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A New Identity for Generalized Hypergeometric Functions and Applications

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Abstract: We establish a new identity for generalized hypergeometric functions and apply it for first- and second-kind Gauss summation formulas to obtain some new summation formulas. The presented identity indeed extends some results of the recent published paper (*Some summation theorems for generalized hypergeometric functions*, Axioms, 7 (2018), Article 38).

Keywords: generalized hypergeometric functions; Gauss and confluent hypergeometric functions; summation theorems of hypergeometric functions

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1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers and z be a complex variable. For real or complex parameters a and b, the generalized binomial coefficient

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \binom{a}{a-b} \qquad (a, b \in \mathbb{C}),$$

in which

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

denotes the well-known gamma function for $\mathrm{Re}(z) > 0$, can be reduced to the particular case

$$\binom{a}{n} = \frac{(-1)^n (-a)_n}{n!},$$

where $(a)_b$ denotes the Pochhammer symbol [1] given by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & (b = 0,\ a \in \mathbb{C}\backslash\{0\}), \\ a(a+1)...(a+b-1) & (b \in \mathbb{C},\ a \in \mathbb{C}). \end{cases} \tag{1}$$

By referring to the symbol (1), the generalized hypergeometric functions [2]

$${}_pF_q\left(\begin{array}{c} a_1, \ \dots \ , a_p \\ b_1, \ \dots \ , b_q \end{array} \middle| \, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \tag{2}$$

are indeed a Taylor series expansion for a function, say f, as $\sum_{k=0}^{\infty} c_k^* z^k$ with $c_k^* = f^{(k)}(0)/k!$ for which the ratio of successive terms can be written as

$$\frac{c_{k+1}^*}{c_k^*} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}.$$

According to the ratio test [3,4], the series (2) is convergent for any $p \leq q+1$. In fact, it converges in $|z| < 1$ for $p = q+1$, converges everywhere for $p < q+1$ and converges nowhere ($z \neq 0$) for $p > q+1$. Moreover, for $p = q+1$ it absolutely converges for $|z| = 1$ if the condition

$$A^* = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) > 0,$$

holds and is conditionally convergent for $|z| = 1$ and $z \neq 1$ if $-1 < A^* \leq 0$ and is finally divergent for $|z| = 1$ and $z \neq 1$ if $A^* \leq -1$.

There are two important cases of the series (2) arising in many physics problems [5,6]. The first case (convergent in $|z| \leq 1$) is the Gauss hypergeometric function

$$y = {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

with the integral representation

$${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,$$
$$(\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ |\arg(1-z)| < \pi), \quad (3)$$

Replacing $z = 1$ in (3) directly leads to the well-known Gauss identity

$${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \mathrm{Re}(c-a-b) > 0. \quad (4)$$

The second case, which converges everywhere, is the Kummer confluent hypergeometric function

$$y = {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!},$$

with the integral representation

$${}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} dt,$$
$$(\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ |\arg(1-z)| < \pi).$$

In this paper, we explicitly obtain the simplified form of the hypergeometric series

$${}_pF_q\left(\begin{array}{ccc} a_1, \dots, a_{p-1}, & m+1 \\ b_1, \dots, b_{q-1}, & n+1 \end{array} \middle| z\right),$$

when m, n are two natural numbers and $m < n$.

2. A New Identity for Generalized Hypergeometric Functions

Let m, n be two natural numbers so that $m < n$. By noting (1), since

$$\frac{(m+1)_k}{(n+1)_k} = \frac{\Gamma(k+m+1)\Gamma(n+1)}{\Gamma(k+n+1)\Gamma(m+1)} = \frac{n!}{m!}\frac{1}{(k+m+1)(k+m+2)...(k+n)},$$

so, we have

$$\frac{(m+1)_k}{k!(n+1)_k} = \frac{\Gamma(k+m+1)\Gamma(n+1)}{k!\,\Gamma(k+n+1)\Gamma(m+1)} = \frac{n!}{m!}\frac{(k+1)_m}{(k+n)!}. \tag{5}$$

Hence, substituting (5) into a special case of (2) yields

$$\begin{aligned} {}_pF_q\left(\begin{matrix} a_1, & ..., a_{p-1}, & m+1 \\ b_1, & ..., b_{q-1}, & n+1 \end{matrix} \middle| z\right) &= \frac{n!}{m!}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_{p-1})_k}{(b_1)_k...(b_{q-1})_k}z^k\frac{(k+1)_m}{(k+n)!} \\ &= \frac{n!}{m!}\sum_{j=n}^{\infty}\frac{(a_1)_{j-n}...(a_{p-1})_{j-n}}{(b_1)_{j-n}...(b_{q-1})_{j-n}}z^{j-n}\frac{(j+1-n)_m}{j!}. \end{aligned} \tag{6}$$

In [7], two particular cases of (6) for $m = 0$ and $m = 1$ were considered and other cases have been left as open problems. In this section, we wish to consider those open problems and solve them for any arbitrary value of m. For this purpose, since

$$(a)_{j-n} = \frac{\Gamma(a-n)}{\Gamma(a)}(a-n)_j = (-1)^n\frac{(a-n)_j}{(1-a)_n},$$

relation (6) is simplified as

$$\begin{aligned} {}_pF_q\left(\begin{matrix} a_1, & ..., a_{p-1}, & m+1 \\ b_1, & ..., b_{q-1}, & n+1 \end{matrix} \middle| z\right) &= \frac{n!}{m!}\frac{(-1)^{n(p-q)}}{z^n}\frac{(1-b_1)_n...(1-b_{q-1})_n}{(1-a_1)_n...(1-a_{p-1})_n} \\ &\times\sum_{j=n}^{\infty}\frac{(a_1-n)_j...(a_{p-1}-n)_j}{(b_1-n)_j...(b_{q-1}-n)_j}\frac{z^j}{j!}(j+1-n)_m. \end{aligned} \tag{7}$$

It is clear in (7) that

$$\sum_{j=n}^{\infty}\frac{(a_1-n)_j...(a_{p-1}-n)_j}{(b_1-n)_j...(b_{q-1}-n)_j}\frac{z^j}{j!}(j+1-n)_m = \sum_{j=0}^{\infty}(.) - \sum_{j=0}^{n-1}(.) = S_1^* - S_2^*. \tag{8}$$

To evaluate $S_1^* = \sum_{j=0}^{\infty}(.)$, we can directly use Chu-Vandermonde identity, which is a special case of Gauss identity (4), i.e.,

$${}_2F_1\left(\begin{matrix} -m, & q-p \\ & q \end{matrix} \middle| 1\right) = \frac{(p)_m}{(q)_m}. \tag{9}$$

Now if in (9), $p = j-n+1$ and $q = -n+1$, we have

$$(j-n+1)_m = (1-n)_m {}_2F_1\left(\begin{matrix} -m, & -j \\ & 1-n \end{matrix} \middle| 1\right) = (1-n)_m\sum_{k=0}^{m}\frac{(-m)_k(-j)_k}{(1-n)_k k!}. \tag{10}$$

Hence, replacing (10) in S_1^* gives

$$\begin{aligned} S_1^* &= \sum_{j=0}^{\infty} \frac{(a_1-n)_j ... (a_{p-1}-n)_j}{(b_1-n)_j ... (b_{q-1}-n)_j} \frac{z^j}{j!} (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k (-j)_k}{(1-n)_k k!} \\ &= (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k}{(1-n)_k k!} \left(\sum_{j=k}^{\infty} \frac{(a_1-n)_j ... (a_{p-1}-n)_j}{(b_1-n)_j ... (b_{q-1}-n)_j} z^j \frac{(-j)_k}{j!} \right). \end{aligned} \tag{11}$$

It is important to note in the second equality of (11) that $(-j)_k = 0$ for any $j = 0, 1, 2, \ldots, k-1$. Therefore, the lower index is starting from $j = k$ instead of $j = 0$. Now since

$$\frac{(-j)_k}{j!} = \frac{(-1)^k}{(j-k)!},$$

relation (11) is simplified as

$$\begin{aligned} S_1^* &= (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k}{(1-n)_k k!} \left(\sum_{j=k}^{\infty} \frac{(a_1-n)_j ... (a_{p-1}-n)_j}{(b_1-n)_j ... (b_{q-1}-n)_j} z^j \frac{(-1)^k}{(j-k)!} \right) \\ &= (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k (-z)^k}{(1-n)_k k!} \left(\sum_{r=0}^{\infty} \frac{(a_1-n)_{r+k} ... (a_{p-1}-n)_{r+k}}{(b_1-n)_{r+k} ... (b_{q-1}-n)_{r+k}} \frac{z^r}{r!} \right). \end{aligned} \tag{12}$$

On the other hand, the well-known identity

$$(a)_{r+k} = (a)_k (a+k)_r,$$

simplifies (12) as

$$\begin{aligned} S_1^* &= (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k (a_1-n)_k ... (a_{p-1}-n)_k}{(1-n)_k (b_1-n)_k ... (b_{q-1}-n)_k} \frac{(-z)^k}{k!} \\ &\quad \times \left(\sum_{r=0}^{\infty} \frac{(a_1-n+k)_r ... (a_{p-1}-n+k)_r}{(b_1-n+k)_r ... (b_{q-1}-n+k)_r} \frac{z^r}{r!} \right) \\ &= (1-n)_m \sum_{k=0}^{m} \frac{(-m)_k (a_1-n)_k ... (a_{p-1}-n)_k}{(1-n)_k (b_1-n)_k ... (b_{q-1}-n)_k} \frac{(-z)^k}{k!} \\ &\quad \times {}_{p-1}F_{q-1} \left(\begin{matrix} a_1-n+k, & ... & a_{p-1}-n+k \\ b_1-n+k, & ... & b_{q-1}-n+k \end{matrix} \middle| \, z \right). \end{aligned}$$

To compute the finite sum $S_2^* = \sum_{j=0}^{n-1} (.)$ in (8), we can directly use the identity

$$(j-n+1)_m = \frac{(-n+1)_m (-n+1+m)_j}{(-n+1)_j},$$

to get

$$
\begin{aligned}
S_2^* &= \sum_{j=0}^{n-1} \frac{(a_1-n)_j \ldots (a_{p-1}-n)_j}{(b_1-n)_j \ldots (b_{q-1}-n)_j} \frac{z^j}{j!} (j+1-n)_m \\
&= (1-n)_m \sum_{j=0}^{n-1} \frac{(a_1-n)_j \ldots (a_{p-1}-n)_j}{(b_1-n)_j \ldots (b_{q-1}-n)_j} \frac{z^j}{j!} \frac{(-n+1+m)_j}{(-n+1)_j} \\
&= (1-n)_m \, {}_pF_q \left(\begin{array}{ccc} a_1-n, & \ldots\, a_{p-1}-n, & -(n-1-m) \\ b_1-n, & \ldots\, b_{q-1}-n, & -(n-1) \end{array} \middle| \, z \right). \qquad (13)
\end{aligned}
$$

Finally, by noting the identity

$$
\frac{(-n+1)_m}{m!} = (-1)^m \binom{n-1}{m},
$$

the main result of this paper is obtained as follows.

Main Theorem. *If m, n are two natural numbers so that $m < n$, then*

$$
\begin{aligned}
&{}_pF_q \left(\begin{array}{ccc} a_1, & \ldots\, a_{p-1}, & m+1 \\ b_1, & \ldots\, b_{q-1}, & n+1 \end{array} \middle| \, z \right) = n! \binom{n-1}{m} \frac{(-1)^{n(p-q)+m}}{z^n} \frac{(1-b_1)_n \ldots (1-b_{q-1})_n}{(1-a_1)_n \ldots (1-a_{p-1})_n} \\
&\times \left\{ \begin{array}{r} \sum_{k=0}^{m} \frac{(-m)_k (a_1-n)_k \ldots (a_{p-1}-n)_k}{(1-n)_k (b_1-n)_k \ldots (b_{q-1}-n)_k} \, {}_{p-1}F_{q-1} \left(\begin{array}{ccc} a_1-n+k, & \ldots & a_{p-1}-n+k \\ b_1-n+k, & \ldots & b_{q-1}-n+k \end{array} \middle| \, z \right) \frac{(-z)^k}{k!} \\ -\, {}_pF_q \left(\begin{array}{ccc} a_1-n, & \ldots\, a_{p-1}-n, & -(n-1-m) \\ b_1-n, & \ldots\, b_{q-1}-n, & -(n-1) \end{array} \middle| \, z \right) \end{array} \right\}, \qquad (14)
\end{aligned}
$$

where $\{a_k\}_{k=1}^{p-1} \notin \{1, 2, \ldots, n\}$ and $\{b_k\}_{k=1}^{q-1} \notin \{n, n-1, \ldots, n-m+1\}$.

Note that the case $m > n$ in (14) leads to a particular case of Karlsson-Minton identity, see e.g., [8,9].

3. Some Special Cases of the Main Theorem

Essentially whenever a generalized hypergeometric series can be summed in terms of gamma functions, the result will be important as only a few such summation theorems are available in the literature. In this sense, the classical summation theorems such as Kummer and Gauss for ${}_2F_1$, Dixon, Watson, Whipple and Pfaff-Saalschutz for ${}_3F_2$, Whipple for ${}_4F_3$, Dougall for ${}_5F_4$ and Dougall for ${}_7F_6$ are well known [1,10]. In this section, we consider some special cases of the above main theorem to obtain new hypergeometric summation formulas.

Special case 1. Note that if $m = 0$, the first equality of (13) reads as

$$
S_2^* = \sum_{j=0}^{n-1} \frac{(a_1-n)_j \ldots (a_{p-1}-n)_j}{(b_1-n)_j \ldots (b_{q-1}-n)_j} \frac{z^j}{j!}.
$$

Hence, the main theorem is simplified as

$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \ldots\, a_{p-1}, & 1 \\ b_1, & \ldots\, b_{q-1}, & n+1 \end{array}\middle|\, z\right) = n!\frac{(-1)^{n(p-q)}}{z^n}\frac{(1-b_1)_n\ldots(1-b_{q-1})_n}{(1-a_1)_n\ldots(1-a_{p-1})_n} $$
$$ \times\left({}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-n,\ \ldots,\ a_{p-1}-n \\ b_1-n,\ \ldots,\ b_{q-1}-n \end{array}\middle|\, z\right) - \sum_{j=0}^{n-1}\frac{(a_1-n)_j\ldots(a_{p-1}-n)_j}{(b_1-n)_j\ldots(b_{q-1}-n)_j}\frac{z^j}{j!}\right), $$

which is a known result in the literature [10] (p. 439).

Special case 2. For $n = m+1$, relation (13) gives $S_2^* = (-1)^m m!$ and the main theorem therefore reads (for $m+1 \to m$) as

$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \ldots\, a_{p-1}, & m \\ b_1, & \ldots\, b_{q-1}, & m+1 \end{array}\middle|\, z\right) = (-1)^{m(p-q+1)}\frac{m!}{z^m}\frac{(1-b_1)_m\ldots(1-b_{q-1})_m}{(1-a_1)_m\ldots(1-a_{p-1})_m}\times $$
$$ \left\{1-\sum_{k=0}^{m-1}\frac{(a_1-m)_k\ldots(a_{p-1}-m)_k}{(b_1-m)_k\ldots(b_{q-1}-m)_k}\,{}_{p-1}F_{q-1}\left(\begin{array}{ccc} a_1-m+k, & \ldots & a_{p-1}-m+k \\ b_1-m+k, & \ldots & b_{q-1}-m+k \end{array}\middle|\, z\right)\frac{(-z)^k}{k!}\right\}. $$

For instance, we have [7]

$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \ldots\, a_{p-1}, & 2 \\ b_1, & \ldots\, b_{q-1}, & 3 \end{array}\middle|\, z\right) = \frac{2}{z^2}\frac{(1-b_1)_2\ldots(1-b_{q-1})_2}{(1-a_1)_2\ldots(1-a_{p-1})_2} $$
$$ \times\left(\frac{(a_1-2)\ldots(a_{p-1}-2)}{(b_1-2)\ldots(b_{q-1}-2)}z\,{}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-1,\ \ldots,\ a_{p-1}-1 \\ b_1-1,\ \ldots,\ b_{q-1}-1 \end{array}\middle|\, z\right)\right. $$
$$ \left.-\,{}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-2,\ \ldots,\ a_{p-1}-2 \\ b_1-2,\ \ldots,\ b_{q-1}-2 \end{array}\middle|\, z\right)+1\right). $$

As a very particular case, replacing $p = 3$ and $q = 2$ in the above relation yields

$$ {}_3F_2\left(\begin{array}{c} a,\ b,\ 2 \\ c,\ 3 \end{array}\middle|\, 1\right) $$
$$ = \frac{2}{(a-2)_2(b-2)_2}\left((c-2)_2+\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}(ab-a-b-c+3)\right). $$

Special case 3. For $p = q = 1$, the main theorem is simplified as

$$ {}_1F_1\left(\begin{array}{c} m+1 \\ n+1 \end{array}\middle|\, z\right) $$
$$ = n!\binom{n-1}{m}\frac{(-1)^m}{z^n}\left(e^z\,{}_1F_1\left(\begin{array}{c} -m \\ -(n-1) \end{array}\middle|\, -z\right) - {}_1F_1\left(\begin{array}{c} -(n-1-m) \\ -(n-1) \end{array}\middle|\, z\right)\right). $$

For instance, by referring to the special case 1, we have [7,10]

$$ {}_1F_1\left(\begin{array}{c} 1 \\ m \end{array}\middle|\, z\right) = \frac{(m-1)!}{z^{m-1}}\left(e^z - \sum_{j=0}^{m-2}\frac{z^j}{j!}\right). $$

Special case 4. For $p=2$ and $q=1$, the main theorem is simplified as

$$ {}_2F_1\left(\begin{array}{c} a,\ m+1 \\ n+1 \end{array}\middle|\, z\right)=n!\left(\begin{array}{c} n-1 \\ m \end{array}\right)\frac{(-1)^{n+m}}{z^n}\frac{1}{(1-a)_n} \times\left\{(1-z)^{n-a}{}_2F_1\left(\begin{array}{c} a-n,\ -m \\ -(n-1) \end{array}\middle|\, \frac{z}{z-1}\right)-{}_2F_1\left(\begin{array}{c} a-n,\ -(n-1-m) \\ -(n-1) \end{array}\middle|\, z\right)\right\}, $$

in which we have used the relation ${}_1F_0\left(\begin{array}{c} a \\ - \end{array}\middle|\, z\right)=(1-z)^{-a}$. For instance, by referring to the special case 1, we have [7,10]

$$ {}_2F_1\left(\begin{array}{c} a,\ 1 \\ m \end{array}\middle|\, z\right)=\frac{(m-1)!}{z^{m-1}}\frac{\Gamma(1-a)}{\Gamma(m-a)}\left((1-z)^{m-a-1}-\sum_{j=0}^{m-2}(a-m+1)_j\frac{z^j}{j!}\right). $$

Special case 5. For $p=3$ and $q=2$, the main theorem is simplified as

$$ {}_3F_2\left(\begin{array}{c} a_1,\ a_2,\ m+1 \\ b_1,\ n+1 \end{array}\middle|\, z\right)=n!\left(\begin{array}{c} n-1 \\ m \end{array}\right)\frac{(-1)^{n+m}}{z^n}\frac{(1-b_1)_n}{(1-a_1)_n(1-a_2)_n} \times\left\{\begin{array}{r} \sum_{k=0}^{m}\frac{(-m)_k(a_1-n)_k(a_2-n)_k}{(1-n)_k(b_1-n)_k}{}_2F_1\left(\begin{array}{c} a_1-n+k,\ a_2-n+k \\ b_1-n+k \end{array}\middle|\, z\right)\frac{(-z)^k}{k!} \\ -{}_3F_2\left(\begin{array}{c} a_1-n,\ a_2-n,\ -(n-1-m) \\ b_1-n,\ -(n-1) \end{array}\middle|\, z\right) \end{array}\right\}. \quad (15) $$

As a particular case and by noting the first kind of Gauss formula (4), if $z=1$ is replaced in (15) then we get

$$ {}_3F_2\left(\begin{array}{c} a_1,\ a_2,\ m+1 \\ b_1,\ n+1 \end{array}\middle|\, 1\right)=(-1)^{n+m}n!\left(\begin{array}{c} n-1 \\ m \end{array}\right)\frac{(1-b_1)_n}{(1-a_1)_n(1-a_2)_n} \times\left\{\begin{array}{r} \sum_{k=0}^{m}\frac{(-m)_k(a_1-n)_k(a_2-n)_k}{(1-n)_k(b_1-n)_k}\frac{\Gamma(b_1-n+k)\Gamma(b_1-a_1-a_2+n-k)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)}\frac{(-1)^k}{k!} \\ -{}_3F_2\left(\begin{array}{c} a_1-n,\ a_2-n,\ -(n-1-m) \\ b_1-n,\ -(n-1) \end{array}\middle|\, 1\right) \end{array}\right\}. $$

Therefore, we get

$$ {}_3F_2\left(\begin{array}{c} a_1,\ a_2,\ m+1 \\ b_1,\ n+1 \end{array}\middle|\, 1\right)=\left(\begin{array}{c} n-1 \\ m \end{array}\right)\frac{(-1)^m n!}{(1-a_1)_n(1-a_2)_n} \times\left\{\begin{array}{r} (b_1-a_1-a_2)_n{}_2F_1\left(\begin{array}{c} a_1,\ a_2 \\ b_1 \end{array}\middle|\, 1\right){}_3F_2\left(\begin{array}{c} a_1-n,\ a_2-n,\ -m \\ 1-n+a_1+a_2-b_1,\ 1-n \end{array}\middle|\, 1\right) \\ -(-1)^n(1-b_1)_n{}_3F_2\left(\begin{array}{c} a_1-n,\ a_2-n,\ -(n-1-m) \\ b_1-n,\ 1-n \end{array}\middle|\, 1\right) \end{array}\right\}. \quad (16) $$

As a numerical example for the result (16), we have

$$ {}_3F_2\left(\begin{array}{c} 1/5,\ 3/10,\ 2 \\ 4/5,\ 5 \end{array} \middle| 1 \right) = \frac{72}{(4/5)_4(7/10)_4} \times \left((1/5)_4 \sum_{k=0}^{2} \frac{(-2)_k(-19/5)_k(-37/10)_k}{(-3)_k(-16/5)_k k!} - (3/10)_4 \frac{\Gamma(4/5)\Gamma(3/10)}{\Gamma(3/5)\Gamma(1/2)} \sum_{k=0}^{1} \frac{(-1)_k(-19/5)_k(-37/10)_k}{(-3)_k(-33/10)_k k!} \right). $$

It is clear that the right-hand side of this equality can be easily computed and therefore the infinite series in the left-hand side has been evaluated.

Similarly, by noting the second kind of Gauss formula [1]

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ (a+b+1)/2 \end{array} \middle| \frac{1}{2} \right) = \frac{\sqrt{\pi}\,\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}, $$

relation (15) takes the form

$$ {}_3F_2\left(\begin{array}{c} a_1,\ a_2,\ m+1 \\ b_1,\ n+1 \end{array} \middle| \frac{1}{2} \right) = (-1)^{n+m} 2^n n! \binom{n-1}{m} \frac{(1-b_1)_n}{(1-a_1)_n(1-a_2)_n} \times \left\{ \sqrt{\pi} \sum_{k=0}^{m} \frac{(-m)_k(a_1-n)_k(a_2-n)_k}{(1-n)_k(b_1-n)_k} \frac{\Gamma(-n+k+b_1)}{\Gamma((a_1-n+k+1)/2)\Gamma((a_2-n+k+1)/2)} \frac{(-1)^k}{2^k k!} - {}_3F_2\left(\begin{array}{c} a_1-n,\ a_2-n,\ -(n-1-m) \\ b_1-n,\ -(n-1) \end{array} \middle| \frac{1}{2} \right) \right\}, $$

where $b_1 = (a_1 + a_2 + 1)/2$.

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