In: Educational studies in mathematics, Editor: Dörfler etc., Volume 22,
Kluver Academic Publishers, 1991, Dordrecht, S. 183-203

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PREFORMAL PROVING: EXAMPLES AND REFLECTIONS[1]

ABSTRACT. The starting point of our reflections is a classroom situation in grade 12 in which it was to be proved intuitively that non-trivial solutions of the differential equation $f' = f$ have no zeros. We give a working definition of the concept of preformal proving, as well as three examples of preformal proofs. Then we furnish several such proofs of the aforesaid fact, and we analyse these proofs in detail. Finally, we draw some conclusions for mathematics in school and in teacher training.

1. A CLASSROOM SITUATION AND RESULTING PROBLEMS

In a school practical course in January 1988 (supervised by the first author) a student teacher introduced the exponential function in a grade 12 class of a Kassel school (18-year-olds) as the solution of the differential equation $f' = f$ (with $f(0) = 1$). The class had been previously introduced to differential calculus in the classical school way (i.e. geometrically oriented) and had dealt with some intra-mathematical applications, especially curve sketching, with polynomial functions.

After the idea of directional field was acquired in that lesson, the field to $f' = f$ was plotted (the lower half plane was excluded; see Fig. 1). The

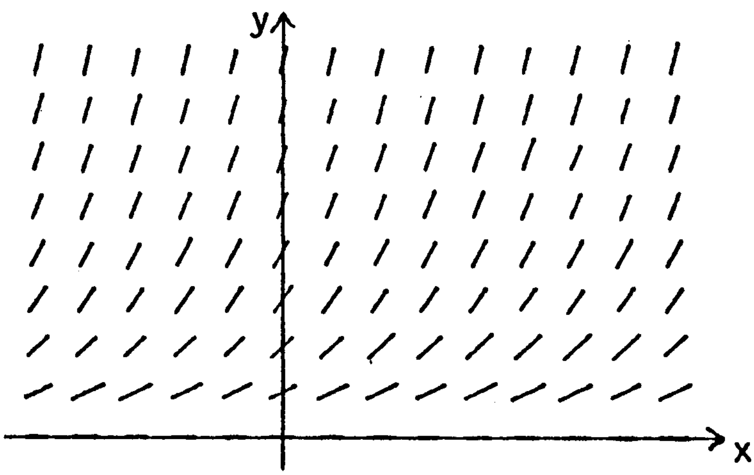

Fig. 1.

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trivial solution $f = 0$ was soon found by the students, and non-trivial solutions were sketched in the directional field. After only a short while, students asked: "*Is it possible for such a (non-trivial) function to have zeros?*"

In his lesson plan, the teacher had intended to play down this problem. Hence he gave the students only the following *intuitive argument*: "Imagine us being on the x-axis. There, the slope is equal to 0 everywhere. When we now move to the right in the directional field, we are always forced to move horizontally, i.e. we can never get away from the x-axis. Therefore . . . ". Most of the students were satisfied with these arguments.[2] With some students, though, a certain unease remained, which resulted in various inquiries. The teacher, however, succeeded then in reassuring the students (essentially by repeating his arguments) and in turning the students' attention towards a new question, namely how to find numerically a concrete (approximate) solution of $f' = f$.

On the day after that lesson, as it happened, we attended a lecture given by E. Wittmann at Paderborn University entitled "When is a proof a proof?" ("Wann ist ein Beweis ein Beweis?"; see Wittmann/Müller, 1988). In accordance with Branford (1908), Wittmann distinguishes between *three levels of proving*:

- Experimental "proofs"
- "Inhaltlich-anschaulich"[3] (Branford: "intuitional") proofs
- Formal ("scientific") proofs

In characterizing the second step, Branford stated (p. 97): "This kind of evidence establishes general and accurate truths, but appeals implicitly to postulates of sense-experience whenever necessary: founds the truth on an independent basis of its own by direct appeal to first principles." Wittmann stresses once again that the borderline between "proofs" (which are none) and (real) proofs does not run between the second and the third level, i.e. between "inhaltlich-anschaulich" and "purely logical" proofs, but between the first and the second, i.e. between a mere "verification of a finite number of examples" and substantial argumentation on a non-formal basis.

With convincing arguments (see also Wittmann, 1989), he pleads for an emphasizing of "inhaltlich-anschaulich" proofs in school and in teacher training. We too have demanded several times that in mathematics instruction students should "practise and develop the ability to argue. Here we do not think primarily of formal proofs but of a meaningful arguing which, by all means, should be correct and intellectually honest" (Blum/Kirsch 1979, p. 7).

The intuitive argumentation of the teacher described above seems to be (and also seemed to Wittmann in the discussion after his lecture) a good example of how to redeem such demands.

However, the *problematic nature* of the example (and thereby the heart of the students' unease as described) suddenly becomes clear on transferring literally the argumentation to (for instance) the differential equation $f' = \sqrt{f}$ (or, more generally, to $f' = f^{1-\varepsilon}$ with $0 < \varepsilon < 1$). (Again we confine ourselves to $y \geqslant 0$, i.e. to the upper half plane including the first axis.) The directional field looks quite similar here (see Fig. 2), in particular the slope on the x-axis is again equal to zero everywhere. But now, as is known, the non-trivial solutions also have zeros. In fact, for each $c \in \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x < c \\ \frac{1}{4}(x-c)^2 & \text{if } x \geqslant c \end{cases}$$

(more generally[4]: $(\varepsilon(x-c))^{1/\varepsilon}$) is a solution, and these are all non-trivial ones. Therefore all points of the x-axis are branching points. From there solutions do indeed come away from the x-axis.

The existence of branching points shows that the intuitive understanding, that the directional field uniquely indicates how to continue, is not viable. The quoted "proof" is not in order, and this not only in the sense that a formally arguing mathematician could given pathological counterexamples. Nor can the "proof" in any way be justified to a physicist, for instance, arguing with common sense, who would classify such counterexamples as

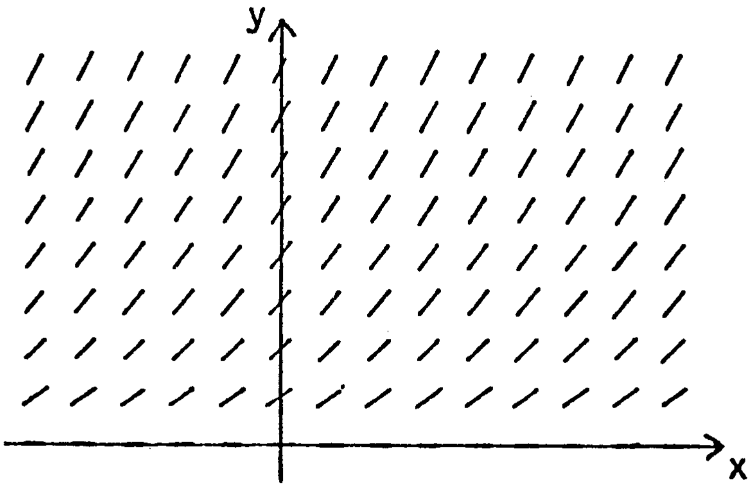

Fig. 2.

irrelevant.[5] And our counterexample is not pathological, rather it is geometrically evident (and not only formally verifiable) that the described (quadratic) solutions fit the new directional field.
Herewith we are faced with the following tasks:

- To describe more accurately the concept of non-formal proving.
- To find a "real" non-formal proof of our assertion that a non-trivial solution of $f' = f$ has no zeros.

2. AIMS OF THIS PAPER

More precisely, we have three essential aims in this paper.

First, our reflections and examples are meant to be a *plea for* doing *mathematics on a preformal level*. For, like many others, we too deplore the fact that all too often in the classroom a formalistic understanding of mathematics prevails, and we too keep on making suggestions as to how mathematics should be learned and taught in a sensible and meaningful way. Here, it should also become clear once again that working on a preformal level is not intended to make mathematics easier for students in a superficial way. Certainly, our preformal proofs are meant to be as obvious and natural as possible especially for the mathematically less experienced learner. They require, however, a substantial engagement in the topic in question, for instance by referring to geometric-intuitive basic conceptions or to basic ideas meaningful in reality. In this respect, these preformal proofs are not "simpler" than the usual formal proofs, in particular for the experienced mathematician. To give such formal proofs, moreover the shortest and most elegant possible ones, is by no means our aim.

Second, we wish to point to certain *didactical problems* in connection with preformal proofs, which seemingly are not seen in the same way by some of those who support such proofs. Of course, we do not wish to argue against working on a preformal level. We think, however, that teachers as well as learners should be aware of these problems, and that they should also have consequences for instruction.

Third, we wish to indicate through the example mentioned in section 1 what we mean by *preformal proving*. In doing so, we do not aspire to give a definition of this concept which is substantiated according to scientific theory. Rather we wish to contribute to the *didactical discussion* on doing mathematics at school on a preformal level, as it has been conducted, especially in the German speaking area, for a long time (essential aspects of this discussion can be found, for example, in Kirsch, 1977 or Wittmann, 1989). Such tasks can, in our opinion, neither be completed by a scientist who is purely a mathematician nor by a psychologist or a pedagogue. Rather, such activities genuinely belong to the heart of mathematics education.

3. ON THE CONCEPT OF PREFORMAL PROVING

Without any claim to a methodological substantiation we shall try to give a pragmatic working definition of the concept of proof discussed here.

In accordance with Z. Semadeni's concept of "action proofs" (see Semadeni, 1984), we mean by a *preformal proof*[6] a chain of correct, but *not formally represented conclusions* which refer to valid, *non-formal premises*. Particular examples of such premises include concretely given real objects, geometric-intuitive facts, reality-oriented basic ideas, or intuitively[7] evident, "commonly intelligible", "psychologically obvious" statements (the latter in loose accordance with Thom 1973). The conclusions should succeed one another in their "psychologically natural" order. For us – in contrast to Semadeni – , inductive arguments ("etc.") and indirect arguments ("imagine that . . ." or "what would happen if . . .") should not be excluded in this context. The conclusions must be capable of being generalized directly from the concrete case. If formalized, they have to correspond to correct formal-mathematical arguments. To accept a preformal proof it is, however, not necessary for such a formalization to be actually effected or even recognizable. Occasionally, the consensus within the mathematical scientific community is quite sufficient. (For this, formal rigour is by no means necessary, as has been conclusively shown by Hanna, 1983, for example; compare also Hanna, 1989.)

What are "intuitive" or "obvious" bases for argumentation, has to be decided in each individual case by the persons involved on the basis of their knowledge. Such bases can, of course, be changed in course of time, in particular by learning or experience. So far "preformal" is necessarily not a precisely defined property. In any case, however, preformal proofs have to be *valid*, *rigorous* proofs. Thus we wish to emphasize explicitly that for us "rigorous" is by no means equivalent to "formal".

According to how the premises or the conclusions are represented, we get different kinds of preformal proofs. In the following we give *three examples*.

An *action* proof (in a narrow sense) consists, in short, of certain concrete actions (actually carried out or only imagined) with a concretely given paradigmatic, generic example, where the actions correspond to correct mathematical arguments. An example is the "red wine proof" of the theorem "The hexagon generated by the centres of the edges of a cube is planar and regular" (formulations according to Heidenreich, 1987). Here, a hollow cube is set up, so that one spatial diagonal is perpendicular, and is filled completely with red wine. Then one lets the wine slowly flow out on an opening at the bottom of the cube, and observes the surface of the wine. In doing so, one can comprehend immediately (and not only verify by experiment) that the hexagon in question is planar and regular (for details see Heidenreich, 1987). Further examples of action proofs can be found in Kirsch (1979).

A *geometric-intuitive* proof refers to basic geometric conceptions and to intuitively evident facts such as areas and their properties (see Kirsch 1977, p. 109). An example: If definite integrals are interpreted as areas, then the monotonicity of the integral function of a non-negative integrand can – as is well-known – be proved by immediately obvious geometric arguments (see Fig. 3): For functions $f \geqslant 0$ we have

$$(a \leqslant) x \leqslant y \Rightarrow \int_a^x f \leqslant \int_a^y f.$$

In a *reality-oriented* proof, basic ideas meaningful in reality and easily accessible for learners are used, such as the derivative as a local rate of change. A well-known example: The theorem "If $f' = 0$ in I then $f = \text{const}$ in I" can be made evident immediately by interpreting f as a distance-time-function and f' as the instantaneous velocity. ("If the speedometer is showing zero all the time then the car has always been standing still".)

The examples just mentioned are meant to make the concept of preformal proving somewhat clearer. We do not claim that we have succeeded in accomplishing that task satisfactorily. We even doubt whether it is possible at all – at least in the given framework – to make this concept more

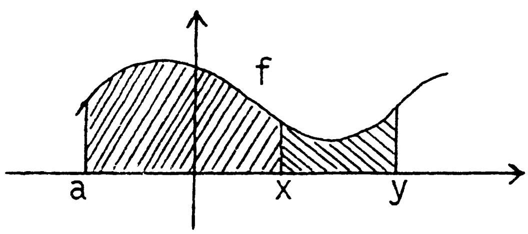

Fig. 3.

precise, for ultimately we believe (to paraphrase Semadeni)[8] that it requires a competent mathematician to judge whether a given preformal proof is acceptable. By "competence" we mean here possessing a certain mathematical maturity and sophistication – notions which defy attempts to define.

From preformal proofs we must separate, on the one hand, merely *experimental* verifications or merely *heuristic* argumentations guided by isolated cases, or "incomplete induction" after a confirmation in special cases, and, on the other hand, purely *formal* proofs (or – even continuing further in this direction – calculized proofs in the sense of formal logic and proof theory; this is not of importance for our purposes). So we distinguish – like Wittmann does – between three levels of proving[9]: "Proofs"; Preformal proofs; Formal proofs. Of course, these levels cannot be strictly separated from one another, rather there are fluent transitions. Of particular interest is the question as to how far the borderline between "proofs" and proofs is shifted towards the first level by the possibilities of experimenting with computers.[10]

Now, what could a preformal proof for our problem of section 1 look like? In the next section we give different proofs which we regard as geometric-intuitive or as reality-oriented.

4. PREFORMAL PROOFS OF THE POSITIVENESS OF NON-TRIVIAL SOLUTIONS OF $f' = f$

In the following again we will confine ourselves to the upper half plane including the first axis, i.e. we will consider only solutions f of $f' = f$ with $f(x) \geqslant 0$ for all $x \in \mathbb{R}$. This is justified by the fact that there are certainly no solutions having both positive and negative values. (This fact is immediately clear after extending the directional field to the whole plane $\mathbb{R}^2$: The part of the graph above the x-axis would have to be increasing, the part below the x-axis would have to be decreasing.)

Because $f(x) \geqslant 0$ we also have $f'(x) \geqslant 0$ for all $x \in \mathbb{R}$, i.e. all solutions are increasing, in any case (at least weakly); cf. Fig. 1. Thus it is clear: If a solution is somewhere positive, i.e. above the x-axis, then all the more so on the right; if a solution is somewhere zero, i.e. on the x-axis, then all the more so on the left. We shall now show, independently of each other:

A: If a solution f of $f' = f$ is somewhere positive, then it can never have been zero; i.e. if a solution is somewhere above the x-axis, then it can never come down to it on the left either. Formally; If $f(x) > 0$ for $x = x_0$ (thus for $x \geqslant x_0$) then also for $x < x_0$ and therefore for all $x \in \mathbb{R}$.

B: If a solution f of $f' = f$ is somewhere zero, then it can never become positive; i.e. if a solution is somewhere on the x-axis, then it can never come off on the right either. Formally: If $f(x) = 0$ for $x = x_0$ (thus for $x \leqslant x_0$) then also for $x > x_0$ and therefore for all $x \in \mathbb{R}$.

4.1. *Two Geometric-Intuitive Proofs*

From the directional field it is obvious at once that all solution curves are convex, above the x-axis even strictly convex, i.e. left-hand curves. This immediately follows from $f''(x) = f'(x) = f(x) \geqslant 0$ resp. $f(x) > 0$.

Proof of A: Let the solution f of $f' = f$ be somewhere positive: $f(x_0) > 0$. The tangent of f at $(x_0|f(x_0))$ has the slope $f'(x_0) = f(x_0)$. It therefore intersects the x-axis at $x_0 - 1$, regardless of the value $f(x_0)$ (Fig. 4). Because f is a left-handed curve it runs strictly above the tangent everywhere except at x_0. Consequently f is positive in the whole interval $[x_0 - 1; x_0]$. Now we repeat this argumentation until $x_0 - 2$, then until $x_0 - 3$, "and so on". Herewith assertion A is proved.[11]

Proof of B: Let the solution f of $f' = f$ initially be zero: $f(x) = 0$ for $x \leqslant x_0$. We imagine that somewhere it becomes positive. Then we determine the smallest integer n where this is the case: $f(n) > 0$, $f(n-1) = 0$. For the sake of simplicity we take $n - 1$ as the new zero point of the x-axis,[12] so we have $f(0) = 0 < f(1)$. Now the slope $f'(1)$ of the graph at the right end of the interval $[0; 1]$ is certainly bigger than the slope $f(1)/1$ of the chord over this interval (Fig. 5). For the slope $f'(0) = f(0)$ at the left end is smaller than this slope of the chord, so the convex graph of f between these two points runs strictly below the chord and intersects it (in the narrow sense, i.e. does not touch it). Therefore

(1) $\quad f'(1) > f(1)$

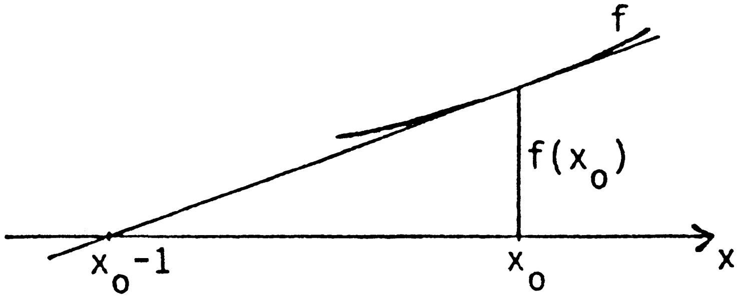

Fig. 4.

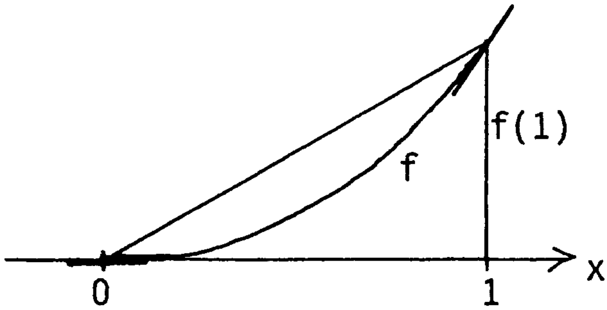

Fig. 5.

must be the case, which from the premise $f' = f$ is impossible. Herewith assertion B is proved.

Both proofs remain valid after slight modification even for the analogous statement for the solutions of the more general differential equation $f' = kf$ with $k > 0$. Then in case of A the tangent in $(x_0|f(x_0))$ intersects the x-axis at $x_0 - 1/k$, and we consider the intervals $[x_0 - n/k;\ x_0]$ $(n = 1, 2, 3, \ldots)$ instead of $[x_0 - n;\ x_0]$. In case of B we determine the smallest number n/k with integer n and $f(n/k) > 0$, and instead of (1) we get (after a suitable translation of the zero point) the inequality $f'(1/k) > f(1/k)/(1/k) = kf(1/k)$.

4.2. *A Further Geometrical Proof*

The following proof of B leads us by another way to our goal. It was found many years ago (in 1946) during lessons and – perhaps because of its preformal character – has not been forgotten over the years. Its intrinsic connection with the proof given in 4.1 will become visible in 4.3.

We consider continuous and monotonic functions $f \geqslant 0$ which initially, but not everywhere, are zero. So we can assume as discussed that $f(0) = 0 < f(1)$ holds. Now obviously the content $\int_0^1 f$ of the area below the graph of f in $[0;\ 1]$ is less than the rectangular area $1 \cdot f(1)$ (Fig. 6). Therefore, if a function of the described type rises from the x-axis, then

$$(2) \qquad f(1) > \int_0^1 f$$

must hold. Now, however, one sees immediately that this is impossible for solutions of $f' = f$ (which are continuous and monotonic). For $f' = f$ leads from (2) – because $f(0) = 0$ – to $f(1) > \int_0^1 f' = f(1)$ which obviously cannot be the case. Herewith assertion B is once more proved.

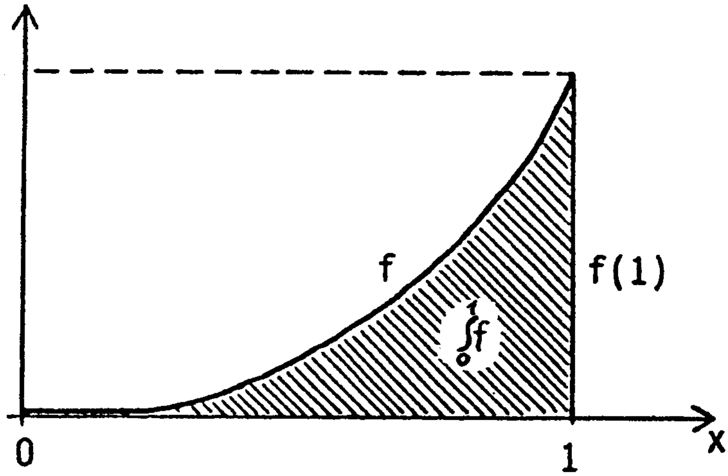

Fig. 6.

The *intuitive ideas* which at that time after some efforts led to this proof can only be reconstructed rather vaguely: The function f is an antiderivative of itself, i.e. its value equals (because $f(0) = 0$) at each point the content of the area to its left. Why can such a function never come up? Because the small area which it would generate beside itself upon coming up, would from the first moment on have to have the same content as the rectangle of the same height on the right and the width 1. This, however, is obviously impossible. Very roughly speaking: The function does not succeed in coming up, because it has to "drag its whole past history along with it", so to speak.

Of course, from (2) and $f' = f$ it also follows at once that $f'(1) > \int_0^1 f' = f(1)$ and thus (1), which has already been seen in 4.1 to be impossible.

4.3. *Kinematic Interpretations*

The proofs in 4.1 and 4.2 can also be interpreted kinematically and then require no previous geometric knowledge. Also our assertions then appear in a new light. This is elaborated only for B.[13]

Let $s \geqslant 0$ be the distance covered as a function of the time t, and $v = \dot{s}$ the velocity. For reasons of dimension we now consider the more general differential equation $\dot{s} = k \cdot s$, with a constant $k > 0$ of the dimension 1/time. For every solution we then have $v \geqslant 0$, v continuous and v increas-

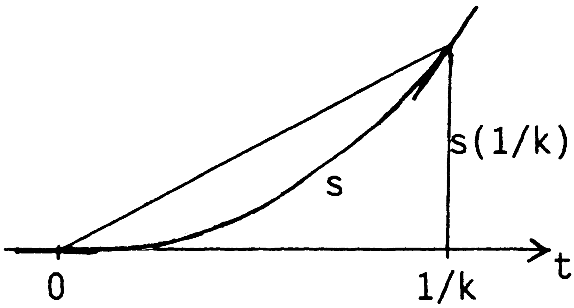

Fig. 7.

ing just like s. The equation says that the respective terminal velocity is proportional to the distance covered, and our assertion B says that under these conditions a vehicle, standing still at some time (at the initial point of distance measuring, $s(0) = v(0) = 0$), can never really move off.

In order to prove this, let us again imagine that the vehicle does move off. This now leads, by geometric arguments as in 4.1 (see Fig. 7), to

$$(1') \qquad \dot{s}(1/k) > \frac{s(1/k)}{1/k},$$

in words: The terminal velocity at $t = 1/k$ is greater than the quotient distance/time, i.e. than the average velocity in the time interval $[0; 1/k]$. But this is impossible from the premises $\dot{s} = k \cdot s = s/(1/k)$ and therefore proves the assertion B as in 4.1.

Now, however, if the concepts involved are familiar, the preceding statement (1′) is immediately evident because of its kinematic meaning, without reference to the geometric argumentation in 4.1. Thus, the kinematic interpretation has provided a *third proof* of our assertion B, now a reality-oriented one instead of a geometric one.

The kinematic interpretation suggests representing graphically (as a function of time) not the distance s, but the velocity $\dot{s} = v$, which under the given conditions is proportional to s. Thus, we come to Fig. 8, where, as well-known from physics, the distance $s(1/k)$ covered by time $t = 1/k$ is represented by the area below the graph in $[0; 1/k]$. Now, the geometric argumentation from 4.2 leads to

$$(2') \qquad 1/k \cdot v(1/k) > s(1/k),$$

in words: The product of time and terminal velocity is greater than the distance covered. But this essentially means nothing else but the kinematic statement just discovered, and naturally this is just as immediately evident.

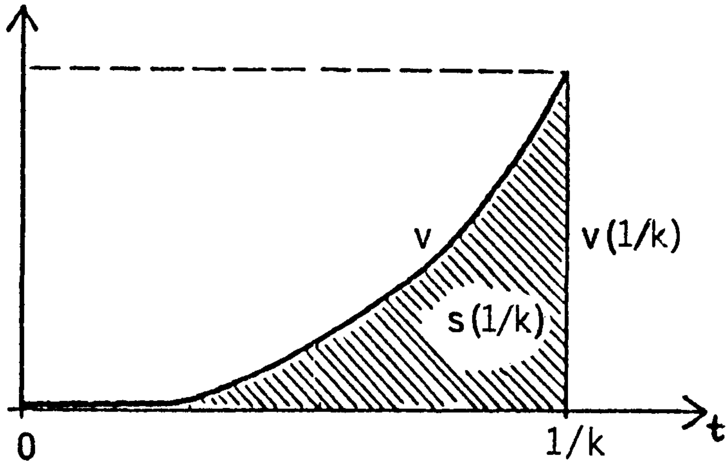

Fig. 8.

In formal respects also, the inequalities (1′) and (2′) are apparently equivalent, for both say that $\dot{s}(1/k) > k \cdot s(1/k)$. So, just like (1′), (2′) provides a kinematic proof of the assertion B.

Thus at the same time the close connection between (1) and (2) and thereby between the two geometric proofs of B becomes clear.

4.4. *Supplement*: *Drive-Off Processes*

In other words, we have showed in 4.3: A drive-off process with terminal velocity proportional to the distance covered is impossible.[14]

This fact was pointed out in another context in Stahel (1985). The author rightly criticizes an advertisement of the BMW company (see Fig. 9) in which different drive-off processes are compared. In these, the velocity is initially proportional to the distance, which gives a line through zero in the distance-velocity diagram. To prove the impossibility of such drive-off processes, Stahel presupposes the well-known unique solvability of $\dot{s} = ks$, $s(0) = 0$ (as just now proved by us).

Further he asks about the form of *possible drive-off processes*, in which, consequently, the vehicle really starts moving at a time $t = t_0$: $v(t) = 0$ for $t \leqslant t_0$, $v(t) > 0$ for $t > t_0$. (Such a time t_0 always exists for continuous and monotonic $v \geqslant 0$, $v \neq 0$ with $v(0) = 0$. This is intuitively clear and can be proved formally, e.g. with Dedekind cuts.) Without loss of generality we can set $t_0 = 0$. Stahel now remarks that each such drive-off process is reflected in the distance-velocity diagram as a curve having a vertical

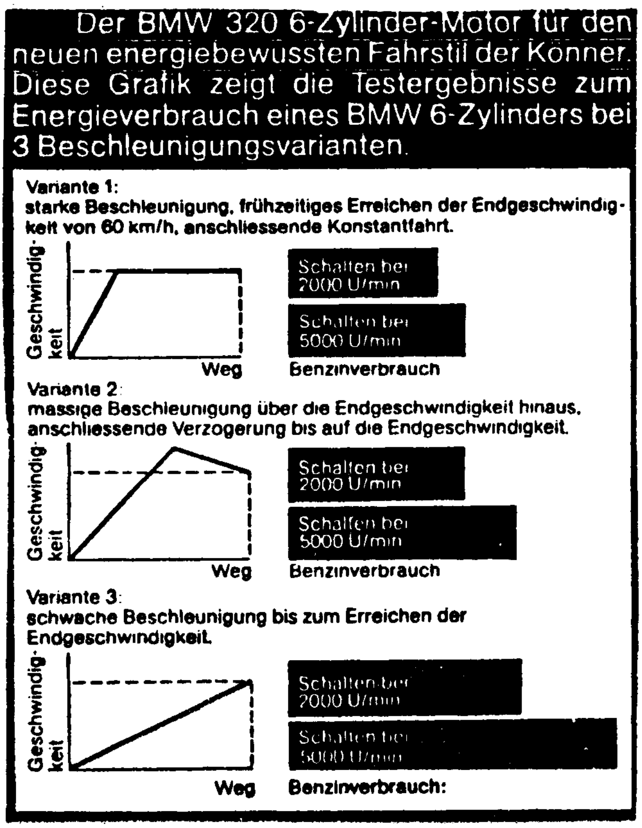

Fig. 9.

tangent at the zero point. After our preceding considerations, this likewise is evident at once. Because for all $t > 0$ we have

$$t \cdot v(t) > s(t), \quad \text{therefore} \quad \frac{v(t)}{s(t)} > \frac{1}{t}$$

(notice that now $s(t) > 0$ for all $t > 0$), and immediately it follows that $\lim_{t \to 0^+} v(t)/s(t) = +\infty$.

Stahel gives a formal proof of this by means of the rule of de l'Hospital and derivatives of any high order, which is not valid, however. He uses, namely, the assertion that one of the derivatives of $\dot{v}$ is positive ("because our vehicle is to move off"). However it is known that there are functions that increase from zero even though all their derivatives vanish at zero (e.g. f with $f(t) = e^{-1/t^2}$ for $t \neq 0$ and $f(0) = 0$). So in this case the preformal mode of arguing has even protected us from making a mistake.

5. ANALYSIS OF THE PROOFS PRESENTED

In what respect are the proofs presented in section 4 *preformal*? In the answer to this question following, it will at the same time become clear what mathematics instruction in our opinion should achieve inter alia, namely that the basic conceptions and ideas used in these proofs are actually obvious and familiar to learners.

The two *proofs in 4.1* are *geometric-intuitive* in so far as they are built up in quite a natural way (we will come back to this shortly) and use only the well-known geometric interpretations of the basic concepts of differential calculus as well as evident facts expressed by these, such as the monotonicity criterion in the form "If all tangents go upwards then the graph goes uphill".

An important role is played by the convexity criterion: "If the tangents become steeper and steeper then the graph curves to the left". What is essential here is the use of evident but formally not very easily provable properties of convex curves. In A: The tangent to a strictly convex curve runs completely on one side of the curve, except for the tangential point. In B: The chord of a convex curve intersects the curve (in the narrow sense) at both extreme points, unless they coincide.

The inductive argumentation "$f(x_0) > 0$, thus $f(x) > 0$ in $[x_0 - 1; x_0]$, thus $f(x) > 0$ in $[x_0 - 2; x_0 - 1]$, and so on" then following in A may certainly be termed obvious and natural and requires no formalization as mathematical induction.

The same holds for the indirect argumentation in B. There we presume as known that every solution $f \geqslant 0$ of $f' = f$ is differentiable and convex. Then we show, geometrically and without referring to $f' = f$, that for every differentiable convex function $f \geqslant 0$ with $f(0) = 0 < f(1)$ the inequality (1) $f'(1) < f(1)$ holds,[15] hence $f' = f$ cannot be fulfilled. This shows in quite a natural way that there cannot be a solution $f \geqslant 0$ of $f' = f$ with $f(0) = 0 < f(1)$, that is a non-trivial solution with zeros.

The *proof in 4.2* is *geometric-intuitive*, too. It again uses the well-known geometric interpretations of the basic concepts of calculus, now particularly the area of the ordinate set of a non-negative function as an interpretation of the definite integral (which is calculated in the usual manner by means of the fundamental theorem of calculus) and the absence of leaps in the graph as a property of continuous functions.

Further, an essential and equally apparent property of area is used: If M_1, M_2 are (measurable) point sets in the plane with $M_1 \subseteq M_2$ and if $M_2 \backslash M_1$ contains inner points, then the content of the area of M_1 is less than that of M_2. The last-mentioned fact exists if for M_1 we take the hatched ordinate set of f and for M_2 the sketched rectangle in Figure 6. For, the continuous function f cannot leap immediately from the value 0 at $x = 0$ to the value 1 and therefore leaves blank a part of the area (with inner points) in the upper left corner.

All the facts used here (including the fundamental theorem) are intuitively accessible. Their formalization is quite obvious for the expert (probably easier than in 4.1) but not necessary for a substantial comprehension of the proof – which therefore turns out to be typically preformal just at this point.

The indirect argumentation in the proof in 4.2 is analogous to that in 4.1. Now we only presume as known that every solution $f \geqslant 0$ of $f' = f$ is continuous and monotonic. Then we show, here also geometrically and without referring to $f' = f$, that for every continuous and monotonic function $f \geqslant 0$ with $f(0) = 0 < f(1)$ the inequality (2) $f(1) > \int_0^1 f$ holds. The way to the recognition that $f' = f$ cannot be fulfilled then is now somewhat longer (insertion of $f' = f$ and use of the fundamental theorem) than with the indirect conclusion in 4.1. This small disadvantage is confronted with the advantage that inequality (2) can probably be inferred even easier from Figure 6 than (1) from Figure 5, since the comparison of areas is certainly even more directly evident than are statements about convexity.

We consider the indirect argumentation in both proofs analyzed as natural, too, since the chain of reasoning goes like this:
First step: We realize what form the solutions of the differential equation $f' = f$ have to have.
Second step: We imagine that a function of this form rises from the first axis and reach a conclusion
Third step: ... which proves to be incompatible with the differential equation.

In the *proof in 4.3,* the concepts used in 4.1 and 4.2 and there geometrically interpreted, are now interpreted kinematically, that is

x as time t

$f(x)$ as distance $s(t)$, $f'(x)$ as velocity $\dot{s}(t)$ (Fig. 7), resp.

$g(x) = f'(x)$ as velocity $v(t)$, $\int_0^x g$ as distance $\int_0^t v$ (Fig. 8).

If substantial, appropriate conceptions are now linked to these kinematic concepts, the statements (1′) and (2′) prove to be immediately evident (under the premises there). In this sense the proof is now *reality-oriented.* For the concluding indirect argumentation, the above holds in the same way.

All the proofs just analyzed are, in our opinion, typically preformal. The use of symbols like f' or $\int f$ should by no means be misunderstood as formalism. Such signs serve primarily here for better communication. In any case, the accompanying concepts and the argumentations conducted with these, always have a meaning related to the content, as has been shown.

Let us briefly consider how the preformal proofs for assertion B in 4.1 to 4.3 can be *made more precise*: We move away from the geometric or kinematic interpretations of the quantities $x, f(x), f'(x), \int_0^x f$ and prove the inequality (1) or the formally equivalent inequality (2) merely by the well-known tools of calculus. With (2), this is obvious just by using known properties of the Riemann integral of continuous functions.

For the proof of (1), on using the above preconditions $f \geqslant 0$, f differentiable, f' increasing (convexity), and $f(1) > 0 = f(0)$ (hence $f'(0) = 0$) alone, for the experienced mathematician the use of the mean value theorem suggests itself, for instance as follows: Because $f'(0) = 0$ there is a number c with $0 < c < 1$ and $0 \leqslant f(c)/c < f(1)/1$. This yields $f(1) - f(c) > (1 - c)f(1)$, hence $(f(1) - f(c))/(1 - c) > f(1)$. Further, according to the mean value theorem there is a number d with $c < d < 1$ and $f'(d) = (f(1) - f(c))/(1 - c)$. Because of the monotonicity of f' we now deduce $f'(1) \geqslant f'(d) > f(1)$.

Here – in order to preserve the structure of the indirect argumentation – we have formalized only the second step (proof of (1) from the identified geometric properties of f) of the three-step proof of assertion B in 4.1. This turned out to be rather complicated. The proof of B becomes far shorter and more elegant if, starting from the precondition $f' = f$, $f \geqslant 0$, $f(0) = 0 < f(1)$, we aim directly for a contradiction. This is done in the following proof which was recommended to us as a simplification of our proofs and which we look upon as typically *formal* – in contrast to the proofs analyzed above:

According to the mean value theorem, there is a number $c \in (0;1)$ with $f'(c) = (f(1) - f(0))/(1 - 0) = f(1)$, hence $f(c) = f(1)$ because $f' = f$. As a result of the monotonicity of f, f must be constant on $[c;1]$, which (because of the differentiability of f) implies especially $f'(1) = 0$. From $f' = f$ we conclude $f(1) = 0$ which is contradiction to the precondition.

Of course, this proof can be interpreted geometrically or kinematically subsequently. By this, however, its character does *not* become *preformal* in our sense. For, to us this proof seems not to be natural for learners, due to the use of the mean value theorem and to its logical structure. According to our experiences, learners who are not formally trained scarcely come to think of using the mean value theorem (in spite of its doubtless existing geometric or kinematic substance). This is essentially due to the fact that the mean value theorem is a pure, non-constructive existence theorem, the relevance of which is, on the whole, legitimated intra-mathematically. (It is well-known that it plays a crucial role in a deductive construction of real analysis; J. Dieudonné has even characterized it as "probably the most useful theorem in analysis".)

The last example in particular shows that it is presumably not possible to characterize proofs as preformal or as formal without presupposing a certain conception of the desirable knowledge and abilities of learners as well as of the learning of mathematical topics in general. We hope that our didactical ideas have become a little clearer from the preceding examples and explanations (for more details see Blum/Kirsch, 1979).

6. SOME DIDACTICAL CONCLUSIONS

The following questions are obvious.

1) How can learners *judge for themselves* the *validity* or non-validity of a given or a self-discovered preformal proof?[16]

The conclusions and the premises in such proofs have to be correct after all, and this can actually be judged only by someone who has at his disposal "higher order" knowledge and abilities, i.e. who has enough mathematical maturity (compare section 3) to recognize the correctness or incorrectness, if necessary after a formalization (which is often obvious to a mathematician). Such knowledge and abilities are, however, generally *not* available to learners. Among others, H.-N. Jahnke has pointed to that problem. He has – referring to scientific theory according to Sneed – formulated it as follows (cp. Jahnke, 1978, p. 250ff.): Proving has always to be understood also as "proving in terms of the future", i.e. from the point of view of the "more elaborated system". In our opinion a difficulty in principle comes to light here which appears with every acquisition of knowledge and which possibly diminishes the didactical significance of preformal proofs.

2) How can learners *find for themselves* preformal proofs?

For this, it is certainly not sufficient to simply demonstrate several such proofs to learners and to hope for transfer; of course, this also holds true, by analogy, for formal proofs. Necessary (but presumably not sufficient) *preconditions* for students to understand, judge, and in some cases even find independently preformal proofs are[17]:

- to place value on manifold kinds of representations of mathematical content, especially to stress reality-oriented basic ideas and to impart geometric intuitive basic conceptions (for, as we pointed out in sections 3–5, these ideas and conceptions can be essential constituents of preformal proofs).
- to frequently furnish preformal proofs in the classroom (in order to convey a broad reservoir of experience and to develop the necessary competence mentioned in section 3);
- to furnish examples of formal proofs, too (also and particularly as a contrast to preformal proofs);
- to formalize non-formal arguments and to ikonicize or enactivize formal arguments, or to interpret them in the real world (in our opinion, such translation abilities also belong to this notion of necessary competence);
- to speak with students about proofs and proving, also and especially about different levels thereby, i.e. to reflect upon what is going on in the classroom (since the construction of such a meta-knowledge is surely an indisputable precondition for a transfer of knowledge).

In doing so, it may be quite fascinating and profitable if – as in our example – students pass through different levels which also contain incorrect arguments and which can then be judged in retrospect. This also corresponds, as we know, to the development of knowledge from a historic-genetic and an epistemological point of view as in Lakatos (1976). In this sense, originally incorrect arguments by no means have to be didactically disadvantageous,[18] on the contrary: Mistakes may, for different reasons (compare, for example, Fischer/Malle, 1985, p. 76ff.), play a fruitful and constructive role in the learning process.

Everything said so far holds all the more for *teacher training*. In order to enable the future teacher to realize the aforesaid aspects in his (or her) instruction he himself has to be educated at university correspondingly, i.e. – among other things – he has to get to know proofs on different levels, and he has to learn to conduct such proofs for himself as well as to reflect upon that. By no means would it be sufficient if – as is sometimes suggested for the training of primary or lower secondary school teachers – a total restriction to preformal proofs took place. Of course it is just as insufficient if – as is unfortunately the rule – future upper secondary school teachers get to know almost exclusively formal proofs. Once again it becomes apparent how demanding teacher training is or should be.

NOTES

[1] Revised and extended version of a conference proceedings contribution in German: Warum haben nicht-triviale Lösungen von $f' = f$ keine Nullstellen? Beobachtungen und Bemerkungen zum inhaltlich-anschaulichen Beweisen. In: Kautschitsch, H. and Metzler, W. (eds.): Anschauliches Beweisen. Schriftenreihe Didaktik der Mathematik, Band 18. Wien/Stuttgart 1989, pp. 199–209. We would like to thank Prof. K. Heidenreich (Reutlingen) as well as the editors of ESM for several valuable hints.

[2] As, by the way, were most of the students at our university, too, to whom we presented those arguments. For many of these students this argumentation was particularly suggestive, owing to the fact that they saw an analogy to the well-known intuitive proof of the theorem "If $f' = 0$ in an interval I then $f = \text{const}$ in I" (see section 4) which was familiar to them.

[3] Seemingly, this term cannot be translated into English in an adequate manner; compare T. Fletcher's arguments in ESM 19 (1988), p. 269. A rough translation is "intuitive-meaningful" proofs.

[4] The case $f' = f$ obviously is the limit case of $f' = f^{1-\varepsilon}$ for $\varepsilon \to 0$. The respective biggest zero moves for $\varepsilon \to 0$ more and more to the left, and the limit function $\lim_{\varepsilon \to 0} (\varepsilon x + 1)^{1/\varepsilon} = e^x$ no longer has a zero.

[5] Should one wish to reject the occurring of branching points as unnatural per se, as contradicting a physical feeling of causality, for instance, then no proof at all would be required for the non-existence of non-trivial solutions with zeros. For, the constant function $f = 0$ obviously would be the only solution with zeros.

[6] The term "preformal" is, in our opinion, better than "non-formal" or "informal" because the prefixes "non" and "in" represent negations and therefore suggest something not of full value, in sharp contrast to our intentions. By the way, Semadeni's original term was "premathematical proofs", which is even further from what we intend.

[7] "Intuitive" is meant as defined by Fischbein (1983), for example, i.e. intuitive knowledge as immediate, self-evident, unquestionable, certain, coercive, global knowledge.

[8] See Semadeni (1984, p. 33). Compare also Lakatos (1978, p. 65): "In a genuine low-level pre-formal theory proof cannot be defined .... There is no method of verification."

[9] Compare also with the levels (even more intensely oriented towards the learning process) in Balacheff (1987), who distinguishes between "preuves pragmatiques", "preuves intellectuelles" and "démonstrations", where the last-named kind means formal proofs in the usual codification of mathematics.

[10] According to a discussion statement by Dr. R. Schaper (Kassel).

[11] By the way, we immediately see where and why this argumentation fails for $f' = \sqrt{f}$, for instance (more generally: $f' = f^{1-\varepsilon}$). But it holds all the more for $f' = f^2$, for instance (more generally: $f' = f^{1+\varepsilon}$ with $\varepsilon > 0$), though in this case there are no solutions defined for all x because f, for increasing x, goes too steeply upwards. So in this respect also $f' = f$ is the limit case (cf. footnote 4).

[12] This is permissible, for a translation of the zero point does not alter the premise $f' = f$, which means that at each point the slope of the tangent equals the value of the function. (This is a preformal argument!) This simplification is dispensible for the following proof. It will, however, effect a marked reduction of paperwork in later considerations.

[13] Naturally, assertion A with proof is also kinematically interpretable, sensibly again for the more general differential equation $v = k \cdot s$. The obtained result is: If a distance has already

been covered by a certain time t_0, then a distance must also have been covered already by the time $t_0 - 1/k$. In other words: If a vehicle runs in such a way that the distance covered is proportional to the respective velocity, then it is impossible for it ever to have been standing still (or: it has already covered a distance at any chosen time).

[14] It is known that, in investigating the laws of free fall, Galileo initially made the formulation attempt $v(t) = k \cdot s(t)$ and later rejected it as being "false and impossible". In refutation Galileo naturally starts from the ascertainment that all bodies (let go at a definite time) fall, i.e. that there is a problem solution s with $s(0) = 0$, $s(t) > 0$ for $t > 0$. His argumentation was rejected by E. Mach as invalid and rehabilitated by G. Polya (see Polya, 1968, p. 208). In our mode of expression it means that, if the law of free fall were $v(t) = k \cdot s(t)$, then $2 \cdot s(t)$ would describe the same process of falling as $s(t)$, which of course is impossible.

[15] From $f \geqslant 0$, $f(0) = 0$ and differentiability it follows, by the way, without using $f' = f$, that $f'(0) = 0$.

[16] Compare thereto the example in Kirsch (1979, p. 270).

[17] Of course, each of the following suggestions can be found repeatedly, in this or a similar way, in didactical publications, recently in Neubrand (1989), for example.

[18] According to a discussion statement by Prof. R. Fischer (Klagenfurt), who further pointed out that a system to exclude mistakes from the beginning would necessarily have formal features, in marked contrast to the intentions of working on preformal levels.

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