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19

The Problem of the Graphic Artist

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SUMMARY

'The problem of the graphic artist' is a small example of applying elementary mathematics (divisibility of natural numbers) to a real problem which we ourselves have actually experienced. It deals with the possibilities for partitioning a sheet of paper into strips. In this contribution we report on a teaching unit in grade 6 as well as on informal tests with students in school and university. Finally we analyse this example methodologically, summarise our observations with pupils and students, and draw some *didactical* conclusions.

1. THE PROBLEM

The letterhead of Kassel University was designed several years ago by a graphic artist (Fig. 19.1). His task was not only to devise the letterhead but also various official forms for the university. In the course of his work, a problem arose for which he produced piles of paper full of calculations and drawings. One day he came to the Faculty of Mathematics, with a suitcase full of these products, asking for help. There, by chance, he met us. From that time on we have called his problem *The Problem of the Graphic Artist*. It runs as follows.

A sheet of paper is to be divided into strips so that there are various possibilities to partition the sheet into columns of the same width (Fig. 19.2). Between every two columns one strip must be left blank. For example, let us take 11 as the number of strips. There are, among others, the following possibilities:

- 3 columns consisting of three strips each;
- 2 columns consisting of five strips each;
- 4 columns consisting of two strips each.

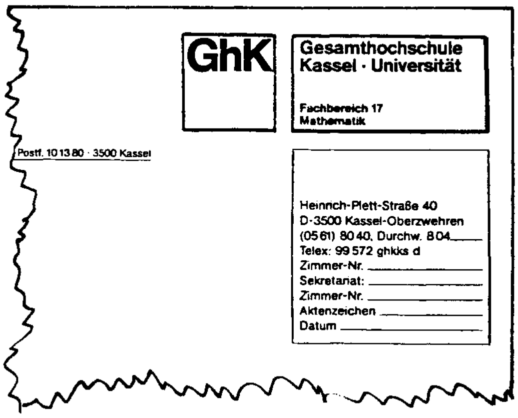


Fig. 19.1

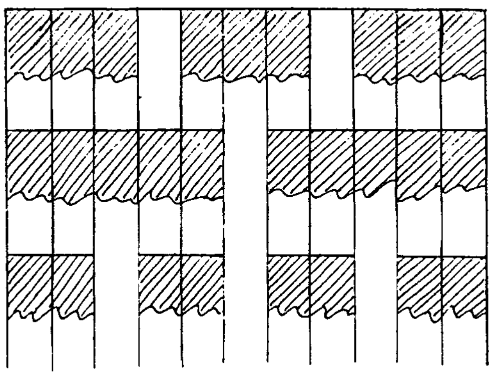


Fig. 19.2

Problem: How many strips should one choose to achieve a particularly large number of such partitioning possibilities and consequently an exceptionally variable lay-out of the sheet?

We gave the graphic artist a *solution* by telling him a general rule how to find all the partitioning possibilities for a given number of strips without experimenting. Today we would probably say we *mathematised* the problem of the graphic artist, and from the resulting mathematical model we deduced a simple method which solved the problem.

In short, we solved the problem of the graphic artist as follows. Let n be the number of strips, A the respective number of columns and B the width of each column (i.e. the number of strips per column). Then clearly (let us imagine an additional blank strip on the right!)

$$n + 1 = A(B + 1) \qquad (n, A, B \in \mathbb{N})$$

Therefore, the partitioning possibilities correspond exactly to all divisors, that is multiplicative decompositions, of $n + 1$ (except for $(n + 1)1$).

We are not quite sure whether the graphic artist actually designed the writing paper of Kassel University according to this solution. However, we think he really did, for the writing paper fits the numbers $n = 35, A = 6, B = 5$ very well.

Since then, we have often given this problem to pupils and students, for in spite of or perhaps because of its simplicity it contains some interesting and substantial *methodological and didactical* aspects (see section 4). In the following (sections 2 and 3) we report on some experiences with regard to this.

2. A TEACHING SEQUENCE IN GRADE 6†

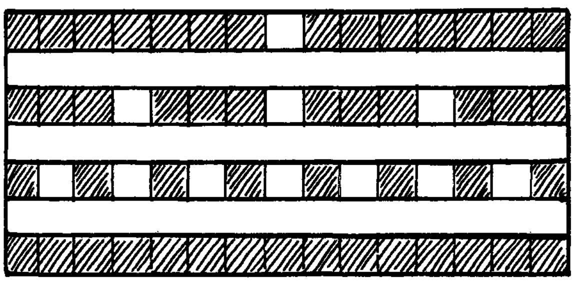
In *preparation* for the topic the divisibility of natural numbers is repeated in the class (34 pupils, aged 11–13). The pupils are set the question 'Which numbers have many divisors?', using as example the distribution of chocolate bars.

After that the teacher presents *the problem of the graphic artist*, as formulated in section 1. He asks specifically for possible 'partitions with blanks' (*Lücken-Zerlegungen*) of a 14-part-bar of the following kind shown in Fig. 19.3. This provides strong motivation for the pupils. They see this immediately as a *mathematical problem* and no longer consider the specific concern of the graphic artist.

Fig. 19.3 –

Here are some *observations*.

- The problem is grasped after only *one* example.
- *Experimental* solutions for special bar lengths n(14, 15, 17, 12, 13, 16, etc.) cause no difficulties.
- The partitions in questions are given *graphically* as well as in *tabular* form, e.g. for $n = 15$ as in Fig. 19.4.



A number of pieces	B width of each piece
2	7
4	3
8	1
1	15

Fig. 19.4

- Parallel to that *general statements* are made, and conjectures are expressed and either proven or refuted. For instance: 'With even numbers n it doesn't work with two pieces'; 'If it works with four pieces of width 3 then it also works with three pieces of width 4'.

† Three lessons, planned and realised by A. Kirsch, together with V. Dippel, Gesamtschule Kaufungen.

- The pupils are surprised that the 'lovely' numbers 12 and 16 allow only *one* partition.
- Pupils speculate on a general solution of the problem, especially 'Prime numbers have many partitions', and make the criticism '13 has only three partitions, just as 9'.

Then the teacher gives *clues*:

- 'Which numbers have very few partitions? What strikes you about it?'
- 'What do you notice about the numbers of pieces 1, 2, 4, 5, 10, and 19?'

In spite of these clues and obvious examples there still was not one pupil who saw the point. Only the very strong clue

- 'What if we added another (artificial) blank on the right?'

lead to a breakthrough and to *statements* from the pupils such as

- 'It has something to do with the divisors of the next biggest number. If *it* has many divisors then there are many partitions.'

This is confirmed from the bar lengths already examined. Then it 'rained' pupils' suggestions for suitable bar lengths (in this order): 59, 29, 35, 47, 83.

The teacher concludes the unit with a *summary of the results*: 'The possible numbers of pieces for a given n are the divisors of the next biggest number $n + 1$, except $n + 1$ itself. The corresponding widths of pieces are always smaller by one than the complementary divisors.'

The *interpretation* of these results in the original problem was evident to the pupils.

3. INFORMAL TESTS WITH HIGH SCHOOL AND UNIVERSITY STUDENTS

The following test paper (originally in German, of which this is an extract) was given to students without any preparation (Fig. 19.5).

3.1 Some *results*†

(1) 15 students, basic course (Grundkurs) grade 12

For question (c) ten students start a solution with variables (four of them use the letters x, y instead of the column headings A, B). Eight of them try to translate the situation into a formula; only five arrive at an adequate equation such as $(n - A + 1)/A = B$, which is a direct translation of the situation in question. However then they make to transformation, and no student achieves a usable result. The same holds for the attempts of the remaining students (two with case distinction n even/odd, two with the rudiments of an equation of a function or a graph in the x–y-plane, one with the help of a table). For question (d) nine students simply express the supposition that odd numbers result in more partitionings, the others say nothing.

† Our thanks to H. Kammer for evaluating the test papers for (1) and (3).

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A newspaper page is to be divided into strips so that
there are various possibilities to partition the sheet
into columns of the same width. Between every two
columns one strip must be left blank.

Example:  14  strips
.................................................

a) Examine in the same way:  11  strips  ............

b) Make up similar  tables for some further numbers  n
   of strips, chosen by yourself, instead of  n = 14
   or  n = 11.

c) Formulate as a general rule: How you arrive at
   all  partitioning possibilities for a given number
   n  of strips without experimenting?

d) Which numbers  n  would you choose to achieve a
   particularly large number of partitioning
   possibilities?
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Fig. 19.5

(2) 7 primary school student teachers
Here a strong experimental procedure can be noticed. On the basis of observations to question (b), students try in vain to find general answers to questions (c) and (d).

One student produces a table (essentially correct) of all partitions for numbers up to $n = 19$. Another arrives – without proof – at the result 'The numbers $n = m \cdot 11 + m - 1$ are favourable' (which is favourable indeed because the multiples of 12 have many divisors). Only one student makes an attempt with variables: $n = A \cdot B + x$, $x = A - 1$, but doesn't transform this and therefore doesn't arrive at a usable result.

The rest give at best further solutions to question (b).

(3) 26 secondary school student teachers
20 students start with variables. 11 of them, by way of equations like $A \cdot B + A - 1 = n$, achieve essentially a solution to questions (c) and (d) by correctly using the calculus of transformation; but only seven formulate this solution completely clearly. The remaining nine students come close to a solution but then give up.

The first attempts of the other 6 students (especially with case distinction n even/odd or with an extension of the observations to (b)) don't lead to a usable result.

For question (d) the students, including those mentioned previously, often suppose prime numbers or odd numbers to be favourable.

In general it can be stated that there were no cases of conscious model building. The setting up of an equation for a description of the situation can, however, be interpreted as an unconscious but adequate model building (see section 4). In few cases only (and then only with secondary school student teachers) were there some mathematical conclusions by schematic transformations. In even fewer cases could these results be interpreted reasonably and pursued to a completely clear solution of the given problem.

It is interesting to note that question (d) was obviously frequently misunderstood in the sense that an absolute answer was required instead of (as the problem demanded) a relative one: 'Those numbers where the next biggest number has many divisors.'

4. SOME METHODOLOGICAL AND DIDACTICAL REMARKS

If we take as a basis one of the usual simplified schemata for the complex relationship between the real world and mathematics (see Blum, 1985) (Fig. 19.6), our example can be analysed as follows.

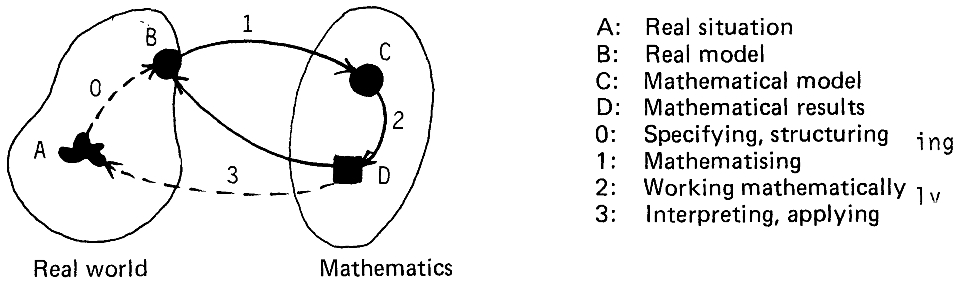

Fig. 19.6

The problem of the graphic artist is already stated as a *real model* close to mathematics. Just as in many word problems this immediate closeness to mathematics leads pupils and students to consider this problem at once as a mathematical one and to forget the real context.

This does not mean, however, that by this the actual mathematisation has already been carried out. This consists here in identifying the relevant variables, in finding an appropriate correlation between these variables, as a formula or also in words, as well as in translating the original question into a mathematical one. Depending on the way the situation is observed, various equations can be set up, all meaningful, for example $n = A \cdot B + (A - 1)$ or $(n - 1 + A)/A = B$. Every such equation – together with determining the meaning of the variables are formulating the question – can be considered as an appropriate *mathematical model* of the problem of the graphic artist. This mathematisation apparently represents the *first* great *difficulty* for pupils and students. Comparable difficulties are well-known when learners are working on word problems that they 'haven't had'. Here, obviously, intellectual abilities are necessary which in today's mathematics instruction in school and university are seldom required and hardly encouraged.

The next step consists in transforming the established equation and especially in drawing suitable *conclusions*, particularly in gaining the essential insight 'Everything depends on the divisors of $n + 1$', which also answers questions such as (c) and (d) in the test paper. Here, the *second* great *difficulty*, even for university mathematics students, can be seen. After all, some mathematics students have achieved some usable mathematical results by schematically transforming the established equation. So, to this end, mathematical studies seem to qualify best of all.

The final step, *interpreting* the achieved mathematical results in the original problem, is not difficult here. However if – as in the case of real problems – the solution is to be translated back into the original real situation and if practical conclusions are to be drawn, a *third difficulty* can be seen – an impressive example of which we ourselves could observe with our graphic artist.

In passing, the simplicity of the mathematics used seems to have been of little help to the individuals confronted with the problem of the graphic artist. On the contrary, the *discrepancy between simplicity and efficiency* seems rather to have caused additional difficulties. Presumably this is connected with the picture of mathematics transmitted in school and university. In any case, the problem of the graphic artist makes impressively clear that what are important for successful applied problem solving are not beauty and profundity of the mathematics involved nor elegance and refinement of the conclusions but the *effect* of the mathematisation (according to H. Dinges; see Kirsch, 1983).

A *fourth difficulty* shows up on a more exact analysis of the behaviour of the pupils and students. All have problems in proceeding towards a goal, in keeping things in focus, in organising their own actions in a meaningful fashion, and so on. In short, they lack general *problem solving strategies*, a consciousness for steps to be taken within a methodological schema. This, too, seems to be too little encouraged in mathematics instruction.

Some *didactical conclusions* are evident. If mathematics in school and university is to be taught and learned in close connection with the real world (for which there are various reasons; see, e.g., Blum, 1985) then it is necessary firstly to specifically *foster* the relevant *abilities* (especially 'translation qualifications') by means of suitable examples and secondly not only to take the essential steps in the circular process of applied problem solving but also to make the students *aware* of them.

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