Value Theory and Social Relations of Production:

The Fundamental Theorem Reconsidered

by

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(The Final Version will elaborate sections 2 and 3 in more depth)

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1. A Simple Analytical Framework

As is well known, Karl Marx published the first volume of Das Kapital in 1867. There he based his analysis of the capitalist production process on the provisional assumption that all commodities - including the commodity labor-power - are exchanged at their labor values although he knew quite well at this time that in Capitalism commodities are not exchanged as products of labor, but of capital.[1] In the posthumous third volume he considered the fact that, due to the tendency of equal profit rates on capital expended, competitive prices systematically deviate from their resprective values. In fact, Marx attempted to establish an algorithm transforming values into production prices in order to substantiate his claim that, in the last instance, prices are determined by labor values (1894, 177).[2]

Considerable progress in clarifying the concepts and problems involved in Marx's value analysis has been achieved in the frame of linear economic models of the Leontief-Sraffa and the von Neumann type.[3] The purpose of this paper is to give first an overview of the main results of this discussion within the limits of a simple Leontief-type model without durable capital goods. Then we proceed to the question whether the so-called Fundamental Theorem establishing an equivalence between a positive rate of surplus value and a positive rate of profit[4] provides not only a logically more satisfactory link between the price and the value system but also reveals the appropriateness of value analysis for the investigation of the social relations of production. We give an alternative proof of this theorem

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1 In fact, Marx developed his 'solution' as early as in 1862 in a letter to Engels. This proves that he was well aware of the problem of a systematic deviation of prices from values when he wrote the first volume.

2 Marx inherited this problem from Ricardo (1817); but despite of its Ricardian roots this question dominated the academic discussion of Marx's economic writings by Marxists and Non-Marxists alike. A useful review of the formal aspects of this discussion is provided by Samuelson (1971).

3 For this, see among others Okishio (1954, 1963), Weizsäcker (1971), Wolfstetter (1973) ans especially Morishima (1973, 1974 a,b,c).

4 See Okishio (1954, 1963), Morishima (1973, 1974a) and Wolfstetter (1973).

that makes explicit the role of a physical surplus product for the validity of the theorem: Marx's subsistence wage assumption in the price system - not derived from the model itself, but externally given - is shown to be crucial for his argument that profit has to be considered as the phenomenal form of surplus value.

After this, we consider the joint production case in the von Neumann-Morishima model. Steedman's (1974, 1978) counter-example against the Fundamental Theorem is rejected because it presupposes the application of inefficient techniques whereas Morishima's generalization of the Fundamental Theorem is shown to be based on the comparison of a fictitious stationary state, characterized by the minimum labor time necessary for the production of workers' consumption goods, and the actual labor time performed in order to produce a physical surplus. We contrast Morishima's analysis with the Marxian view that a non-exploitative socialist society is not defined by the claim that workers should receive the whole net product of the society (Marx, 1975). Finally we discuss the modifications and explanations put forth by Morishima and Catephores (1978) in order to apply the Fundamental Theorem to a socialist society.

We start now our analysis in the frame of a simple static input-output model.[5] Consider an idealized capitalist[6] economy with one primary input, labor, measured in standardized time units[7] which we shall call working units. Let $c \geq 0$ be the (average)

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5 For those input-output models see e.g. Dorfman, Samuelson, Solow (1958), Gale (1960) and Lancaster (1968).

6 Apart from this subsistence wage assumption, there is nothing in the model which could identify the formal system as an idealized capitalist economy. For the consequences of this over-abstraction, see Section 3 below.

7 For the so-called reduction problem arising from different types of labor, see e.g. Nutzinger/Wolfstetter (1974, Vol. 2).

commodity basket per working unit or the "real wage". The money wage rate per working unit is w, and as workers are assumed to spend their whole wages in buying their means of subsistence we have

(1) $wL = p'cL$ or, equivalently,

(1a) $w = p'c$

where L is the number of working units paid and p is the nonnegative and so far unknown price vector of the economy.

In order to simplify the exposition we first consider the technology (A, a_0) as given where $A = (a_{ij})$ is the semipositive and indecomposable matrix of material input requirements of good i per unit of good j $(i,j = 1,\dots,n)$ and $a_0 = (a_{0j})$ is the semipositive vector of direct labor requirements per output unit of good j. This means that every good enters directly or indirectly the production of every other good which in turn implies that labor is needed for the production of every good even if there are no direct labor requirements for a particular output. The physical relationships between the inputs and the outputs can now be summarized in the quantity system.

(Ia) $x = Ax + y$

(Ib) $L = a_0'x$

where x is the vector of gross outputs and y the vector of net outputs. It is assumed that there is full use of the existing labor force L, and that the stock of inputs available at the beginning of the uniform period of production is sufficient to produce the semipositive net output vector y.[8] This in turn, however, presupposes that the gross output vector x allows not only for the replacement of the used up inputs but also for a net product available for workers' and capitalists' consumption

8 This allows only for a stationary reproduction of the system. If we consider growing economies, as we will do later on, then we must impose stronger conditions, such as the steady state requirement that, in the absence of technical progress, both inputs and the labor force L must grow at the same rate as the final demand. See also Weizsäcker (1971, Part I).

and for accumulation.[9] We call those systems productive, and the necessary and sufficient condition for its productivity demands that the series of the input requirements of each 'round' of the production process must converge, i.e., there exists a strictly positive, finite sum

$$(2) \sum_{k=0}^{\infty} A^k = (I-A)^{-1},$$

the so-called Leontief Inverse.[10]

This allows us to calculate the gross product vector necessary to produce the final demand y:

$$(3) \quad x = (I - A)^{-1} y \gg 0$$

Similarly we determine the whole amount of labor necessary to produce a given net product y as

$$(4) \quad L(y) = L(x(y)) = a_0'(I - A)^{-1}y.$$

So far we have established that both gross product x and labor units L are linear functions of the final demand y. Obviously, our system is only viable, if the means of subsistence do not exceed the total final demand. This yields the viability condition

$$(5) \quad y - cL \geq 0.$$

If it were the purpose of Marxian value analysis to determine the rate of exploitation (or of surplus value), e, then everything could stop here as our quantity system contains already all necessary information for that. Since (4) holds for any feasible net output vector y, so it must hold, because of (5), for the subsistence vector cL as well. So we can calculate both the actual labor performed for the production of y, and the necessary labor for producing the means for workers' subsistence, cL; the difference between both is called surplus labor. Hence

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[9] By using part of the net output vector y for accumulation it is possible to increase the gross output, and hence the final demand, in the next period (provided that the existing labor force permits this increase in production). But this will only mean the transition to another stationary state. Long-run growth is only possible if there is a permanent increase in inputs and working units in every period (and/or technical progress). Cf. footnote 8 above.

[10] See the proof by Weizsäcker (1971, 6).

we have our first calculation of the rate of exploitation solely from the data of the quantity system as follows (cf. Marx, 1867, ch. 18):

$$(6)\quad e = \frac{\text{surplus labor}}{\text{necessary labor}} = \frac{a_0' (I - A)^{-1} (y - cL)}{a_0' (I - A)^{-1} cL}$$

Using the properties of linear mappings we can formulate

Lemma 1: Workers are exploited if and only if they do not get the whole net product, i.e. if there is semipositive difference between final demand and means of subsistence: $(y - cL) \geq 0$.

Proof: By assumption, $a_0' \geq 0$ and A is indecomposable and productive; therefore $x(y)$ and $x(cL)$ are both strictly positive. Hence $e = 0$ implies $y = cL$, and $e > 0$ implies $y \geq cL$, taking into account eq. (3) and the viability restriction (5). On the other hand, $y = cL$ implies $e = 0$, and $y \geq cL$ leads to $e > 0$, because of (3).

Remark: Our lemma is in general true for isotonic functions between gross output and final demand and between gross output and direct labor requirements. It simply says that all final outputs not contained in the basket of the means of workers' subsistence imply surplus labor and hence exploitation.

In other words: we have established the well-known equivalence between a positive surplus produce (or, more precisely, semipositive surplus product vector) and positive surplus labor and hence a positive rate of exploitation. Now Marx's aim was to demonstrate that in capitalism surplus produce and surplus labor take the form of surplus value. This, of course, holds by definition if we adopt Marx's provisional assumption that commodities are exchanged at their values: just evaluate surplus produce with the vector of total labor requirements to get the desired result. But this is not really interesting and, above all, it does not allow us to consider profits as a phenomenal form of surplus value and/or prices as systematical deviations of values. Therefore, we have to leave the quantity system and to analyze what happens on the level of equilibrium prices in our simple model.

Interestingly enough, there are two different ways to get to the same formula for the price system. The first considers prices as determined by <u>redistribution</u> of surplus value (or, as we have shown, equivalently by redistribution of surplus produce) appropriated by each single capitalist so as to obtain a uniform profit rate in all branches. This is the peculiar viewpoint adopted by both Marx (1894) and Sraffa (1960). In our simple model, profits in the j-th industry are given by the difference between revenue $p_j x_j$ and costs, namely labor costs $w\, a_{oj} x_j$ and non-labor costs $\sum_{i=1}^{n} p_i a_{ij} x_j$. We have solely circulating capital $\sum_{i=1}^{n} p_i a_{ij} x_j$.[11] Consider now any two industries j, k whose profit rates r_j and r_k must be equal and must equal the general profit rate of the economy, r. So we have

$$(7) \quad r_j = \frac{p_j x_j - \sum_{i=1}^{n} p_i a_{ij} x_j - w\, a_{oj} x_j}{\sum_{i=1}^{n} p_i a_{ij} x_j} = \frac{p_k x_k - \sum p_i a_{ik} x_k - w a_{ok} x_k}{\sum_{i=1} p_i a_{ik} x_k} = r_k = r.$$

Since, by our assumptions, $x \gg 0$ we can divide both fractions by the respective quantities x_j, x_k to get the equation

$$(8) \quad p' - p'A - w\, a_o' = r\, p'A \text{ or}$$

$$(8a) \quad p' = w\, a_o' + (1 + r)p'A.$$

Another way to arrive at this equation is by looking at the <u>temporal structure</u> of the production process (Weizsäcker (1971, ch.I.2)): Both material inputs and workers must be available at the beginning of the uniform production period. We assume that workers are paid at the end of this period.[12] Since entrepreneurs must buy the material inputs before the production process has taken place while revenue accrues only

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[11] As the reader will easily check nothing will change if we include labor costs into the whole 'capital' of our economy. We only would arrive at equation (IIa') in footnote 14 .

[12] Marx (1867) normally assumes that workers are prepaid but emphasizes that in reality they are usually paid after they have worked for the capitalist. Our exposition could be easily modified to deal with this case. Cf. Wolfstetter (1975a) and our following footnotes.

after that, these inputs must be financed by them at the interest rate r. Assuming immediate market clearing at the end of the production period we get hence the same formula as above. Neoclassical reasoning emphasizing intertemporal allocation an the Marx-Sraffa view of redistribution of surplus value (or, equivalently, surplus produce) lead both to an identical result. Summarizing equations (8a) and (1a) we have thus the <u>price system</u>

$$\text{(IIa)} \quad p'(r) = w\, a_0' + (1 + r)\, p'A$$

$$\text{(IIb)} \quad w = p'c.$$

We ask now if there exists a positive configuration (p, r) satisfying (IIa). Taking again the series expansion (2) of the semipositive and irreducible technology matrix A we know already that any feasible p must be strictly positive. From the Frobenius-Perron theorem we can determine the maximum profit or interest rate consistent with A: If domA is the dominant eigenvalue of A we get

$$\text{(9)} \quad r_{max} = \frac{1 - \text{domA}}{\text{domA}}$$

and because of $r > 0$ also

$$\text{(10)} \quad \text{dom A} < 1$$

which is just another formulation for our earlier requirement that we must have a semipositive net product y.[13] So the feasible range for r is given by

$$\text{(10a)} \quad 0 \quad r \quad \frac{1\text{-domA}}{\text{domA}} \ .^{14}$$

For any feasible r^* we can hence solve (IIa) to get

$$\text{(11)} \quad p'^* = p'(r^*) = w\, a_0' \, (I - (1 + r^*)A)^{-1}$$

as the unique and strictly positive price vector.

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13 For proofs see Lancaster (1968), ch. 6 and Rev. 7; Nikaido (1968), ch. 2. To prove sufficiency it is convenient to consider p as the left-hand eigenvector of A to get the equation $p(I - (1+r)A)=0$, and to take y as the right-hand eigenvector of A which establishes the equivalence of the convergence of the series $(I + A + A^2 + \ldots.)y$ and of $(\lambda^0 + \lambda^1 + \lambda^2 + \ldots)y$; this in turn leads to the requirement r_{max} = dom A < 1.

14 If we assume that workers are prepaid, then (II a) changes to (IIa') $p' = (1+r)wa_0' + (1+r)p'A$ which restricts the range of permissible r to (10') $0 \leq r < \text{domA}^{-1} - 1$. The general reasoning, however, remains the same.

It is important to note that both the real wage basket c and the wage rate $w = p'c$ are not independent from the specific r^* chosen. If, for instance $r^* = r_{max}$, then the whole net product is appropriated by the capitalists, which means that $c = 0$ and $w = 0$. On the other hand, if r equals zero, then cL equals y, and the real wage rate $\bar{w} = w/p'c$ achieves its maximum. This is easily derived from Lemma 1. Moreover we can establish $\bar{w}(r)$ as an antitonic function of r, as $w^{-1}p(r)$ is an isotonic function of the interest (or profit) rate (cf. eq.(11)).[15]

We derive now the basic accounting identity between the price system (II) and our earlier quantity system (I). Let x be a feasible output vector and p a feasible price vector. Then multiplying (Ia) by p and (IIa) by x we obtain

(12) $p'x = pAx + p'y = w\, a_0'x + (1 + r)p'Ax$

Using (Ia) and (Ib) we get from this the accounting identity.

(13) $p'(y - cL) = rp'Ax$

On this identity we will comment later on.

Values, Substitution and the Transformation problem

So far we have not used any notion of labor value or of a value system. Marx (1867, p.47) defines his labor values per unit of commodity j as the total amount of labor time that is socially necessary for the production of one net output unit of this commodity. As Marx's definition was implicitly based on a stationary reference system without growth it is better to follow Wolfstetter (1975a) in replacing the perhaps misleading term "production" in the definition above by "reproduction". So, the value of means of production is defined as "the labor-time currently performed for providing future replacement of means of production at a sufficient rate to allow a stationary consumption level to be maintained" (Wolfstetter (1973,p.795).

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15 In fact we obtain the antitonic wage-interest curve

(12a) $\bar{w}(r) = \dfrac{1}{a_0'(I-(1+r)A)^{-1}c}$

This definition leads to the so far tautological determination of values (per output unit) as the direct labor requirements plus the value of the means of production in the sense specified above:

(III) $z' = a_0' + z'A$

Since a_0 is strictly positive and A is productive, semipositive and indecomposable we can solve (III) to get the unique,strictly positive vector of values

(14) $z' = a_0' (I - A)^{-1}$.

As is easily seen, the value system is the mathematical dual of the quantity system. It allows us to calculate not only the total amount of actual labor and of necessary labor but also the labor performed in each industry or contained in each commodity. It should be noted that those labor values of every commodity presuppose, in Marx's view, production for exchange. Of course our model could apply to a non-market model economy as well and we could calculate the specific labor contents of each good; but we would not be entitled to call them 'values'.

Since we considered the technology (A, a_0) as given,there is another way to calculate values. We may ask: How much labor is to be performed in the present period if one unit of commodity j is to be supplied at the end of each succeeding period? In order to get one unit of j at the end of the period one needs A^j inputs at the beginning of each period (where A^j is the j-th column of A). But to sustain this stream in the future, one needs also A^2u^j inputs at the beginning, and so on. Summing-up, we get the vector of gross outputs for the production of one unit of j

(15) $x^j = (I + A + A^2 + \ldots)u^j = (I - A)^{-1}u^j$

where u^j is the unit vector with 1 at the j-th place and zero components elsewhere. Multiplying this with the vector of direct labor requirements we get again our value equation

(16) $a_0'x^j = a_0' (I - A)^{-1}u^j = z_j$

as before. This calculation of today's labor requirements for a stationary stream of one unit of good j at all futre dates presupposes, however, that no technical progress occurs (cf. Wolfstetter (1975a)).

So far we have not dealt with the problem of substitution which we excluded by assuming the technology as "given". Only on the basis of a given technology we can, for instance, determine labor values. But if we have more than one technology (A, a_0) what is the criterion for the selection of the optimal technology? Only a naive view of value analysis as a theory relative prices (that occurs in some Marxists and non-Marxists alike) would suggest that they are selected on the base of minimization of total labor requirements. This would only happen in a hypothetical stationary reference system where $y = cL$ and hence r would have to be zero as is easily derived from equations (7), (Ib) and (IIb). In this system, obviously prices and values coincide, and so do cost and labor minimization. Now those technologies are selected that minimize unit costs. Since we have only one non-produced input (labor), constant returns to scale and no joint production the composition of any feasible y has no influence on the choice of technology.[16] Yet, the problem seems to be circular as values are determined, <u>inter alia</u>, by the technology (A, a_0), and on the other hand, the technology is determined on the basis of input values. But using the properties of Shepard's (1951) unit cost functions one can establish the existence of a market equilibrium (z^*, A^*, a_0^*) independent of any feasible y and cL (cf.Weizsäcker (1971, ch. I.2)).

This allows us, even with substitution, to derive the equation

(11') $p'^* = w\, a_0'^* \, (I - A^*)^{-1} = z'^*$.

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[16] This is the famous Nonsubstitution Theorem. For a proof of this property and references see e.g. Lancaster (1968, ch. 6.7).

Only in this special stationary case values can meaningfully serve as equilibrium prices.[17] The transformation problem hence is "solved" by identifying values and prices. An important property of this case is the equivalence of a definition of values independent of prices and the independence of these values from the composition of final demand (nonsubstitution theorem).[18]

In general, this property clearly does not apply. First we show that even without substitution the transformation from values into prices depends on the rate of interest (or profit) which is not determined by the value system. In order to solve the transformation problem, just consider the price system and the value system (III). Combining equations (11) and (14) we get the following mapping

$$(17) \quad w^{-1}p'(r^{*}) = z' \, (I - A) \, [I - (1+r^{*}) \, (I - A)^{-1}]$$

which is unique <u>for any given r^{*}</u>. But without knowledge of r, there is no unique mapping from values into prices but instead a correspondence. This is clearly less than Ricardo and Marx had in mind.[19]

There is an obvious objection against our transformation procedure. As we have seen, prices were determined directly without any reference to the value system. So the transformation from values into prices was in any case an unnecessary detour. In order to get our algorithm we presupposed already both the value and the price system. And even in our simple case, we needed one additional information, namely the rate of interest (or profit)

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17 Even less meaningful are those special cases where there is no influence of the rate of profit on equilibrium prices. These are special requirements for A, such as, for instance, an equal organic composition of capital, which come close to the single commodity case of vulgar neoclassical theory. There is an obvious analogy between the transformation problem and the re-switching debate which we will not discuss here. For a discussion of these less interesting special cases cf. Morishima (1973, ch. 7).

18 For an extension and a proof of this equivalence see v. Weizsäcker (1971, pp. 76-78.

19 Nothing important changes if we start from the alternative price equation (IIa'). We would get from (IIa') and (14)

$$(17') \quad w^{-1}p^{*} = (1 \; r^{*})z'(I-A)[I - (1-r)A]^{-1} = z'\{I + r^{*}(I-(1+r^{*})A)^{-1}\}$$

(cf. Wolfstetter (1975a, p.9)).

from outside the value system. That is, we needed already one "price" in order to get from values to prices before we can start the transformation process.

It is clear that our objection would not hold if we assumed that the means of subsistence are given by technology. We then would have $c = 0$ and $r = r_{max}$ as in equation (9). All final demand would go to the capitalists, the workers would be considered as ordinary inputs which have to be fed according to the scale of production, and the rate of profit would be given by the productivity of the system, measured in terms of the dominant eigenvalue of A. This is Sraffa's (1960, ch. 2) model with surplus. But it really does not describe a capitalist economy with 'free' wage-workers whose means of subsistence are determined by sociological, historical and political forces and may change over time (cf. Marx (1867, p.168)). This model would only apply to a slave economy where labor is owned and not rented by the masters as under capitalism.

Coming back to the transformation problem we have to look at the additional problems that may arise from substitution. Clearly, with more than one technology (A, a_0) and $r \;\; 0$, the choice of techniques would be made on the basis of prices. Under idealized capitalist competition the unit cost minimizing technology (A^*, a_0^*), evaluated at equilibrium prices p^*, would have been chosen. For a given rate of profit (or a given distribution between wages and profits) "competitive equilibrium leads to the choice of that production system which minimizes prices simultaneously in terms of the wage unit" (Wolfstetter (1973, p. 802)).

This augurs little good for the transformation problem: In order to calculate values according to (14) we have to determine the cost-minimizing technology and hence need equilibrium prices. This means that the explanandum becomes part of the explanans which is quite an unsatisfactory situation. There are two possible ways out of this difficulty: first, to revise Marx's definition, as proposed by Samuelson and v. Weizsäcker (1971), or to look for another qualitative link between the value and the price system that is provided by the so-called "Fundamental Theorem" of Okishio (1954, 1963), Morishima (1973) and Wolfstetter (1973).

First we discuss briefly the Weizsäcker-Samuelson notion of 'rational values' or 'synchronized labor costs'. The authors start from the observation that in the stationary case where the steady state growth rate g is zero by definition, Marxian labor values reflect the social opportunity costs of production. Since in this case prices and values coincide if $g = r = 0$, they ask for a more general concept of values as equilibrium prices if there is steady state growth at the rate $g > 0$.

This concept exists, and it is easily derived from our earlier considerations. Consider an economy with population and labor force growing at the rate $g > 0$. In order to maintain a constant per capita consumption (technical progress being absent) clearly the vector of inputs must grow at the same rate, due to the linear mapping from inputs into outputs. Let v_{t-1} be the vector of inputs produced in t-1 that are available for the production of gross output x_t in the following period. So we have

(18) $v_t = (1+g)v_{t-1} = (1+g)Ax_t$ and

(19) $x_t - v_t = x_t - (1+g)Ax_t = c_t L_t = c\ L_t$

as the vector of per capita consumption is assumed to be constant over time. Now define a new matrix $B \equiv (1+g)A$ and remember that not all positive growth rates are feasible (cf. footnote 13 and equation 9 above).[20] As $c \geq 0$, we have an analogous restriction on g as we had in equation (1o) on r, namely

(20) $0 \leqq g < \frac{1-\text{dom } A}{\text{dom } A}$;

note the strict inequality on the right hand. On the other hand we obtain from our definition of B

(21) dom B = dom (1+g)A = (1+g) dom A.

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[20] That is, we are treating B as if it was a purely technological matrix to obtain the new evaluations with respect to B, instead of A.

Substituting this into the right-hand inequality (20) we get

(22) dom $B < 1$.

But this completes the proof of our proposition since B has all the same relevant properties: inflating A by (1+g) cannot affect semipositivity and indecomposability, and through our restriction on g also the productivity of B is maintained as shown in (22). So we can define *revised* values in terms of B as follows

(III') $\bar{\bar{z}}' = \bar{\bar{z}}'B + a_0'$

and solve the equation to get

(23) $\bar{\bar{z}}' = a_0' (I - B)^{-1}$

This is clearly the dual of our quantity system (19), and it has all the *mathematical* properties of the value system (III). Defining the quantity system in terms of B we have

(24) $\bar{\bar{x}} = B\bar{\bar{x}} + cL$ or $\bar{\bar{x}} = (I - B)^{-1}cL$ (24a)

(25) $L = a_0' \bar{\bar{x}}$.

We have shown that all relevant properties of the nonsubstitution theorem are fulfilled, and in the same way as indicated above (p.11 seq.) we can establish the existence of a market equilibrium ($\bar{\bar{z}}^*$, B^*, $a_0{}^*$) for any given feasible growth rate g. Another way to prove the cost-minimizing property of these *re-vised* values is taken by Wolfstetter (1973, Appendix).

This would be quite a successful solution of the transformation problem as we could again identify these values and equilibrium prices *provided* that we were allowed to identify these revised values with Marx's concept. The identity between *revised* values and prices in a golden-rule situation is easily shown:

(26) $\bar{\bar{z}}' = a_0'(I - (1+g)A)^{-1} = w^{-1}p'(r=g)$.

What happened was nothing but an appropriate change of the measuring units by relating "values" and quantities to B, instead of A, while B had all important *mathematical* characteristics of A. We can define on this new base revised concepts of surplus product, surplus value and exploitation as has been done by v. Weizsäcker (1973c). Retransforming these concepts into our original system (A, a_0) we will get, as the reader can easily verify[21], the following properties of the revised measure of exploitation, $\bar{\bar{e}}$:

(a) $\bar{\bar{e}} = 0$, whenever $r=g$, and hence the whole surplus product (in terms of the Marxian system) is used for accumulation and there is no consumption for capitalists, while there is no surplus product at all in terms of the revised system.

(b) $\bar{\bar{e}} > 0$ is sufficient but not necessary for e to be positive;

(c) Our Lemma 1 still holds in terms of the revised quantity system, but we would have to reformulate it with respect to the original system as follows: According to the revised measure $\bar{\bar{e}}$, workers are exploited whenever the capitalists consume;

(d) $\bar{\bar{e}} > 0$, whenever $r > g$ and hence not the whole surplus product (in the Marxian sense) is used for accumulation.[22]

(e) If we consider values as employment multipliers then the units of the revised system have to be taken.[23]

Now, Wolfstetter (1973) has convincingly argued that these revised values do not grasp the social relations of production. Marx always emphasizes the important difference between a feudal surplus product which is consumed by feudal lords, and not reinvested, and the capitalist's surplus value which has to be reinvested, according to the capitalistic 'compulsory laws of accumulation'. For him the social role of the entrepreneur was precisely to accumulate and

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21 We only prove the last assertion: In order to maintain a steady state per capital vector c with L growing at the rate g we need inputs v_t at any time growing at the same rate. This leads to the revised quantity system (24),(25):

(27) $L = a_0'\bar{\bar{x}} = a_0'(I-B)^{-1}cL \equiv a_0'(I-(1+g)A)^{-1}cL = \bar{\bar{z}}'\ cL$,

taking into account (26).

22 We note that this statement is equivalent with the so-called Fundamental Theorem to be analyzed in the next section if and only if g=0.

23 This has been correctly observed by v.Weizsäcker (1973a, pp.1248 - 1250). Morishima's (1974b, p. 389) objection that the 'dynamic' employment multipliers $\bar{\bar{z}}$ can be derived from the Marxian z is not really convincing since this 'transformation' has just the same weaknesses that we discussed already with respect to the traditional transformation problem.

for that reason he was forced to extract surplus value from the workers by submitting them to his command. Since Marx investigated the power relations in the sphere of production, the important fact for him was not that accumulation was also the source of future consumption (which is the neoclassical and Austrian view of the process), but that the capitalist was the social agent to make the investment decision. Hence, in his view, surplus value was at the capitalist's disposal, even and above all if he had to put it back into the process of production.

Without investigating here the merits and weaknesses of these two conflicting views of the production process in greater depth, we may add that the Weizsäcker-Samuelson revision of the theory of value and exploitation can be related to the corresponding Marxian concepts. Their rate of exploitation, for instance, can be interpreted as a measure of the capitalists' efficiency to perform their social role as the agents of accumulation. It should, however, be noted that these revised or rational values are appropriate shadow prices for socialist planning as Wolfstetter (1973) admits because they lead to the choice of the optimal techniques that simultaneously minimize the synchronized labor costs and hence ensure an efficient intertemporal allocation.[24] We could, in the frame of this model, not distinguish between an efficient capitalist system with $\bar{\bar{e}} = 0$ (and hence optimal accumulation without consumption for the capitalists) but still e 0 (and hence exploitation) on the one hand, and an idealized socialist system with the same formal characteristics (where by definition no exploitation exists[25]). Perhaps one could extend the Marxian notion of bourgeoisie to the planning bureaucracy and then evaluate its social efficiency in terms of the revised concept of exploitation?

Now some authors, especially Baumol (1974), have argued that Marx was not really interested in the transformation problem as a means to link the value and the price system. While we certainly agree

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[24] For an extended discussion of the applicability of "rational values" to socialist planning see Baisch (1976).

[25] More on this will be said in section 3 below.

that the importance of the transformation problem for Marx and his system has been heavily axaggerated by Marxists and non-Marxists alike, it seems to us not correct to assert "that Marx did not intend his transformation analysis to show how prices can be deduced from values" (Baumol, 1974, p. 52). It is certainly true that Marx's main interest in the transformation problem was "to describe how non wage incomes are produced and then how this aggregate is redistributed" (p. 53); but considering Marx's frequent statements in Das Kapital and his Theories of Surplus Value that prices have to be derived from values,it is difficult to maintain that his "interest in the transformation analysis as a sequel to his value theory was not a matter of pricing" (l.c.). But this is only of minor interest and therefore we turn to the qualitative link between the value and the price system that is provided by the so-called Fundamental Theorem of Okishio (1954, 1963), Morishima (1973, ch.5; 1974a) and Wolfstetter (1973, Appendix).

2. The Fundamental Theorem Reconsidered

If we want to relate the value und the price system on an aggregate basis we do not have to derive individual prices from individual values. As is now more or less commonly accepted Marx's main aim was to demonstrate that behind freedom and equality of commodity-owners at the marketplace (including those who only own their labor-power, i.e. the workers),there is a fundamental sphere of production where workers are subject to the capitalist's command after they have sold their specific commodity labor-power. This makes good sense in explaining the internal organization of the firm as a non-market relationship as we have tried to show elsewhere (Nutzinger, 1976a). But for this purpose one can dispose of value analysis. Now according to Morishima, Okishio and Wolfstetter, Marx suggested a fundamental theorem that establishes

(1) a sufficient link between the value and the price system,
(2) the relevance of value analysis for the investigation of the (capitalist) production process, and
(3) an explanation of the origin of profits through surplus value.

In its simplest form, the <u>Fundamental Theorem</u> asserts the following:

If the assumptions of the price system, the quantity system and the value system are fulfilled, <u>a positive rate of exploitation is a necessary and sufficient condition for the profit rate to be positive</u>.

<u>Proof:</u> In contrast to earlier proofs (Okishio, 1963; Morishima, 1973; Wolfstetter, 1973), our proof makes explicit the crucial role of a semipositive surplus product over the means of subsistence.[26]

If $e > 0$, then from (16) and (4) we obtain $z'(y-cL) = (1-z'c)z'y > 0$ and because of $z'y > 0$ also $1 - z'c > 0$. From the subsistence wage hypothesis we get $w^{-1}p'(r)c = 1$ and hence

(IV) $w^{-1}p'(r)c - z'c = (w^{-1}p'(r) - z')c > 0$.

Now, $w^{-1}p(r)$ is a strictly isotonic vector function of r and $z \underline{=} w^{-1}p(r=0)$; hence $r > 0$ in order to allow for $c\ 0$ in (IV).

On the other hand, if $(1-\text{dom}A)/\text{dom}A > r > 0$, then $rp'Ax = p'(y-cL) > 0$. By the viability condition (5), then $y - cL \geq 0$. But evaluating this by $z \gg 0$ we obtain $z'(y-cL) > 0$ and accordingly $e > 0$.

<u>Remark:</u> By means of our basic accounting identity and the viability condition of the system, we relate both surplus labor and profits to the existence of a surplus product over the means of subsistence. We also note that a proof could be given without (explicit) calculation of values or the knowledge of the value system as Morishima (1974a) suggests.[27] The quantity system where we derive necessary and surplus labor would do it as well. This is easily seen by looking at equation (6), derived from the quantity system. Then the fundamental theorem is similarly proved using the fact that the series $I+A+A^2+ \ldots = (I-A)^{-1}$ is a strictly positive matrix.

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[26] Okishio (1963) proved solely the necessity part, but it is only a short step to show sufficiency as well, as our own proof indicates.

[27] Cf. Morishima (1974a, p. 615):"It is important to note that this proof requires value calculation."

First we note the crucial role of two implicit assumptions of the theorem which follow from the subsistence wage assumption and from the viability of the system. From them we get the accounting identity in the price system which in turn permits us to establish the semipositivity of the surplus product vector. Now we see that the equivalence between positive rates of surplus value and positive rates of profit is established through a common third, namely a semipositive surplus product. Evaluating this semipositive vector with different but strictly positive vectors, namely the vector of values and the vector of prices, must end-up in positive scalar products. So we are left with the question (to be discussed in section 3): What is the advantage of evaluating economic activities by means of labor values, instead of prices?

Some short comments on earlier proofs are necessary: First we observe that the theorem has been derived by looking at the working day. But this is no real difference: As Marx assumes that the working day in all industries is fixed and identical through the competitive process, we can just change the time unit to a working day of length (uniform in all industries) to get

$$(28)\quad \frac{1 - z'V^{*}}{z'V^{*}} = e$$

as a third definition of the exploitation rate. The working time necessary for the means of subsistence is then given as a percentage of the whole working day, and workers are exploited whenever $z'V^{*} < 1$.

Second, it is important to note that the equality of necessary working time and value of means of subsistence is not an independent assumption within the value system, but follows from Marx's definition of exchange values and the technological properties of our simple model. [28] This stems from the fact that the value vector z plays a double role:

(1) It evaluates the means of subsistence per day and worker, and
(2) it measures the necessary working time.

Hence we have in terms of values, by definition, an identity between 'value of labor-power' and 'value of wage goods'.

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[28] But in the more general Marx - v.Neumann-Morishima model similar properties can be established (for this see the second part of this section).

In the price system, no such identity holds. But now we can see the crucial role of the subsistence wage assumption within the price system: Since wages per any time unit, say per day, are completely spent for buying the respective bundle of subsistence goods, an important property of the value system that holds by definition, namely the equality of the value of labor-power and the value of the means of subsistence, is preserved by assumption in the price system, namely the equality of money wages and the means of subsistence, evaluated at prices. This at least for Marx's time not all too unrealistic assumption leads immediately to the equality of profits and the surplus-product in terms of prices. Precisely this identity, however, is needed for the fundamental theorem, and it is effected through Marx's subsistence wage assumption.[29]

A final and perhaps a little bit unfair remark can be added: As technology does not change over time, we also could use one generation as the length of the period of production. But then the definition of exploitation by means of the surplus product would sound rather myopic and selfish. Yet, there is nothing in the model that would prevent this 'egoistic' choice of the time unit.

The Case of Intrinsic Joint Production

In his final evaluation of Marx's labor theory of value Morishima (1973, ch.14) concluded that the changes brought forth by substitution and by intrinsic joint production " ... shake and destroy the foundations of the labour theory of value, in the original form in terms of simultaneous equations which was developed by Marx. Thus the recognition of joint production and alternative manufacturing processes, as a result of which the von Neumann revolution arises, encourages us to sacrifice Marx's own formulation of the labour theory of value and to extend (or modify) it to meet the spirit of the von Neumann revolution" (p. 180). Morishima's central idea is the application of inequalities instead of equations in order to rule out nonprofitable techniques, and

[29] The importance of the subsistence wage assumption is also emphasized by Ellerman (1976).

the replacement of actual values by optimal values. These two conditions cannot ensure uniqueness of those values, but at least an unique 'value' of necessary labor and the nonnegativity of optimal values.[30] And this is all what is needed for the modified fundamental theorem, as is easily verified.

The assumptions of Morishima's (1974a) generalization of the theorem can be stated as follows:

In order to allow for substitution (or alternative processes) which tends to make the number of processes larger than the number of goods, and to allow for intrinsic joint production which leads to the opposite tendency we have to replace our square matrix I-A by the difference of the output matrix B and the input matrix A, i.e. B-A=T. Because of joint products and alternative techniques, T will not be necessarily square. The important advantage of this approach is, of course, the consideration of fixed capital: a machine that produces in t some units of certain good and depreciates in this process can be viewed as producing a joint product, namely the good and the machine at t+1. Of course, the rows of A and B will be semipositive which means that every commodity is produced by at least one process and that at least one input (except labor) enters the production of any good. The sign of the elements of T is, however, indeterminate, depending on the numerical magnitudes of the respective a_{ij}'s and b_{ij}'s. Instead of the product vector we have now x, the vector of the intensities of operation of the processes, or briefly the <u>operation vector</u>. As before, a_0 is the column vector of direct labor requirements. So we can define the necessary labor to produce a given basket of wage goods cL as the solution of the linear program: Minimize

(34) $Z = a_0'x \quad S.T. \quad Tx \geqq cL, \; x \geq 0$

The dual of this minimum problem is correspondigly: Maximize

(35) $\bar{Z} = z'cL \quad S.T. \quad z'T \leqq a_0', \; z \geq 0.$

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[30] For a discussion of the possibility of negative values and negative surplus value in the <u>actual</u> system see Wolfstetter (1976).

Both the operation vector x^* minimizing Z and the optimal value vector z^* maximizing $\bar{Z}$ will not be necessarily unique. But because of the duality relationship between (34) and (35) the following equality must be fulfilled for any $z^* \in Z^o$ and $x^* \in X^o$ (Z^o and X^o are the sets of optimal solutions for (34) and (35) respectively):

(36) $Z^* = a_o'x^* = z^{*\prime}cL = \bar{Z}^*.$

So we have unique and identical magnitudes of "necessary labor" and of "(optimal) value of the means of subsistence", as in the single output case. Now we can compare this stationary situation with the amount of labor actually expended. Let x^a be the actual operation vector. Then the rate of exploitation, defined as the relative difference between labor actually performed and necessary labor for the optimal production of the means of subsistence, is given by

(37) $e = (a_o'x^a - a_o'x^*)/a_o'x^*.$

First we note that e will be positive unless $x^a \in X^o$. Although we cannot give a clear relation between any two semipositive vectors $x^a \in X$ and $x^* \in X^o$ (some components of x^a may be smaller than those of x^*), any x^a which allows to produce more than cL must give a higher value $a_o'x^a$ than $a_o'x^*$.[31] Otherwise x^* is not a minimal operation vector. The actual activity vector x^a will differ from the optimal for two reasons: Actual intensities are selected on the basis of profit maximization, not of minimal labor employed, and they will be positive for those processes which serve the production of investment goods and 'luxury goods' for the capitalists' consumption. So we can reformulate our Lemma 1 for the general case as follows:

Lemma 1': Workers are exploited if they perform more labor than necessary for the production of their means of subsistence.

This is, apart from surpluses over the means of subsistence which occur even in the optimal program and have to be 'thrown away', exactly identical to our earlier statement (Lemma 1),

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31 More precisely: any x^a which allows to produce more than cL and that surplus of goods which is unavoidable even in the optimal program (34) due to the fact that some of the weak inequalities in the constraints cannot be fulfilled as equalities.

How can we relate this Lemma to a positive interest (or profit) rate? From the constraints of the optimization problems (34) and (35) it is easily proved by contradiction, that both the growth rate and the profit or interest rate must be zero in the optimum situation, if they exist at all. For this, consider the fact that in order to meet workers' total demand in t, cL_t, and the production prerequisites in this period, Ax_t, gross output Bx_{t-1} must fulfil the following condition:

(38) $Bx_{t-1} - Ax_t \gneq cL_t$.

Now, if g is the smallest of the growth rates of the processes $i=1,\dots,r$, then (omitting the time subscript) the following weak inequality must hold

(39) $Bx - (1+g)Ax \geqq (1+g)cL$.

Similarly, we arrive at the profitability condition

(40) $P'B \leqq (1+R)(P'A + a_0')$

where $P = w^{-1}p$ and R is the maximal profit rate of the system. Combining conditions (39) and (40) with the constraints of (34) and (35) we find that any $x^* \in X^0$ and $z^* \in Z^0$ are not consistent with $g > 0$ and $R > 0$. Hence we have $e = g = r = 0$, and again we can relate this absence of exploitation in the sense of Morishima to the absence of a marketable surplus product in the sense of footnote 31.

In a similar vein, one can show[32] that any $x^a \notin X^0$ must be associated with a positive profit rate, <u>provided</u> that there exists a semipositive price vector P(r). But this is ensured by the existence of a von Neumann equilibrium. As Morishima shows, we can associate to any feasible $x^a \notin X^0$ the von Neumann type optimization problem

(41) Maximize the growth factor $G = (1+g)$

S.T. $[B - G(A + CA_0)]x \geqq 0, \quad x \geq 0, \quad G \gneq 0,$

where the first constraint is obtained from (39) by replacing cL by ca_0x because of the identity $L^a = a_0x^a$.

Correspondingly we transform (35) into

[32] Cf. Morishima (1974a) or Nutzinger (1976b).

(42) Minimize the interest factor $R'' = (1+r)$

S.T. $P(B - R''(A + CA_0) \leqq 0,\ P \geqslant 0,\ R'' \geqslant 0$

whereby the restriction for the price vector is obtained from (40), taking into account that $P'c = 1$ because of the subsistence wage hypothesis and the normalization of prices, $P' = p'w^{-1}$.

Using this von Neumann type approach, the Fundamental Theorem is maintained in the general case: There exists a von Neumann equilibrium with optimal solutions for the problems (41) and (42) such that $G^* \geq R''^*$ and hence $g^* \gtrsim r^*$. Moreover, if the system is indecomposable, then $g^* = r^*$. As any operation vector $x^a \notin X^o$ is associated with $P(r^* > 0)$, and by Lemma 1' with $e > 0$ as well, $e > 0$ (i.e. $x^a \notin X^o$) is necessary and sufficient for a positive rate of growth and a posituve rate of interest. This is Morishima's generalization of the Fundamental Theorem.

3. What Remains from the Fundamental Theorem?

Morishima's Generalized Fundamental Marxian Theorem (GMFT[33]) is a true generalization of the 'simple' Fundamental Theorem presented at the beginning of the last section: In the frame of our simple model we were free to replace actual values with techniques selected on the basis of profit maximization by optimal values, determined by minimization of labor requirements to produce the means of subsistence. Obviously we can replace the 'given technology assumption' of our simple quantity, price and value systems by the following optimization problem: Determine the technology (A^*, a_0^*) which minimizes $a_0' x$ subject to the constraints $x \gtrsim (I - A)^{-1} cL$, $a_0' x \gtrsim L$.

In this case the 'original' Fundamental Theorem still holds, as the necessary labor for producing the means of subsistence, $a_0^{*\prime} x(cL)$, derived from this optimization problem, obviously cannot exceed the labor amount $a_0' x(cL)$ for any other technology (A, a_0). Hence, the 'original' Theorem which presupposed actual instead of optimal values holds *a fortiori*. There are other good reasons to consider Morshima's (1974a) version as the final formulation of the

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[33] This notion and the corresponding abbreviation are from Morishima/Catephores (1978).

Fundamental Theorem, not at least because it is the only way of maintaining an equivalence between surplus labor and profits: It allows us to compare the labor actually expended (to be determined in the actual quantity system) and the minimum amount of labor necessary to produce the means of subsistence (to be determined in the stationary state where workers minimize their labor time to produce solely their means of subsistence).

But there are two interrelated objections against the economic meaningfulness of this approach:

First, what Morishima does is simply the following: He compares a stationary state with $x^a \in X^o$ and $P \subsetneq Z^o$ with a non-stationary state and shows that in the second case more labor is expended than in the former. This is an obvious consequence of labor being the sole primary input of his system. Then he calls this fact "surplus labor". Of course, Morishima/Catephores are aware of the fact that socialism - as a non-exploitative society - should not be conceived as a stationary state. In order to overcome this complication they redefine the notion of necessary product and necessary labor to allow for 'collective consuption' and for 'the expansion of output'. With these and similar redefintions they end up in defining all labor performed "necessary labor" because under socialism the citizens decide themselves how much labor to perform, and they are the beneficiaries of the output produced by their own labor. The problem with this redefinition of concepts in the case of socialism is the mere fact that the underlying decision-making processes are nowhere inlcuded in their model but are introduced from outside as an addendum in order to let vanish surplus labor and the rate of exploitation.

Second, and more important, Morishima develops the notion of 'optimal values' for his GFMT; but there is no special need for using values at all in comparing statinary and non-stationary situations. Using actual prices one can get the same: just evaluate any surplus product in terms of equilibrium (or current) prices. As long as labor is the only primary input, we always should get the desired result.

What then is left from value theory, apart from some insights into the properties of linear economic models and from nice connections between labor values and eigenvalues? I am afraid that further progress will not be achieved unless we go beyond Matrix Marx into a direct analysis of the social relations of production.[34]

[34] For interesting approaches, see e.g. Braverman (1974) and Edwards (1978).

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