x) = \sum_{k=0}^{n-1} p_k^{(n+1)}(x) D_q^k f(x) \\
&= p_0^{(n+1)}(x) f(x) + \sum_{k=1}^{n-1} p_k^{(n+1)}(x) D_q^{k-1} h(x) \\
&= \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)} \left(D_q^{n-1} h(x) - \sum_{k=0}^{n-2} p_{k+1}^{(n)}(x) D_q^k h(x) \right) + \sum_{k=0}^{n-2} p_{k+1}^{(n+1)}(x) D_q^k h(x).
\end{aligned}$$

Hence,

$$D_q^n h(x) = \sum_{k=0}^{n-1} P_k(x; h) D_q^k h(x),$$

where

$$\begin{aligned}
P_k(x; h) &= p_{k+1}^{(n+1)}(x) - \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)} p_{k+1}^{(n)}(x), \quad k = 0, 1, \dots, n-2, \\
P_{n-1}(x; h) &= \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)}.
\end{aligned}$$

By multiplying with the common denominator of the rational functions $\{P_k(x; h), k = 0, 1, \dots, n-1\}$, we can conclude that $h(x)$ satisfies the equation

$$\sum_{k=0}^n \tilde{P}_k(x; h) D_q^k h(x) = 0,$$

i.e it is a q -holonomic function of order $\leq n$. \diamond

Example 3.1. In Example 2.2, for the q -Pochhammer symbol we proved that it satisfies

$$f(x) = (x; q)_\infty \quad \Rightarrow \quad (1-x)D_q f(x) + \frac{1}{1-q} f(x) = 0.$$

Now, we have

$$h_m(x) = D_q^m((x; q)_\infty) \quad \Rightarrow \quad (1-q^m x)D_q h_m(x) + \frac{q^m}{1-q} h_m(x) = 0 \quad (m \in \mathbb{N}_0).$$

Theorem 3.2. If $u(x)$ and $v(x)$ are q -holonomic functions of order n and m respectively, then the functions $u(x) + v(x)$ are q -holonomic functions of order at most $m+n$ and there is an algorithm to compute the corresponding q -differential equations.

Proof. If $u(x)$ and $v(x)$ are q -holonomic functions of order n and m respectively, they satisfy holonomic equations

$$\sum_{k=0}^n \tilde{p}_k(x) D_q^k u(x) = 0, \quad \sum_{j=0}^m \tilde{r}_j(x) D_q^j v(x) = 0, \quad (8)$$

where $\tilde{p}_k(x)$ and $\tilde{r}_j(x)$ are polynomials and $\tilde{p}_n \neq 0$, $\tilde{r}_m \neq 0$. According to Lemma 2.1, $D_q^l u(x)$ and $D_q^l v(x)$ can be represented as

$$D_q^l u(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) D_q^k u(x), \quad D_q^l v(x) = \sum_{j=0}^{m-1} r_j^{(l)}(x) D_q^j v(x), \quad (9)$$

where $p_k^{(l)}(x)$ and $r_j^{(l)}(x)$ are rational functions as mentioned lemma.

Let $h(x) = u(x) + v(x)$. Then, according to (9), we have

$$D_q^l h(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) D_q^k u(x) + \sum_{j=0}^{m-1} r_j^{(l)}(x) D_q^j v(x), \quad l = 0, 1, \dots, m+n. \quad (10)$$

Taking the values for $l = 0, 1, \dots, m+n-1$ in the above identities and expressing q -derivatives of $u(x)$ and $v(x)$ by q -derivatives of $h(x)$, we get

$$\begin{aligned} D_q^k u(x) &= \sum_{l=0}^{m+n-1} a_k^{(l)}(x) D_q^l h(x), \quad k = 0, 1, \dots, n-1, \\ D_q^j v(x) &= \sum_{l=0}^{m+n-1} b_j^{(l)}(x) D_q^l h(x), \quad j = 0, 1, \dots, m-1. \end{aligned}$$

By eliminating $D_q^k u(x)$ ($k = 0, 1, \dots, n-1$) and $D_q^j v(x)$ ($j = 0, 1, \dots, m-1$) from the last identity ($l = m+n$) of (10), we get

$$D_q^{m+n} h(x) = \sum_{l=0}^{m+n-1} c_l(x) D_q^l h(x),$$

where

$$c_l(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) a_k^{(l)}(x) + \sum_{j=0}^{m-1} r_j^{(l)}(x) b_j^{(l)}(x).$$

By multiplying with the common denominator of $\{c_l(x), l = 0, 1, \dots, m+n-1\}$, we get the holonomic equation for $h(x)$

$$\sum_{l=0}^{m+n} \tilde{c}_l(x) D_q^l h(x) = 0.$$

This proves that the q -holonomic order of $u(x) + v(x)$ is at most $m+n$, but can be less. An iterative version of the given algorithm will determine the q -holonomic equation of lowest order for $u(x) + v(x)$. \diamond

Note that the algorithm given in Theorem 3.2 finds a q -differential equation which is not only valid for $u(x) + v(x)$, but also for every linear combination $\lambda_1 u(x) + \lambda_2 v(x)$, in particular for $u(x) - v(x)$.

Example 3.2. The small q -exponential function from Example 2.3 is q -holonomic of first order and satisfies

$$u(x) = e_q(x) \Rightarrow D_q^k u(x) = \frac{1}{(1-q)^k} u(x) \quad (k = 0, 1, \dots).$$

Also, the q -sine from Example 2.5 is q -holonomic of second order and satisfies

$$v(x) = \sin_q(x) \Rightarrow D_q^{k+2} v(x) = \frac{-1}{(1-q)^2} D_q^k v(x) \quad (k = 0, 1, \dots).$$

Now, by the algorithm given in the proof of Theorem 3.2, the function $h(x) = u(x) + v(x)$ satisfies

$$D_q^3 h(x) = \frac{1}{1-q} D_q^2 h(x) - \frac{1}{(1-q)^2} D_q h(x) + \frac{1}{(1-q)^3} h(x).$$

i.e., it is q -holonomic of third order.

Theorem 3.3. *If $u(x)$ and $v(x)$ are q -holonomic functions of order n and m respectively, then the function $u(x) \cdot v(x)$ is q -holonomic of order at most $m \cdot n$ and there is an algorithm to compute the corresponding q -differential equation.*

Proof. If $u(x)$ and $v(x)$ are q -holonomic functions of order n and m respectively, they satisfy holonomic equations (8), and their q -derivatives (9).

Let $h(x) = u(x) \cdot v(x)$. Then, according to (1.5), we have

$$\begin{aligned} D_q^l h(x) &= \sum_{\nu=0}^l \sum_{\mu=0}^l \alpha_{\nu\mu}(x) D_q^\nu u(x) D_q^\mu v(x) \\ &= \sum_{\nu=0}^l \sum_{\mu=0}^l \alpha_{\nu\mu}(x) \left(\sum_{k=0}^{n-1} p_k^{(\nu)}(x) D_q^k u(x) \right) \left(\sum_{j=0}^{m-1} r_j^{(\mu)}(x) D_q^j v(x) \right), \end{aligned}$$

i.e.

$$D_q^l h(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \beta_{kj}^{(l)}(x) D_q^k u(x) D_q^j v(x) \quad (l = 0, 1, \dots, mn), \quad (11)$$

where

$$\beta_{kj}^{(l)}(x) = \sum_{\nu=0}^l \sum_{\mu=0}^l \alpha_{\nu\mu}(x) p_k^{(\nu)}(x) r_j^{(\mu)}(x).$$

Taking the relations for (11) for $l = 0, 1, \dots, mn - 1$ and expressing the q -derivatives $D_q^k u(x) D_q^j v(x)$ by q -derivatives of $h(x)$, we get

$$D_q^k u(x) D_q^j v(x) = \sum_{l=0}^{mn-1} \gamma_{kj}^{(l)}(x) D_q^l h(x) \quad (0 \leq k \leq n-1; 0 \leq j \leq m-1).$$

Eliminating all those $D_q^k u(x) D_q^j v(x)$ from the last identity ($l = mn$) of (11), it becomes

$$D_q^{mn} h(x) = \sum_{l=0}^{mn-1} \sigma_l(x) D_q^l h(x),$$

where

$$\sigma_l(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \beta_{kj}^{(l)}(x) \gamma_{kj}^{(l)}(x).$$

By multiplying with the common denominator of $\{\sigma_l(x), l = 0, 1, \dots, mn-1\}$, we get the q -holonomic equation for $h(x)$

$$\sum_{l=0}^{mn} \tilde{\sigma}_l(x) D_q^l h(x) = 0.$$

This proves that the q -holonomic order of $u(x) \cdot v(x)$ is at most mn , but can be less. An iterative version of the given algorithm will determine the q -holonomic equation of lowest order for $u(x) \cdot v(x)$. \diamond

Note furthermore that by Lemma 2.2 there is no similar algorithm for the quotient $u(x)/v(x)$.

Example 3.3. We use again $u(x) = e_q(x)$ and $v(x) = \sin_q(x)$. Now, by the given algorithm the function $h(x) = u(x) \cdot v(x)$ satisfies

$$(1-q)^2 D_q^2 h(x) - (1-q^2) D_q h(x) + (qx^2 - (1+q)(x-1)) h(x) = 0,$$

i.e., it is q -holonomic of second order.

Theorem 3.4. *If $u(x)$ is a q -holonomic function of order n , then the function $w(x) = u(x^\nu)$ ($\nu \in \mathbb{N}$) is a q -holonomic function of order at most n and there is an algorithm to compute the corresponding q -differential equation.*

Proof. By assumption $u(t)$ satisfies a q -holonomic equation

$$\sum_{k=0}^n \tilde{p}_k(t) D_q^k u(t) = 0, \quad (12)$$

where $\tilde{p}_k(t)$ are polynomials and $\tilde{p}_n \neq 0$. Then, by Lemma 2.1, $D_q^l u(t)$ can be represented as

$$D_q^l u(t) = \sum_{k=0}^{n-1} p_k^{(l)}(t) D_q^k u(t), \quad (13)$$

where $p_k^{(l)}(t)$ are rational functions determined by that lemma.

Let $t = x^\nu$. Using Lemma 1.3, we have

$$D_q w(x) = D_{q^\nu} u(t) D_q(x^\nu) = \frac{u(t) - u(q^\nu t)}{(1-q^\nu)t} [\nu]_q x^{\nu-1}.$$

According to (4), we get

$$D_q w(x) = \sum_{j=1}^{\nu} e_{j,\nu}(x) D_q^j u(t),$$

where

$$e_{j,\nu}(x) = (-1)^{j-1} (1-q)^{j-1} \begin{bmatrix} \nu \\ j \end{bmatrix}_q q^{\binom{j}{2}} x^{\nu j-1}, \quad j = 1, 2, \dots, \nu. \quad (14)$$

By (13), we can write

$$D_q w(x) = \sum_{j=1}^{\nu} e_{j,\nu}(x) \sum_{k=0}^{n-1} p_k^{(j)}(t) D_q^k u(t) = \sum_{k=0}^{n-1} f_{k,\nu}^{(1)}(x) D_q^k u(t),$$

where

$$f_{k,\nu}^{(1)}(x) = \sum_{j=1}^{\nu} p_k^{(j)}(x^\nu) e_{j,\nu}(x), \quad k = 0, 1, \dots, n-1. \quad (15)$$

Furthermore,

$$\begin{aligned} D_q^2 w(x) &= \sum_{k=0}^{n-1} D_q (f_{k,\nu}^{(1)}(x) D_q^k u(t)) \\ &= \sum_{k=0}^{n-1} D_q f_{k,\nu}^{(1)}(x) D_q^k u(t) + \sum_{k=0}^{n-1} f_{k,\nu}^{(1)}(qx) D_q (D_q^k u(t)) \dots \end{aligned}$$

As before, the second sum in the above term can be transformed to

$$\begin{aligned} \sum_{i=0}^{n-1} f_{i,\nu}^{(1)}(qx) D_q (D_q^i u(t)) &= \sum_{i=0}^{n-1} f_{i,\nu}^{(1)}(qx) \sum_{j=1}^{\nu} e_{j,\nu}(x) D_q^j (D_q^i u(t)) \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) D_q^{i+j} u(t) \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) \sum_{k=0}^{n-1} p_k^{(i+j)}(t) D_q^k u(t). \end{aligned}$$

Hence,

$$D_q^2 w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(2)}(x) D_q^k u(t),$$

where

$$f_{k,\nu}^{(2)}(x) = D_q f_{k,\nu}^{(1)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) p_k^{(i+j)}(x^\nu), \quad k = 0, 1, \dots, n-1.$$

By induction, we obtain the representations

$$D_q^l w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x) D_q^k u(x), \quad l = 0, 1, 2, \dots, n \quad (16)$$

where $f_{k,\nu}^{(0)}(x) = \delta_{k0}$, $f_{k,\nu}^{(1)}(x)$ is given in (15) and

$$f_{k,\nu}^{(l)}(x) = D_q f_{k,\nu}^{(l-1)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(l-1)}(qx) e_{j,\nu}(x) p_k^{(i+j)}(x^\nu). \quad (17)$$

Taking the first n of the identities (16), we can determine

$$D_q^k u(x) = \sum_{l=0}^{n-1} b_{l,\nu}^{(k)}(x) D_q^l w(x), \quad k = 0, 1, \dots, n-1,$$

where $b_{l,\nu}^{(k)}(x)$ are rational functions. Substituting this in identity (16), we get

$$D_q^n w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(n)}(x) \sum_{l=0}^{n-1} b_{l,\nu}^{(k)}(x) D_q^l w(x) = \sum_{l=0}^{n-1} c_{l,\nu}(x) D_q^l w(x),$$

where

$$c_{l,\nu}(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(n)}(x) b_{l,\nu}^{(k)}(x).$$

By multiplying with the common denominator of $\{c_{l,\nu}(x), l = 0, 1, \dots, n-1\}$, we obtain

$$\sum_{l=0}^{n-1} \tilde{c}_{l,\nu}(x) D_q^l w(x) = 0. \quad \diamond$$

Example 3.4. In Example 2.1, it was proved that

$$u(x) = x^s \Rightarrow (q-1)x D_q u(x) - (q^s - 1)u(x) = 0.$$

Hence $n = 1$, $\tilde{p}_1(x) = (q-1)x$ and $\tilde{p}_0(x) = -(q^s - 1)$. By the procedure of Theorem 3.4, we get for $w(x) = u(x)^\nu$

$$D_q w(x) = f_{0,\nu}^{(1)}(x) w(x) \quad \text{where} \quad f_{0,\nu}^{(1)}(x) = \frac{q^{s\nu} - 1}{q-1} \cdot \frac{1}{x}.$$

Finally,

$$w(x) = u(x)^\nu \Rightarrow (q-1)x D_q w(x) - (q^{s\nu} - 1)w(x) = 0.$$

Example 3.5. In Example 2.2, it was proved that

$$u(x) = (x; q)_\lambda \Rightarrow (q-1)(x-1) D_q u(x) - (q^\lambda - 1)u(x) = 0.$$

Using our algorithm we get for $w(x) = u(x^2) = (x^2; q)_\lambda$ the q -holonomic equation $(q-1)(x-1)(x+1)(x^2q-1)D_q w(x) - x(q^\lambda - 1)(x^2q^{\lambda+1} - q - 1 + x^2q)f(x) = 0$ and similar, but more complicated, equations for $(x^\nu; q)_\lambda$ for higher $\nu \in \mathbb{N}$.

Example 3.6. In Example 2.5, for the q -sine function, we got

$$u(x) = \sin_q(x) \Rightarrow (1-q)^2 D_q^2 u(x) + u(x) = 0.$$

Now, for $w(x) = u(x^2)$, we have

$$D_q w(x) = f_{0,2}^{(1)}(x)u(x) + f_{1,2}^{(1)}(x)D_q u(x),$$

with

$$f_{0,2}^{(1)}(x) = \frac{qx^3}{1-q}, \quad f_{1,2}^{(1)}(x) = (1+q)x$$

and

$$D_q^2 w(x) = f_{0,2}^{(2)}(x)u(x) + f_{1,2}^{(2)}(x)D_q u(x),$$

with

$$f_{0,2}^{(2)}(x) = \frac{(qx)^2(-2-q-q^2+q^3x^4)}{(1-q)^2}$$

$$f_{1,2}^{(2)}(x) = \frac{(1+q)(1-q+q^2(1+q^2)x^4)}{1-q}.$$

By eliminating $D_q u(x)$, we get

$$D_q^2 w(x) = c_{0,2}(x)w(x) + c_{1,2}(x)D_q w(x),$$

wherefrom we get for the function $w(x) = u(x^2)$ the following equation

$$xD_q^2 w(x) - \left(1 + q^2 \frac{1+q^2}{1-q} x^4\right) D_q w(x) + qx^3 \left(\frac{1-q^4}{(1-q)^3} + \frac{q^2}{(1-q)^2} x^4\right) w(x) = 0.$$

4 Sharpness of the algorithms

In the previous section we proved that the sum, product and composition with powers of q -holonomic functions are q -holonomic too. In this section we show that the given bounds for the orders are sharp in all algorithms considered.

Example 4.1. The functions $u(x) = x^2$ and $v(x) = x^3$ are q -holonomic of first order. According to Theorem 3.2, the function $h(x) = u(x) + v(x)$ is q -holonomic of order at most two. However, all polynomials are q -holonomic functions of first order, and we find that $h(x)$ satisfies the equation

$$x(1+x)D_q h(x) - ([2]_q + [3]_q x)h(x) = 0.$$

This example shows that the order of the sum of some q -holonomic functions can be strictly less than the sum of their orders. This applies if the two functions $u(x)$ and $v(x)$ are linearly dependent over $\mathbb{K}(q)(x)$.

However, we will prove that for every algorithm given in the previous section there are functions for which the maximal order is attained.

Lemma 4.1. *The functions $E_q(x^\mu)$ ($\mu = 1, 2, \dots, n$) are linearly independent over $\mathbb{K}(q)(x)$.*

Proof. Let us consider a linear combination

$$r_1 E_q(x) + r_2 E_q(x^2) + \dots + r_n E_q(x^n) = 0,$$

where $r_\mu = r_\mu(x)$ ($\mu = 1, 2, \dots, n$) are rational functions and suppose that $r_\nu \neq 0$. Then,

$$r_\nu E_q(x^\nu) = - \sum_{\substack{\mu=0 \\ \mu \neq \nu}}^n r_\mu E_q(x^\mu),$$

i.e.,

$$\sum_{\substack{\mu=0 \\ \mu \neq \nu}}^n \frac{r_\mu}{r_\nu} \frac{E_q(x^\mu)}{E_q(x^\nu)} = -1. \quad (18)$$

Since

$$A(m) = \lim_{x \rightarrow \infty} \frac{\sum_{n=0}^m \frac{q^{\binom{n}{2}}}{(q; q)_n} (x^\mu)^n}{\sum_{n=0}^m \frac{q^{\binom{n}{2}}}{(q; q)_n} (x^\nu)^n} = \lim_{x \rightarrow \infty} x^{m(\mu-\nu)} = \begin{cases} +\infty, & \mu > \nu, \\ 0, & \mu < \nu, \end{cases}$$

we have

$$\lim_{x \rightarrow \infty} \frac{E_q(x^\mu)}{E_q(x^\nu)} = \lim_{m \rightarrow \infty} A(m) = \begin{cases} +\infty, & \mu > \nu, \\ 0, & \mu < \nu. \end{cases}$$

This is a contradiction with (18). Hence, it follows that $r_\mu \equiv 0$ for all $\mu = 1, 2, \dots, n$, i.e. $E_q(x^\mu)$ ($\mu = 1, 2, \dots, n$) are linearly independent over $\mathbb{K}(q)[x]$. \diamond

Lemma 4.2. *The function*

$$F_n(x) = \sum_{\mu=1}^n E_q(x^\mu) \quad (19)$$

is q -holonomic of order n .

Proof. The function $E_q(x)$ satisfies the q -holonomic equation of first order (see Example 2.4)

$$(1-q)(t+1)D_q f(t) - f(t) = 0.$$

With respect to Theorem 3.4, for each $\mu \in \mathbb{N}$, the function $E_q(x^\mu)$ is q -holonomic of first order and one has

$$D_q^l (E_q(x^\mu)) = f_{0,\mu}^{(l)}(x) E_q(x^\mu), \quad l = 0, 1, \dots, \quad (20)$$

where $f_{0,\mu}^{(l)}(x)$ are rational functions given as in (17).

According to Theorem 3.2, the function $F_n(x)$ is q -holonomic of order at most n . Therefore

$$D_q^l F_n(x) = \sum_{\mu=1}^n D_q^l (E_q(x^\mu)) = \sum_{\mu=1}^n f_{0,\mu}^{(l)}(x) E_q(x^\mu).$$

Let us suppose that the function $F_n(x)$ satisfies a q -holonomic equation of order m , i.e.

$$D_q^m F_n(x) + \sum_{i=0}^{m-1} A_i D_q^i F_n(x) = 0. \quad (21)$$

This equation can be represented in the form

$$\sum_{\mu=1}^n \left(f_{0,\mu}^{(m)}(x) + \sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) \right) E_q(x^\mu) = 0.$$

Since $E_q(x^\mu)$ ($\mu = 1, 2, \dots, n$) are linearly independent over $\mathbb{K}(q)[x]$, it follows that

$$f_{0,\mu}^{(m)}(x) + \sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) = 0, \quad \mu = 1, 2, \dots, n.$$

This can be written in the form of the system of equations

$$\sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) = -f_{0,\mu}^{(m)}(x), \quad \mu = 1, 2, \dots, n$$

with unknown rational functions $A_i = A_i(x)$.

If $m < n$, then the system is overdetermined and has no solution. Hence it follows that $m = n$. \diamond

Theorem 4.3. *For each $n \in \mathbb{N}$ there is a function F which is q -holonomic of order n , such that $H = D_q F$ is q -holonomic of order n .*

Proof. The function defined by (19) satisfies the statement. \diamond

Theorem 4.4. *For each $n, m \in \mathbb{N}$ there are functions U and V that are q -holonomic of order n and m respectively, such that $H = U + V$ is q -holonomic of order $n + m$.*

Proof. Consider the functions

$$U(x) = \sum_{\mu=1}^n E_q(x^\mu) \quad \text{and} \quad V(x) = \sum_{\mu=n+1}^{n+m} E_q(x^\mu). \quad (22)$$

According to Lemma 4.2, they are q -holonomic of order n and m respectively, and the function

$$H(x) = U(x) + V(x) = \sum_{\mu=1}^{n+m} E_q(x^\mu)$$

is q -holonomic of order $n + m$. \diamond

Theorem 4.5. For each $n, m \in \mathbb{N}$ there are functions U and V that are q -holonomic of order n and m respectively, such that $H = U \cdot V$ is q -holonomic of order $n \cdot m$.

Proof. The statement is valid for the functions defined by (22), because in the function

$$H(x) = U(x) \cdot V(x) = \sum_{\mu=1}^n \sum_{\nu=n+1}^{n+m} E_q(x^\mu) E_q(x^\nu)$$

there are nm linearly independent summands $E_q(x^\mu) E_q(x^\nu)$ ($\mu = 1, 2, \dots, n$; $\nu = n+1, n+2, \dots, n+m$) over $\mathbb{K}(q)[x]$. The proof of their independence is again based on Lemma 4.1. \diamond

Theorem 4.6. For each $n \in \mathbb{N}$ there is a function F which is q -holonomic of order n , such that $W(x) = F(x^\nu)$ is q -holonomic of order n .

Proof. Starting from the function $F_n(x)$ defined by (19), we can form

$$W(x) = F_n(x^\nu) = \sum_{\mu=1}^n E_q(x^{\mu\nu})$$

which is of the same type as $F_n(x)$. \diamond

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