The Navier-Stokes equations with time discretisation and Lagrangian approximation

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Introduction

In the present thesis, we consider the nonstationary nonlinear NAVIER-STOKES equations in $(0,T)\times\Omega$, where $\Omega\subset\mathbb{R}^3$ is a bounded domain and $0<T\in\mathbb{R}$:

$$\begin{aligned} \partial_t v - \nu\Delta v + \nabla p + v\cdot\nabla v &= f && \text{in } (0,T)\times\Omega, \\ \nabla\cdot v &= 0 && \text{in } (0,T)\times\Omega, \\ v_{|\partial\Omega} &= 0 && \text{in } [0,T), \\ v_{|t=0} &= v_0 && \text{in } \Omega. \end{aligned} \tag{N_0}$$

This system describes the motion of an incompressible, nonstationary, and viscous fluid like water or oil [20, p. 1].

The external force density f, the initial velocity v_0, and the kinematic viscosity constant $\nu>0$ are given data, while the vector v denotes the unknown velocity and p some unknown pressure function.

The system results from the balance of forces, including the conservation of momentum, of mass, and of energy (compare SHINBROT [19, pp. 10–25, 102–111]). Here the nonlinear convective term $v\cdot\nabla v := (v\cdot\nabla)\,v$ results from the material derivative of the velocity field v.

In the case $\Omega\subset\mathbb{R}^2$ there exists a uniquely determined global (in time) solution of the NAVIER-STOKES equations. In the present case $\Omega\subset\mathbb{R}^3$ such result exists only under smallness assumptions on the data. Without such smallness assumptions only the existence of local (in time) strong solutions could be proved up to now. These results were first proved by LADYZHENSKAYA [12]. In addition, there always exist so called weak solutions global in time, see HOPF [11]. Their regularity, however, does not suffice to prove uniqueness.

In the present thesis, we combine the NAVIER-STOKES equations – which correspond to the so-called EULERIAN representation of fluid flow – with the LAGRANGIAN description of fluid flow.

The LAGRANGIAN representation of fluid flow describes the motion of the particles of the fluid. For a single particle in a given velocity field v starting at time $t=s$ in $x_s\in\Omega$, we can describe its trajectory by the mapping

$$t\mapsto x(t) =: X(t,s,x_s).$$

This mapping is a solution of the initial value problem

$$\begin{aligned} \dot{x}(t) &= v(t,x(t)), \\ x(s) &= x_s. \end{aligned}$$

For a stationary fluid flow the trajectories of the particles do not depend on the starting time and coincide with the streamlines of the velocity field. Hence, for a particle starting at time $t = 0$ in $x_0 \in \Omega$, the trajectory is described by

$$t \mapsto x(t) =: X(t, x_0),$$

where $x(t)$ is the solution of the autonomous initial value problem

$$\begin{aligned} \dot{x}(t) &= v(x(t)), \\ x(0) &= x_0. \end{aligned}$$

In the present thesis, we use a time discretisation with stepsize $\varepsilon > 0$ to transfer the NAVIER-STOKES equations into a finite number of steady boundary value problems, the solutions of which approximate $v(t)$ at the grid points of the time grid. In these systems we approximate the nonlinear term by using the LAGRANGIAN representation.

A similar ansatz by combining these two representations was used by VARNHORN [22, pp. 121–155] and by ASANALIEVA, HEUTLING & VARNHORN [5, pp. 213–229]. In both papers, the LAGRANGIAN representation is applied to the non-steady NAVIER-STOKES equations. In contrast to previous works, using a particle method based on unsteady velocity fields (compare e. g. VARNHORN [22,23]), here particle methods based on steady velocity fields are used, which improves the convergence results and simplifies the methods.

Parts of this thesis are based on the investigations of SHINBROT [19, pp. 159–179], who considered a discretized NAVIER-STOKES system with vanishing external forces.

The thesis is organized as follows: In Chapter 1 we define our notation and the function spaces used. We state the most important imbedding results and some elementary inequalities. Finally, we consider weak convergence of functions and describe some important properties of the STOKES operator.

In the second chapter we consider the LAGRANGIAN representation of stationary fluid flow. In particular, for a trajectory $t \mapsto x(t) =: X(t, x_0)$ of a particle starting at time $t = 0$ in x_0 in a given velocity field v, we prove existence, uniqueness, and the conservation of measure, which implies the important properties

$$\begin{aligned} \|f \circ X(t, \cdot)\|_{0,p} &= \|f(\cdot)\|_{0,p}\,, \\ \langle f \circ X(t, \cdot), g \circ X(t, \cdot)\rangle &= \langle f, g\rangle\,, \end{aligned}$$

valid for functions $f \in L^p(\Omega)$ or $f, g \in L^2(\Omega)$, respectively. Here $\|\cdot\|_{0,p}$ denotes the L^p-norm and $\langle \cdot, \cdot \rangle$ the L^2-scalar product.

In Chapter 3, we introduce a time discretisation for the NAVIER-STOKES equations defining an equidistant time grid in $[0, T]$ with stepsize $0 < \varepsilon := \frac{T}{N}$ for some $2 \leq N \in \mathbb{N}$. For $k = 0, \ldots, N$ we denote the grid points by $t_k := k\varepsilon$.

Now, restricting the NAVIER-STOKES equations (N_0) to the time $t = t_k$ for $k = 1, \dots, N$, we modify the resulting steady system by approximating the terms in the following way: We approximate the time derivative $\partial_t v(t_{k+1})$ by a backwards difference quotient

$$\partial_t v(t_{k+1}) \approx \frac{v(t_{k+1}) - v(t_k)}{\varepsilon}$$

and replace $f(t_{k+1})$ by the average

$$f(t_{k+1}) \approx \frac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} f(\tau)\ \mathrm{d}\tau =: f^{k+1}.$$

Using a time delay, we approximate the nonlinear term $v(t_{k+1}) \cdot \nabla v(t_{k+1})$ by the linearization

$$v(t_{k+1}) \cdot \nabla v(t_{k+1}) \approx v(t_k) \cdot \nabla v(t_{k+1}).$$

Then we use the LAGRANGIAN description of fluid flow to further simplify this term. In particular, we use the central total (LAGRANGIAN) difference quotient

$$L_\varepsilon^k v(t_{k+1}, x) := \frac{1}{2\varepsilon} \{v(t_{k+1}, X_k(\varepsilon, x)) - v(t_{k+1}, X_k(-\varepsilon, x))\},$$

where the mapping X_k results from the solution of the initial value problem

$$\begin{aligned} \dot{x}(t) &= v(t_k, x(t)), \\ x(0) &= x_0. \end{aligned}$$

Hence, we obtain the following approximation for the nonlinear term in $t = t_{k+1}$, $k = 0, \dots, N-1$:

$$\begin{aligned} v(t_{k+1}) \cdot \nabla v(t_{k+1}) &\approx v(t_k) \cdot \nabla v(t_{k+1}) \\ &\approx \frac{1}{2\varepsilon} \{v(t_{k+1}, X_k(\varepsilon, x)) - v((t_{k+1}, X_k(-\varepsilon, x))\} \\ &=: L_\varepsilon^k v(t_{k+1}, x). \end{aligned}$$

For $k = 0, \dots, N-1$ this leads to the steady boundary value problem

$$\begin{aligned} v^{k+1} - \varepsilon\nu\Delta v^{k+1} + \varepsilon L_\varepsilon^k v^{k+1} + \varepsilon\nabla p^{k+1} &= \varepsilon f^{k+1} + v^k && \text{in } \Omega, \\ \nabla \cdot v^{k+1} &= 0 && \text{in } \Omega, \\ v^{k+1}_{|\partial\Omega} &= 0, \end{aligned} \tag{N_ε^k}$$

where we set

$$L_\varepsilon^k v^{k+1}(x) := \frac{1}{2\varepsilon} \{v^{k+1}(X_k(\varepsilon, x)) - v^{k+1}(X_k(-\varepsilon, x))\},$$

with mapping X_k resulting from the initial value problem

$$\dot{x}(t) = v^k(x(t)),$$
$$x(0) = x_0.$$

Assuming v^k to be an approximation of the solution $t \mapsto v(t)$ of (N_0) at time $t = t_k$, this system suggests v^{k+1} and p^{k+1} to be approximations of $v(t_{k+1})$ and $p(t_{k+1})$. Thus, starting with $v^0 := v_0$ we obtain, successively for $k = 0, \dots, N-1$, approximative solutions v^{k+1} of $v(t_{k+1})$ (compare Figure 1).

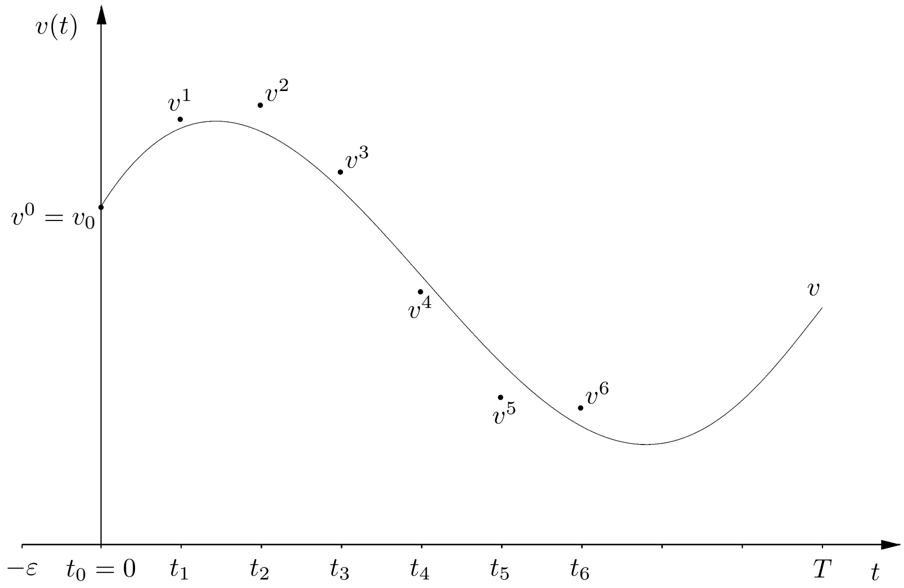

Figure 1: Approximative solutions v^{k+1} of $v(t_{k+1})$ for $k = 0, \dots, 5$

In Chapter 4 we prove existence and uniqueness of a weak solution of (N_ε^k) for fixed $k \in \{0, \dots, N-1\}$. As in HEYWOOD [10, pp. 650–653], we use a GALERKIN ansatz based on the eigenfunctions of the STOKES operator. We can prove that the whole sequence of GALERKIN approximations converges to a uniquely determined weak solution v^{k+1} of (N_ε^k) if $v^k \in C^1(\overline{\Omega})$ is divergence-free with vanishing values on the boundary. Finally, we derive some a-priori estimates and some regularity statements of v^{k+1}.

In the fifth Chapter, we use the steady weak solutions from Chapter 4 to define a non-steady velocity field $v^\varepsilon : [-\varepsilon, T] \to \mathbb{R}^3$ piecewise constant in time by

$$v^\varepsilon(t) := \begin{cases} v_0 & , t \in [-\varepsilon, 0] \\ v^{k+1} & , t \in (t_k, t_{k+1}], \qquad k = 0, \dots, N-1 \end{cases}$$

(compare Figure 2).

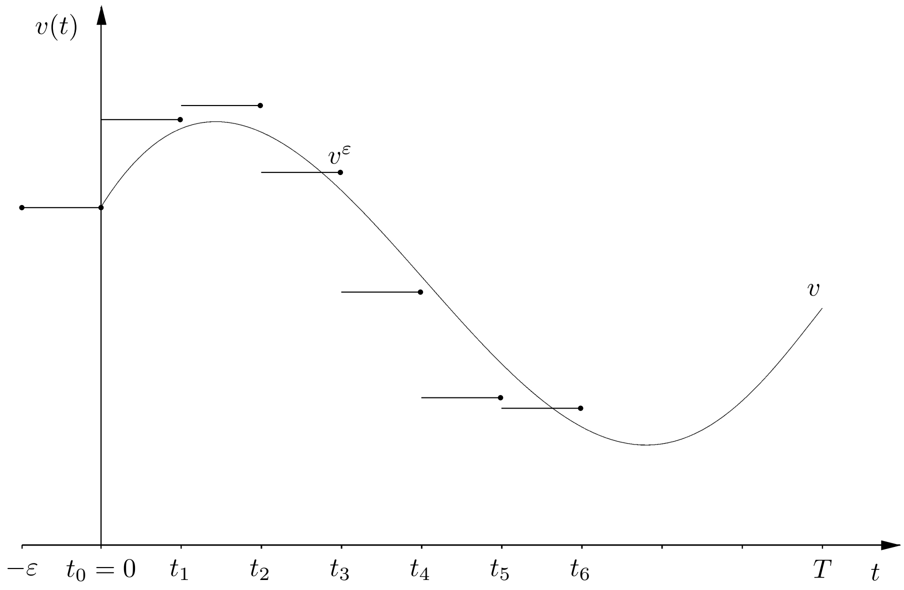

Figure 2: The non-steady velocity field v^ε on $[-\varepsilon, t_6]$

We first prove, that for $v_0 \in C^1(\overline{\Omega})$ being divergence-free with vanishing values on the boundary, v^ε is well defined. Then we derive some regularity properties and prove that v^ε satisfies the energy equality at the gridpoints t_k, $k = 0, \ldots, N$. In addition, we establish some a-priori estimates for v^ε.

In the last Chapter 6 we consider the function v^ε and proceed to the limit as $\varepsilon \to 0$. Here we construct a subsequence $\{v^{\varepsilon_N}\}_N$ satisfying

$$
\begin{aligned}
v^{\varepsilon_N}(t) &\overset{N\to\infty}{\rightharpoonup} v(t) && \text{in } \mathcal{H}^0(\Omega) \text{ for all } t \in [0,T],\\
v^{\varepsilon_N} &\overset{N\to\infty}{\longrightarrow} v && \text{in } L^2(0,T;\mathcal{H}^0(\Omega)),\\
v^{\varepsilon_N} &\overset{N\to\infty}{\rightharpoonup} v && \text{in } L^2(0,T;\mathcal{H}^1(\Omega))
\end{aligned}
$$

for some function

$$
v \in L^\infty\left(0,T;\mathcal{H}^0(\Omega)\right) \cap L^2\left(0,T;\mathcal{H}^1(\Omega)\right).
$$

These convergence properties suffice to prove that v is a weak solution of the Navier-Stokes equations (N_0), satisfying the energy inequality. Here the limit procedure in the nonlinear term is the most crucial point.

1 Preliminaries

At first we outline our notation. All definitions and results are contained in common books about functional analysis, differential equations and NAVIER-STOKES equations. In particular, we mostly use ALT [2], WERNER [25], EVANS [9], SCHWEIZER [18], ADAMS & FOURNIER [1], SOHR [20] and TEMAM [21].

By $\mathbb{N}$ we denote the *natural numbers*, where we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By $\mathbb{Q}$ we denote the *rational* and by $\mathbb{R}$ the *real numbers*.

For some real numbers a, b with $a < b$ we define the intervals

$$\begin{aligned} [a,b] &:= \{x \in \mathbb{R} \mid a \leq x \leq b\}, \\ [a,b) &:= \{x \in \mathbb{R} \mid a \leq x < b\}, \\ (a,b] &:= \{x \in \mathbb{R} \mid a < x \leq b\}, \\ (a,b) &:= \{x \in \mathbb{R} \mid a < x < b\}. \end{aligned}$$

We consider the EUCLIDIAN *space*

$$\mathbb{R}^3 := \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}, \quad i = 1,2,3\}.$$

For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$, a scalar product is defined by

$$x \cdot y := \sum_{i=1}^{3} x_i y_i,$$

and

$$|x| := \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

denotes the EUCLIDIAN *norm*.

Throughout this chapter, let $\Omega \subset \mathbb{R}^3$ denote a bounded domain with smooth boundary $\partial\Omega$ at least of class C^1. We set $\overline{\Omega} := \Omega \cup \partial\Omega$.

We use the same symbols for scalar-valued and vector-valued real functions.

For some function v defined in Ω, by

$$\partial_i v := \frac{\partial v}{\partial x_i} \quad , i = 1,2,3,$$

we denote the *partial derivative with respect to the i^{th} coordinate*. Here $\partial_t v$ means the partial derivative with respect to the time t. We also write

$$\partial_t v = \frac{\partial v}{\partial t} = \dot{v}.$$

For any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with *length*

$$|\alpha| := \sum_{i=1}^{3} \alpha_i,$$

we define

$$\partial^\alpha v := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} v$$

as *partial derivative of order* $|\alpha|$.

Using the *gradient* $\nabla := (\partial_1, \partial_2, \partial_3)$, for a scalar function $p : \Omega \to \mathbb{R}$ we can define the vector

$$\nabla p := (\partial_1, \partial_2, \partial_3)p = (\partial_1 p, \partial_2 p, \partial_3 p),$$

and for vector functions $v = (v_1, v_2, v_3)$, $v : \Omega \to \mathbb{R}^3$, we define the 3×3-matrix

$$\nabla v := (\partial_1, \partial_2, \partial_3)v = (\partial_i v_j)_{ji}, \quad i, j = 1, 2, 3.$$

Here, for 3×3-matrices $A = (a_{ij})_{i,j=1,2,3}$ and $B = (b_{ij})_{i,j=1,2,3}$, the Frobenius *scalar product* is defined by

$$A \cdot B := \sum_{i,j=1}^{3} a_{ij} b_{ij}.$$

The *divergence* of a vector function v is defined by

$$\nabla \cdot v := \sum_{i=1}^{3} \partial_i v_i.$$

If $\nabla \cdot v = 0$ in Ω we call v *divergence-free* or *solenoidal*.

Finally, we define the Laplace *operator* by

$$\Delta v := \nabla \cdot \nabla v = (\nabla \cdot \nabla) v = (\partial_1^2 + \partial_2^2 + \partial_3^2) v.$$

If $v : (t, x) \mapsto v(t, x)$ is a function defined in $[0, T] \times \Omega$, for $t \in [0, T]$ fixed we denote by $v(t) := v(t, \cdot)$ the function defined by $x \mapsto (v(t))(x) := v(t, x)$ in Ω. Similarly, for $x \in \Omega$ fixed, by $v(x) := v(\cdot, x)$ we denote the function defined by $t \mapsto (v(x))(t) := v(t, x)$ in $[0, T]$.

By $v_{|\partial\Omega}$ we denote the restriction of v on $\partial\Omega$ and by $v_{|t=a}$ for $a \in [0, T]$ we mean $v(a)$.

For two functions $f, g : [0, T] \to \mathbb{R}$ and some value $a \in [0, T]$ we write

$$f \in o(g) \text{ as } t \to a \quad \text{if } \lim_{t\to a} \left| \frac{f(t)}{g(t)} \right| = 0,$$

$$f \in \mathcal{O}(g) \text{ as } t \to a \quad \text{if } \limsup_{t\to a} \left| \frac{f(t)}{g(t)} \right| < \infty.$$

For any sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ we write $\{\alpha_i\}_i$.

1.1 Function spaces

1.1.1 Spaces of continuous functions

Let $m \in \mathbb{N}_0$. We define the function space $C^m(\Omega)$ of functions v continuous in Ω, for which all partial derivatives $\partial^\alpha v$ of order $|\alpha| \leq m$ are continuous as well. We set $C(\Omega) := C^0(\Omega)$ and

$$C^\infty(\Omega) := \bigcap_{m=0}^{\infty} C^m(\Omega).$$

For $0 \leq m \leq \infty$, by

$$C_B^m(\Omega) := \{v \in C^m(\Omega) \mid v \text{ is bounded}\}$$

we denote the subspace of bounded functions in $C^m(\Omega)$.

Defining the *support* of some function v by

$$\operatorname{supp} v := \overline{\{x \in \Omega \mid v(x) \neq 0\}},$$

we set

$$C_0^m(\Omega) := \{v \in C^m(\Omega) \mid \operatorname{supp} v \text{ is compact}, \operatorname{supp} v \subset \Omega\}$$

as function space of m-times continuously differentiable functions with compact support in Ω.

By $C^m(\overline{\Omega})$ we denote the space of m-times continuously differentiable functions, for which all partial derivatives of order $|\alpha| \leq m$ can be extended continuously onto $\partial\Omega$. For both spaces $C^m(\overline{\Omega})$ and $C_B^m(\Omega)$, a norm is given by

$$\|v\|_{C^m} := \max_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha v(x)|.$$

Additionally, for $1 \leq m < \infty$ we define the space of m-times continuously differentiable and divergence-free vector functions with compact support in Ω by

$$C_{0,\sigma}^m(\Omega) := \{v \in C_0^m(\Omega) \mid \nabla \cdot v = 0\},$$

and we set

$$C_{0,\sigma}^\infty(\Omega) := \bigcap_{m=0}^{\infty} C_{0,\sigma}^m(\Omega).$$

1.1.2 Lebesgue spaces

For $1 \leq p < \infty$, by $L^p(\Omega)$ we denote the Banach space of Lebesgue-measurable functions in Ω, i. e. functions with finite norm

$$\|v\|_{0,p} := \left(\int_\Omega |v(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

By $L^\infty(\Omega)$ we denote the Banach space of all essentially bounded functions with finite norm

$$\|v\|_{0,\infty} := \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

For $1 \le p \le \infty$ we call $L^p(\Omega)$ a Lebesgue *space*. For a proof that all Lebesgue spaces are Banach spaces, see [1, pp. 29f.].

The elements of $L^p(\Omega)$ are equivalence classes of functions which coincide a. e. in Ω. We make no difference between equivalent functions and write $v \in L^p(\Omega)$ if $\|v\|_{0,p} < \infty$, and $v = 0$ if $v(x) = 0$ a. e. in Ω.
The space $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$ (see [7, p. 77]).

The space $L^2(\Omega)$ is a Hilbert space (compare [1, p. 31]) with scalar product

$$\langle u, v\rangle := \int_\Omega u(x) \cdot v(x)\ \mathrm{d}x$$

and with norm

$$\|v\| := \|v\|_{0,2} = \left(\int\limits_\Omega |v(x)|^2\ \mathrm{d}x\right)^{\frac{1}{2}} = \langle v, v\rangle^{\frac{1}{2}}\,.$$

1.1.3 Sobolev spaces

Let $\alpha \subset \mathbb{N}_0^3$ be a multi-index and let $v \in L^1(\Omega)$. If there exists a function $w \in L^1(\Omega)$ satisfying

$$\int_\Omega v(x)\partial^\alpha\varphi(x)\ \mathrm{d}x = (-1)^{|\alpha|}\int_\Omega w(x)\varphi(x)\ \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\Omega)$, we call w the *weak* or *distributional partial derivative of order* $|\alpha|$ *of* v and set $\partial^\alpha v := w$.
For $m \in \mathbb{N}_0$ and $1 \le p \le \infty$ by $W^{m,p}(\Omega)$ we denote the Sobolev space of all functions $v \in L^p(\Omega)$ for which all weak derivatives of order $|\alpha| \le m$ satisfy $\partial^\alpha v \in L^p(\Omega)$. For $1 \le p < \infty$

$$\|v\|_{m,p} := \left(\sum_{|\alpha|\le m} \|\partial^\alpha v\|_{0,p}^p\right)^{\frac{1}{p}}$$

denotes the corresponding norm and for $p = \infty$ we set

$$\|v\|_{m,\infty} := \max_{|\alpha|\le m} \|\partial^\alpha v\|_{0,\infty}\,.$$

By

$$W_0^{m,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{m,p}}$$

we denote the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, and by

$$H^{m,p}(\Omega) := \overline{C^m(\Omega)}^{\|\cdot\|_{m,p}}$$

we denote the closure of $C^m(\Omega)$ in $W^{m,p}(\Omega)$.
An important result of MEYERS and SERRIN [15, pp. 1055f.] states the equality

$$W^{m,p}(\Omega) = H^{m,p}(\Omega)$$

for $m \in \mathbb{N}_0$, $1 \leq p < \infty$.

All above defined spaces are called SOBOLEV spaces equipped with the norm $\|\cdot\|_{m,p}$ and $\|\cdot\|_{m,\infty}$, respectively.
For $m = 0$ we have $W^{0,p}(\Omega) = L^p(\Omega)$ by definition, and, since $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$, it holds $W_0^{0,p}(\Omega) = L^p(\Omega)$. For all $m \in \mathbb{N}_0$, $1 \leq p \leq \infty$ we find

$$W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega) \subset L^p(\Omega).$$

All SOBOLEV spaces are BANACH spaces. A proof for $W^{m,p}(\Omega)$ can be found in [1, pp. 60f.]. The result holds true for $W_0^{m,p}(\Omega)$, since this is a closed subspace of $W^{m,p}$.

For $m \in \mathbb{N}_0$, $p = 2$ the spaces $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$ are HILBERT spaces as closed subspaces of $L^2(\Omega)$ (compare [1, p. 61]) with scalar product

$$\langle u, v\rangle_{m,2} := \sum_{|\alpha|\leq m} \langle \partial^\alpha u, \partial^\alpha v\rangle$$

and norm $\|v\|_{m,2} = \langle v, v\rangle_{m,2}^{\frac{1}{2}}$. For $m = 0$ we use $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{0,2}$ and $\|\cdot\| := \|\cdot\|_{0,2}$. We also write

$$H^m(\Omega) := H^{m,2}(\Omega).$$

Since Ω is bounded, in $W_0^{1,p}(\Omega)$ we also use the norms

$$\|\nabla v\|_{0,p} := \left(\sum_{i=1}^{3} \|\partial_i v\|_{0,p}^p\right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$\|\nabla v\|_{0,\infty} := \max_{i=1,2,3} \|\partial_i v\|_{0,\infty}$$

for $p = \infty$. These norms are equivalent to $\|v\|_{1,p}$, and it holds

$$\|\cdot\|_{1,p}^p = \|\cdot\|_{0,p}^p + \|\nabla\cdot\|_{0,p}^p \tag{1.1}$$

for $1 \leq p \leq \infty$.

1.1.4 Divergence-free Sobolev spaces

We define the important spaces $\mathcal{H}^0(\Omega)$ and $\mathcal{H}^1(\Omega)$ as closures of $C^\infty_{0,\sigma}(\Omega)$ with respect to the norms $\|\cdot\|$ and the DIRICHLET *norm* $\|\nabla\cdot\| := \|\nabla\cdot\|_{0,2}$, respectively.

All elements of $\mathcal{H}^0(\Omega)$ and $\mathcal{H}^1(\Omega)$ are divergence-free. In particular it holds

$$\begin{aligned}\mathcal{H}^0(\Omega) &:= \left\{v \in L^2(\Omega) \mid \nabla\cdot v = 0 \,\wedge\, v\cdot\nu_{|\partial\Omega} = 0\right\},\\ \mathcal{H}^1(\Omega) &:= \left\{v \in W^{1,2}_0(\Omega) \mid \nabla\cdot v = 0\right\},\end{aligned}$$

where $v\cdot\nu_{|\partial\Omega}$ denotes the trace of the normal component of v (compare [21, pp. 11, 13]).

As closed subspace of $L^2(\Omega)$, the space $\mathcal{H}^0(\Omega)$ is a HILBERT space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.
$\mathcal{H}^1(\Omega)$ is a HILBERT space as closed subspace of $W^{1,2}_0(\Omega)$ with scalar product

$$\langle\nabla u, \nabla v\rangle := \int_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x = \sum_{i,j=1}^{3}\langle\partial_j u_i, \partial_j v_i\rangle$$

and DIRICHLET norm $\|\nabla v\| = \langle\nabla v, \nabla v\rangle^{\frac{1}{2}}$.

1.1.5 Bochner spaces

Now let B denote any BANACH space with norm $\|\cdot\|_B$, and let I denote any interval in $\mathbb{R}$. We consider the function $v: I \to B$, $t \mapsto v(t)$. If

$$\|v(t) - v(t_0)\|_B \xrightarrow{t\to t_0} 0$$

holds true for all $t_0 \in I$, we call v *continuous*. By $C(I;B)$ we denote the space of all such continuous functions. If, in addition, these functions have compact support in I we write $C_0(I;B)$.
$C_0(I;B)$ is a BANACH space [18, p. 200] with norm

$$\|v\|_{C_0(I;B)} := \max_{t\in I}\|v(t)\|_B\,.$$

A function $\dot{v} \in C(I;B)$ is called the *derivative of* v, if it satisfies

$$\left\|\frac{v(t+h)-v(t)}{h} - \dot{v}(t)\right\|_B \xrightarrow{h\to 0} 0$$

for all $t \in I$, and by $C^1(I;B)$ we denote the subspace of $C(I;B)$ consisting of functions for which there exists a derivative $\dot{v} \in C(I;B)$. For the subspace of functions in $C^1(I;B)$

with compact support in I we write $C_0^1(I;B)$.

For $1 \le p < \infty$, we define $L^p(I;B)$ as closure of $C_0(I;B)$ with respect to the norm

$$\|v\|_{L^p(I;B)} := \left(\int_I \|v(t)\|_B^p \, \mathrm{d}t \right)^{\frac{1}{p}} .$$

$L^\infty(I;B)$ is the space of all functions with essentially bounded norm $\|v(t)\|_B$ and we write

$$\|v\|_{L^\infty(I;B)} := \operatorname*{ess\,sup}_{t\in I} \|v(t)\|_{0,q} .$$

For $1 \le p \le \infty$ the Bochner space $L^p(I;B)$ is a Banach space [18, p. 192]. If the interval I has borders $a, b \in \mathbb{R}$, $a < b$, we also write $L^p(a,b;B)$.

1.2 Imbeddings

At first we define the notion of an imbedding as in Adams [1, p. 9]: For two Banach spaces $X \subset Y$ with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, let $i : X \to Y$ be the identity operator satisfying $ix = x$ for all $x \in X$. We say X *is (continuously) imbedded in* Y, and write $X \to Y$, if i is continuous.

If, additionally, i is compact (for each bounded sequence $\{x_n\}_n$ in X there exists a subsequence $\{x_{n_k}\}_k$ such that $\{ix_{n_k}\}_k$ converges in Y) we say X *is compactly imbedded in* Y and write $X \hookrightarrow Y$.

Since the identity operator i is linear, its continuity is equivalent to the existence of some constant M such that

$$\|ix\|_Y \le M \|x\|_X$$

holds true for all $x \in X$ [1, p. 9].

We often use the following well-known imbedding results:

Lemma 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ of class C^1. Then*

1. *for $1 \le q \le p \le \infty$ we have*

$$L^p(\Omega) \to L^q(\Omega),$$

2. *if $\partial\Omega$ is of class C^m, $m \in \mathbb{N}$, and if $k \in \mathbb{N}_0$ with $m \ge k+2$, it holds*

$$W^{m,2}(\Omega) \to C^k(\overline{\Omega}),$$

3. *for* $j, m \in \mathbb{N}_0$ *with* $j < m$ *we have the compact imbedding*

$$W^{m,2}(\Omega) \hookrightarrow W^{j,2}(\Omega).$$

Proof: A proof of the fist and second imbedding can be found in Adams & Fournier [1, pp. 28, 85f.]. The compact imbedding is proved in Wloka [26, p. 118]. □

Now we quote the fundamental Sobolev Imbedding Theorem [1, pp. 85f.]:

Theorem 1.2 (Sobolev Imbedding Theorem) *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary* $\partial\Omega$ *of class* C^1 *and let* $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, $1 \leq p < \infty$.

1. *If either* $mp > 3$ *or* $m = 3$, $p = 1$, *then*

$$W^{j+m,p}(\Omega) \to C_B^j(\Omega).$$

Moreover, it holds

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega) \qquad \text{for } p \leq q \leq \infty,$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega) \qquad \text{for } p \leq q \leq \infty.$$

2. *If* $mp = 3$, *then*

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega) \qquad \text{for } p \leq q < \infty,$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega) \qquad \text{for } p \leq q < \infty.$$

3. *For* $mp < 3$ *and* $\widetilde{p} := \frac{3p}{3-mp}$ *it holds*

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega) \qquad \text{for } p \leq q \leq \widetilde{p},$$

and, in particular,

$$W^{m,p}(\Omega) \to L^q(\Omega) \qquad \text{for } p \leq q \leq \widetilde{p}.$$

1.3 Elementary inequalities

At first, we state the POINCARÉ inequality. A proof can be found in EVANS [9, pp. 279f.].

Proposition 1.3 (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let $1 \leq p \leq \infty$. Then each $v \in W_0^{1,p}(\Omega)$ satisfies*

$$\|v\|_{0,p} \leq c_p \|\nabla v\|_{0,p} . \tag{1.2}$$

Moreover, for $1 \leq p < 3$ and $q \in [1, \widetilde{p}]$ with $\widetilde{p} := \frac{3p}{3-p}$, it holds

$$\|v\|_{0,q} \leq c_p \|\nabla v\|_{0,p} . \tag{1.3}$$

Here the constant c_p is the POINCARÉ *constant, depending only on p, q and Ω.*

Proofs for the following elementary inequalities can be found in EVANS [9, pp. 706–709]. For $p = \infty$ we set $\frac{1}{p} := 0$.

Proposition 1.4 (Hölder inequality) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all scalar functions $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, it holds $uv \in L^1(\Omega)$ with*

$$\|uv\|_{0,1} \leq \|u\|_{0,p} \|v\|_{0,q} . \tag{1.4}$$

The same also holds true for vector functions, and in this case we have

$$\|u \cdot v\|_{0,1} \leq \|u\|_{0,p} \|v\|_{0,q} . \tag{1.5}$$

The following corollary states a general version of the HÖLDER inequality (compare ZANGER [27, pp. 7f.]).

Corollary 1.5 (General Hölder inequality) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $m \in \mathbb{N}$, $1 \leq r < \infty$ and $1 \leq p_k \leq \infty$ for $k = 1, \ldots, m$ with $\displaystyle\sum_{k=1}^{m} \frac{1}{p_k} = \frac{1}{r}$.*

Then for all functions $u_k \in L^{p_k}(\Omega)$, $k = 1, \ldots, m$, it holds $\displaystyle\prod_{k=1}^{m} u_k \in L^r(\Omega)$ with

$$\left\| \prod_{k=1}^{m} u_k \right\|_{0,r} \leq c \prod_{k=1}^{m} \|u_k\|_{0,p_k} . \tag{1.6}$$

Here the constant c only depends on p, q.

The HÖLDER inequality even holds true for matrices (see ZANGER [27, pp. 7f.]):

Proposition 1.6 (Hölder inequality for gradients) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $1 \le r < \infty$, $1 \le p \le \infty$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then for all vector functions $u \in L^p(\Omega)$, $v \in W^{1,q}(\Omega)$, it holds*

$$\|u \cdot \nabla v\|_{0,r} \le c \, \|u\|_{0,p} \, \|\nabla v\|_{0,q} \,. \tag{1.7}$$

If, additionally, $u \in W^{1,p}(\Omega)$, it holds

$$\|(\nabla u)(\nabla v)\|_{0,r} \le c \, \|\nabla u\|_{0,p} \, \|\nabla v\|_{0,q} \,. \tag{1.8}$$

Here the constants c only depend on p, q.

Proposition 1.7 (Cauchy-Schwarz inequality) *Let X denote a normed space with inner product $\langle u, v\rangle_X$ and associated norm $\|v\|_X := \langle v, v\rangle_X^{\frac{1}{2}}$. Then it holds*

$$|\langle u, v\rangle_X|^2 \le \langle u, u\rangle_X \, \langle v, v\rangle_X = \|u\|_X^2 \, \|v\|_X^2 \,. \tag{1.9}$$

Proposition 1.8 (Young inequality) *Let $1 < p < \infty$, $1 < q < \infty$, and let $\frac{1}{p} + \frac{1}{q} = 1$. Then for $a, b \ge 0$ it holds*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.10}$$

In the following we state a simplified version of Gronwall's inequality. A proof can be found in Evans [9, pp. 708f.].

Proposition 1.9 (Gronwall inequality) *Let $I = [t_1, t_2] \subset \mathbb{R}$ denote an interval and let $a \in \mathbb{R}$, $b \in [0, \infty)$. Then for any function $u : I \to \mathbb{R}$ satisfying*

$$u(t) \le a + b \int_{t_1}^{t} u(\tau) \, \mathrm{d}\tau,$$

it holds

$$u(t) \le a + ab \int_{t_1}^{t} e^{b(t-\tau)} \, \mathrm{d}\tau = a e^{b(t-t_1)}$$

for each $t \in I$.

Now we consider the Friedrich inequality to estimate a solenoidal function by its gradient, where the weight of the gradient can be made as small as desired.

Lemma 1.10 (Friedrich inequality) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. There exists a sequence $\{\omega_j\}_j$ of vector functions $w_j \in C_{0,\sigma}^\infty(\Omega)$ depending only on Ω and satisfying the following property: For each $\delta > 0$, there exists an $M_\delta \in \mathbb{N}$ such that the estimate*

$$\|v\|^2 \le \delta \, \|\nabla v\|^2 + \sum_{j=1}^{M_\delta} |\langle v, \omega_j\rangle|^2$$

holds true for all $v \in \mathcal{H}^1(\Omega)$.

For a proof of the case $v \in W_0^{1,2}(\Omega)$ and $w_j \in C^\infty(\overline{\Omega})$, $j \in \mathbb{N}$ see [19, pp. 147f.].

Finally, we state the Green formulas for the Laplace operator [9, p. 712], which are a direct consequence of the Gauss Theorem (see [9, pp. 711f.]).

Proposition 1.11 (Green formulas) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$ of class C^1, let $\nu : \partial\Omega \to \mathbb{R}^3$ denote the exterior unit normal vector and let $\mathrm{d}S$ denote the surface element. Then for $u \in C^1(\overline{\Omega})$ and $v \in C^2(\overline{\Omega})$ it holds*

$$\int_\Omega \Delta v \; \mathrm{d}x = \int_{\partial\Omega} \partial_\nu v \; \mathrm{d}S,$$

$$\int_\Omega u\Delta v + \nabla u \cdot \nabla v \; \mathrm{d}x = \int_{\partial\Omega} u\partial_\nu v \; \mathrm{d}S.$$

If, additionally, $u \in C^2(\overline{\Omega})$ we find

$$\int_\Omega u\Delta v - v\Delta u \; \mathrm{d}x = \int_{\partial\Omega} u\partial_\nu v - v\partial_\nu u \; \mathrm{d}S.$$

Let us remark that on the left hand side of the second equation, by $\nabla u \cdot \nabla v$ we have a Frobenius scalar product of two matrices, if we assume u, v to be vector functions.

Due to density arguments, the Green formulas hold true for Sobolev functions.

For functions u satisfying $u_{|\partial\Omega} = 0$, thus for functions $u, v \in W_0^{1,2}(\Omega)$ with $v \in H^2(\Omega)$, we obtain the often used equality

$$\langle \nabla u, \nabla v \rangle = - \langle u, \Delta v \rangle . \tag{1.11}$$

1.4 Weak convergence

Before we define the weak convergence, we need to define a dual space.

Definition 1.12 (Dual space) *As dual space X' of a normed space X, we denote the space of all bounded linear functionals $x' : X \to \mathbb{R}$. A norm on X' is defined by*

$$\left\|x'\right\|_{X'} = \sup_{\|x\| \leq 1} |x'(x)|.$$

The dual space X' of a normed space X is a Banach space [25, p. 47].

Definition 1.13 (Weak convergence) *Let X denote a normed space with dual space X' and let $x \in X$. A sequence $\{x_n\}_n$ in X converges weakly to x in X if*

$$x'(x_n) \overset{n\to\infty}{\longrightarrow} x'(x) \quad \text{for all } x' \in X'.$$

In this case we write

$$x_n \overset{n\to\infty}{\rightharpoonup} x \quad \text{in } X.$$

1.4.1 Weak convergence in reflexive spaces

Definition 1.14 (Reflexive Banach space) *A* BANACH *space X with dual space X' and with $X'' := (X')'$ is called reflexive, if the mapping*

$$i : X \to X'',$$
$$(i(x))(x') = x'(x)$$

is surjective.

Each closed subspace of a reflexive space is reflexive again, and each BANACH space is reflexive if and only if its dual space is reflexive [25, p. 105].

In any BANACH space – it does not necessarily have to be reflexive – each weakly convergent sequence is bounded [18, pp. 72f.]. For reflexive BANACH spaces, we can state the following central result about the weak convergence of bounded sequences [25, pp. 107f.]:

Proposition 1.15 *Let X denote a reflexive* BANACH *space, let $\{x_n\}_n$ be a bounded sequence in X. Then there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and some element $x \in X$ such that*

$$x_{n_k} \overset{k\to\infty}{\rightharpoonup} x \quad \text{in } X. \tag{1.12}$$

1.4.2 Weak convergence in Hilbert spaces

At first we quote the RIESZ Representation Theorem [7, pp. 30f.] to redefine the weak convergence for HILBERT spaces.

Theorem 1.16 (Riesz Representation Theorem) *Let H denote a* HILBERT *space with scalar product $\langle \cdot, \cdot \rangle_H$ and dual space H'. Then H' can be canonically identified with H. This means that for each $u' \in H'$ there exists an unique $\hat{u} \in H$ satisfying*

$$u'(u) = \langle u, \hat{u} \rangle_H \quad \text{for all } u \in H.$$

Now we can rewrite the definition of weak convergence for HILBERT spaces as follows:

Definition 1.17 (Weak convergence in Hilbert spaces) *Let H denote a* HILBERT *space with scalar product $\langle \cdot, \cdot \rangle_H$. Let $u \in H$. A sequence $\{u_n\}_n$ in H converges weakly to u in H if*

$$\langle u_n, \hat{u} \rangle_H \overset{n\to\infty}{\longrightarrow} \langle u, \hat{u} \rangle_H \quad \text{for all } \hat{u} \in H.$$

The RIESZ Representation Theorem immediately implies that each HILBERT space H is reflexive [2, p.220], hence all results of the last section hold true for HILBERT spaces. In addition, we prove the following statements about weakly convergent sequences, which partly hold true even for BANACH spaces (see [9, p. 723], [7, p. 56]).

Proposition 1.18 *Let H denote a* HILBERT *space with scalar product $\langle\cdot,\cdot\rangle_H$ and associated norm $\|\cdot\|_H := \langle\cdot,\cdot\rangle_H^{\frac{1}{2}}$. Let $\{u_n\}_n$ be a sequence in H and let $u, \widetilde{u} \in H$. Then the following statements hold true:*

$$u_n \overset{n\to\infty}{\rightharpoonup} u \text{ in } H \;\wedge\; u_n \overset{n\to\infty}{\rightharpoonup} \widetilde{u} \text{ in } H \Rightarrow u = \widetilde{u}, \tag{1.13a}$$

$$u_n \overset{n\to\infty}{\longrightarrow} u \text{ in } H \Rightarrow u_n \overset{n\to\infty}{\rightharpoonup} u \text{ in } H, \tag{1.13b}$$

$$u_n \overset{n\to\infty}{\rightharpoonup} u \text{ in } H \Rightarrow \|u\|_H \leq \liminf_{n\to\infty} \|u_n\|_H\,, \tag{1.13c}$$

$$u_n \overset{n\to\infty}{\rightharpoonup} u \text{ in } H \;\wedge\; \|u\|_H \geq \limsup_{n\to\infty} \|u_n\|_H \Rightarrow u_n \overset{n\to\infty}{\longrightarrow} u \text{ in } H. \tag{1.13d}$$

Proof: To prove the uniqueness of the weak limit function (1.13a), assume $u_n \overset{n\to\infty}{\rightharpoonup} u$ in H and $u_n \overset{n\to\infty}{\rightharpoonup} \widetilde{u}$ in H with $u \neq \widetilde{u}$. Then for each $\hat{u} \in H$ it holds

$$\langle u - \widetilde{u}, \hat{u}\rangle_H = \langle u, \hat{u}\rangle_H - \langle \widetilde{u}, \hat{u}\rangle_H = \lim_{n\to\infty} \langle u_n, \hat{u}\rangle_H - \lim_{n\to\infty} \langle u_n, \hat{u}\rangle_H = 0.$$

In particular, this holds true for $\hat{u} := u - \widetilde{u}$. It follows $\langle u - \widetilde{u}, u - \widetilde{u}\rangle_H = \|u - \widetilde{u}\|_H^2 = 0$, hence $u = \widetilde{u}$, which contradicts our assumption.

To prove the second statement (1.13b), assume $u_n \overset{n\to\infty}{\longrightarrow} u$ in H and let $\hat{u} \in H$. The assertion follows immediately from

$$|\,\langle u_n, \hat{u}\rangle_H - \langle u, \hat{u}\rangle_H\,| = |\,\langle u_n - u, \hat{u}\rangle_H\,| \leq \|u_n - u\|_H\, \|\hat{u}\|_H \overset{n\to\infty}{\longrightarrow} 0.$$

For $u_n \overset{n\to\infty}{\rightharpoonup} u$ in H we obtain the third assertion (1.13c), using $\langle u_n, u\rangle_H \overset{n\to\infty}{\longrightarrow} \|u\|_H^2$. Consider a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ satisfying $\|u_{n_k}\|_H \overset{k\to\infty}{\longrightarrow} \liminf_{n\to\infty} \|u_n\|_H$. Then we obtain

$$\|u\|_H^2 = \lim_{k\to\infty} \langle u_{n_k}, u\rangle_H \overset{(1.9)}{\leq} \lim_{k\to\infty} \|u_{n_k}\|_H\, \|u\|_H = \liminf_{n\to\infty} \|u_n\|_H\, \|u\|_H\,,$$

which implies (1.13c).

To prove (1.13d) let $u_n \overset{n\to\infty}{\rightharpoonup} u$ in H and $\|u\|_H \geq \limsup_{n\to\infty} \|u_n\|_H$. It follows

$$\|u\|_H \overset{(1.13c)}{\leq} \liminf_{n\to\infty} \|u_n\|_H \leq \limsup_{n\to\infty} \|u_n\|_H \leq \|u\|_H\,,$$

hence $\|u_n\|_H \overset{n\to\infty}{\longrightarrow} \|u\|_H$, which implies the assertion. □

1.4.3 Weak convergence in Lebesgue, Sobolev and Bochner spaces

To apply the above results, we firstly consider the reflexivity of the function spaces used in this thesis: For $1 < p < \infty$ and $m \in \mathbb{N}_0$ the LEBESGUE spaces $L^p(\Omega)$ and the

SOBOLEV spaces $W^{m,p}(\Omega)$ are reflexive [1, pp. 45, 61]. In addition, if B is reflexive, the BOCHNER space $L^p(I;B)$ is reflexive [18, p. 195].

For completeness, we remind that the spaces $L^2(\Omega)$, $W^{m,2}(\Omega)$, $W_0^{m,2}(\Omega)$, $\mathcal{H}^0(\Omega)$ and $\mathcal{H}^1(\Omega)$ are HILBERT spaces.

For all spaces which are no HILBERT spaces, we need the dual space to consider weak convergence.

For $1 \leq p < \infty$ and $p' := \frac{p}{p-1}$ if $p > 1$, $p' := \infty$ if $p = 1$, the dual space of the LEBESGUE space $L^p(\Omega)$ is $L^{p'}(\Omega)$ [2, pp. 159–161].

Now let $1 < p < \infty$ with p' defined as above. For any interval I and some reflexive BANACH space B with dual space B', the dual space of the BOCHNER space $L^p(I;B)$ is $L^{p'}(I;B')$ [18, p. 195].

If B is a HILBERT space, we can rewrite the definition of weak convergence in the BOCHNER space $L^p(I;B)$ [18, p. 198]:

Definition 1.19 (Weak convergence in Bochner spaces) *Let $1 < p < \infty$ with $p' = \frac{p}{p-1}$, let I be an interval and let H denote a* HILBERT *space with scalar product $\langle \cdot, \cdot \rangle_H$. Then a sequence of functions $\{u_n\}_n$ with $u_n \in L^p(I;H)$ converges weakly in $L^p(I;H)$ to some function $u \in L^p(I;H)$, if*

$$\langle u_n, \hat{u} \rangle \;:=\; \int_I \langle u_n(t), \hat{u}(t) \rangle_H \;\mathrm{d}t \overset{n\to\infty}{\longrightarrow} \int_I \langle u(t), \hat{u}(t) \rangle_H \;\mathrm{d}t \;=:\; \langle u, \hat{u} \rangle$$

holds true for all functions $\hat{u} \in L^{p'}(I;H)$.

Now let $1 \leq p < \infty$ and $j, m \in \mathbb{N}_0$ with $j < m$. Let $\{u_n\}_n$ be a sequence satisfying $u_n \in W^{m,p}(\Omega)$ for all $n \in \mathbb{N}$ and let $u_n \overset{n\to\infty}{\rightharpoonup} u$ in $W^{m,p}(\Omega)$ for some function $u \in W^{m,p}(\Omega)$. Since $\{u_n\}_n$ is bounded in $W^{m,p}(\Omega)$ it is bounded in $W^{j,p}(\Omega)$. Thus, using Proposition 1.15, there exists a subsequence $\{u_{n_k}\}_k$ satisfying

$$u_{n_k} \overset{k\to\infty}{\rightharpoonup} u \quad \text{in } W^{j,p}(\Omega).$$

Due to the compactness of the imbeddings

$$W^{m,2}(\Omega) \hookrightarrow W^{j,2}(\Omega)$$

for $p = 2$ and $j, m \in \mathbb{N}_0$, $j < m$ (compare Lemma 1.1), the following much stronger result holds true [2, p. 244]:

Proposition 1.20 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$ of class C^1, and let $j, m \in \mathbb{N}_0$ with $j < m$. Let $\{u_n\}_n$ be a sequence with $u_n \in H^m(\Omega)$ for all $n \in \mathbb{N}$ and let $u \in H^m(\Omega)$. If $u_n \overset{n\to\infty}{\rightharpoonup} u$ in $H^m(\Omega)$, it follows*

$$u_n \overset{n\to\infty}{\longrightarrow} u \quad \text{in } H^j(\Omega).$$

Obviously, for this sequence it even holds $u_n \overset{n\to\infty}{\rightharpoonup} u$ in $H^j(\Omega)$.

1.5 Navier-Stokes equations and Stokes operator

Let $\Omega \subset \mathbb{R}^3$ denote a bounded domain with sufficiently smooth boundary $\partial\Omega$. For $0 < T \in \mathbb{R}$ we consider the nonlinear and nonstationary NAVIER-STOKES equations, which describe the motion of a viscous and incompressible fluid in $(0,T) \times \Omega$:

$$\begin{aligned} \partial_t v - \nu\Delta v + \nabla p + v \cdot \nabla v &= f && \text{in } (0,T) \times \Omega, \\ \nabla \cdot v &= 0 && \text{in } (0,T) \times \Omega, \\ v_{|\partial\Omega} &= 0 && \text{in } [0,T), \\ v_{|t=0} &= v_0 && \text{in } \Omega. \end{aligned} \tag{N_0}$$

In this system, the external force density $f : (0,T) \times \Omega \to \mathbb{R}^3$, the initial velocity $v_0 : \Omega \to \mathbb{R}^3$, and the kinematic viscosity constant $0 < \nu \in \mathbb{R}$ are given data. The velocity vector $v = (v_1, v_2, v_3) : (0,T) \times \Omega \to \mathbb{R}^3$ as well as the pressure function $p : (0,T) \times \Omega \to \mathbb{R}$ are unknown functions.

The first equation of (N_0) describes the balance of forces. Here $v \cdot \nabla v$ is the nonlinear convective term resulting from the total time derivative of the velocity field v. The second equation is the incompressibility condition and states that the velocity is solenoidal. The third equation is the no-slip boundary condition which ensures that no particle reaches the boundary of the domain. The last equation is the initial condition: At time $t = 0$ the velocity should coincide with a given initial velocity v_0. [5, pp. 213f.]

The nonlinear convective term $v \cdot \nabla v$ is defined as follows [20, p. 5]:

$$\begin{aligned} v \cdot \nabla v &= (v \cdot \nabla)v \\ &:= (v_1\partial_1 + v_2\partial_2 + v_3\partial_3)v \\ &= (v_1\partial_1 v_i + v_2\partial_2 v_i + v_3\partial_3 v_i)_{i=1,2,3} \,. \end{aligned}$$

For divergence-free functions v it can be written as

$$\begin{aligned} v \cdot \nabla v &= (v_1\partial_1 + v_2\partial_2 + v_3\partial_3)v \\ &= (\partial_1(v_1 v) + \partial_2(v_2 v) + \partial_3(v_3 v)) - (\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3)\, v \\ &= (\partial_1(v_1 v) + \partial_2(v_2 v) + \partial_3(v_3 v)) - (\nabla \cdot v)\, v \\ &= (\partial_1(v_1 v) + \partial_2(v_2 v) + \partial_3(v_3 v)) \\ &=: \nabla \cdot (vv). \end{aligned}$$

In this thesis, the mathematical approach for the NAVIER-STOKES equations is based on the use of the STOKES operator $-P\Delta$ in $\mathcal{H}^0(\Omega)$.

To define $-P\Delta$, we first consider the space

$$G(\Omega) := \left\{ w \in L^2(\Omega) \mid w = \nabla q \text{ for some } q \in H^1(\Omega) \right\},$$

which satisfies the so-called HELMHOLTZ decomposition (compare SOHR [20, pp. 81–83])

$$L^2(\Omega) = \mathcal{H}^0(\Omega) \oplus G(\Omega), \tag{1.14}$$

where $\oplus$ denotes the direct sum of vector spaces.
The resulting orthogonal projection operator

$$P : L^2(\Omega) \to \mathcal{H}^0(\Omega) \tag{1.15}$$

is called the HELMHOLTZ *projection*. It is a linear operator, projecting divergence-free functions to itself and eliminating the gradients, i. e. it holds

$$\begin{aligned} Pv &= v, \quad v \in \mathcal{H}^0(\Omega), \\ P\nabla p &= 0, \quad p \in H^1(\Omega). \end{aligned}$$

Now, let $\partial\Omega \in C^2$. The linear operator

$$-P\Delta : H^2(\Omega) \cap \mathcal{H}^1(\Omega) \to \mathcal{H}^0(\Omega) \tag{1.16}$$

is called the STOKES *operator*. Its domain $H^2(\Omega) \cap \mathcal{H}^1(\Omega)$ is dense in $\mathcal{H}^0(\Omega)$ [20, pp. 128–131]. For $u, v \in H^2(\Omega) \cap \mathcal{H}^1(\Omega)$ it satisfies

$$\langle -P\Delta v, v\rangle = \langle -\Delta v, Pv\rangle = \langle -\Delta v, v\rangle \overset{(1.11)}{=} \langle \nabla v, \nabla v\rangle = \|\nabla v\|^2 \geq 0$$

and

$$\langle -P\Delta u, v\rangle \overset{(1.11)}{=} \langle \nabla u, \nabla v\rangle \overset{(1.11)}{=} \langle u, -P\Delta v\rangle ,$$

hence it is positive and self-adjoint.

Now we state a fundamental result of CATTABRIGA [6, p. 311] concerning the STOKES operator.

Proposition 1.21 (Cattabriga Inequality) *Let $m \in \mathbb{N}_0$, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$ of class C^{m+2}, and let $f \in H^m(\Omega)$. Then there exist a unique solution $v \in H^{m+2}(\Omega) \cap \mathcal{H}^1(\Omega)$ with corresponding uniquely determined pressure gradient $\nabla p \in H^m(\Omega)$ satisfying the* STOKES *system*

$$\begin{aligned} -\Delta v + \nabla p &= f && \text{in } \Omega, \\ \nabla \cdot v &= 0 && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and the estimate

$$\|v\|_{m+2,2} \leq c_c \|f\|_{m,2} ,$$

where the constant c_c only depends on m and Ω.

Now we consider the inverse of the STOKES operator

$$(-P\Delta)^{-1} : \mathcal{H}^0(\Omega) \to \mathcal{H}^0(\Omega),$$

which, again, is linear, positive and self-adjoint, and, additionally, bounded (see [20, pp. 128–131]). Its image space is $H^2(\Omega) \cap \mathcal{H}^1(\Omega) \subset \mathcal{H}^0(\Omega)$, hence, due to the compactness of the imbedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$, the operator $(-P\Delta)^{-1}$ is even compact.

Since $\mathcal{H}^0(\Omega)$ is a HILBERT space, we can use the following theorem about the eigenvalues of linear, compact and self-adjoint operators [17, pp. 268f.] on $(-P\Delta)^{-1}$.

Theorem 1.22 (Hilbert-Schmidt Theorem) *Let H be a* HILBERT *space and let $K : H \to H$ denote a linear, compact and self-adjoint operator. Then there is a sequence of real eigenvalues $\{\lambda_i\}_i$, such that*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots > 0$$

with

$$\lim_{i\to\infty} \lambda_i = 0.$$

Furthermore, the sequence $\{\varphi_i\}_i$ of corresponding eigenfunctions is an orthonormal basis for the image space of K, satisfying

$$K\varphi_i = \lambda_i\varphi_i \quad \textit{for } i \in \mathbb{N}.$$

This theorem together with the positivity of $(-P\Delta)^{-1}$ implies the existence of a decreasing sequence of positive, nonzero eigenvalues $\{\mu_i\}_i$ of $(-P\Delta)^{-1}$, tending to 0 as $i \to \infty$.

The corresponding sequence of eigenfunctions $\{e_i\}_i$ is an orthonormal basis of $H^2(\Omega) \cap \mathcal{H}^1(\Omega)$, thus – since $H^2(\Omega) \cap \mathcal{H}^1(\Omega)$ is dense in $\mathcal{H}^0(\Omega)$ – in $\mathcal{H}^0(\Omega)$. It holds

$$(-P\Delta)^{-1}e_i = \mu_i e_i \quad \text{for } i \in \mathbb{N},$$

$$v \in \mathcal{H}^0(\Omega) \Rightarrow v = \sum_{i=1}^{\infty} \langle v, e_i\rangle\, e_i,$$

$$\langle e_i, e_j\rangle = \delta_{ij} \quad \text{for } i, j \in \mathbb{N},$$

where $\delta_{ij} := \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$.

Bootstrapping the result of CATTABRIGA and using the imbedding $H^{m+2}(\Omega) \to C^m(\overline{\Omega})$, we obtain $e_i \in C^\infty(\overline{\Omega})$ for $i \in \mathbb{N}$. In fact, since $e_i \in \mathcal{H}^0(\Omega)$, it even holds

$$e_i \in C^\infty_{0,\sigma}(\Omega) \quad \text{for } i \in \mathbb{N}.$$

Finally, for $-P\Delta$, by $\lambda_i := \frac{1}{\mu_i}$, $i \in \mathbb{N}$ we have a sequence $\{\lambda_i\}_i$ of positive, real eigenvalues, satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and

$$-P\Delta e_i = -P\Delta\lambda_i\mu_i e_i = \lambda_i(-P\Delta)(-P\Delta)^{-1}e_i = \lambda_i e_i \quad \text{for } i \in \mathbb{N}.$$

Thus, the orthonormal basis $\{e_i\}_i$ of $\mathcal{H}^0(\Omega)$ from above consists of the corresponding eigenfunctions of $-P\Delta$, and we shall use ist later for a Galerkin procedure.

2 Trajectories of steady flow

Throughout this section, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and let $u : \overline{\Omega} \to \mathbb{R}^3$ be a given velocity field.

For some fixed $x_0 \in \Omega$, consider the following autonomous initial value problem:

$$\begin{aligned} \dot{x}(t) &= u(x(t)), \\ x(0) &= x_0. \end{aligned} \tag{2.1}$$

If u is continuous in $\overline{\Omega}$, then by the PEANO Existence Theorem (see [24, p. 99]) there exists at least one local solution $x : (-\varepsilon, +\varepsilon) \to \Omega$ for some $\varepsilon > 0$ sufficiently small.

If, in addition, $u = 0$ on $\partial\Omega$, then the solutions $t \mapsto x(t)$ exist globally for all $t \in \mathbb{R}$ (see [16, p. 190]).

If, moreover, $u \in C^1(\overline{\Omega})$, then certainly u satisfies a LIPSCHITZ condition (see [24, p. 97]), which implies uniqueness of the solution of (2.1) for all $x_0 \in \Omega$ (see the PICARD-LINDELÖF Theorem in [24, p. 97]), and it holds $x \in C^1(\mathbb{R})$ in this case.

Now, let $u \in C^1(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. Then, using the above results and denoting the unique solution $t \mapsto x(t)$ also by $t \mapsto X(t, x_0)$ to express the dependence on the initial value x_0, for every $t \in \mathbb{R}$ the mapping

$$X(t, \cdot) : \begin{cases} \Omega & \to \quad \Omega \\ x_0 & \mapsto \quad X(t, x_0) \end{cases} \tag{2.2}$$

is well defined.

Due to the uniqueness, the composition rule

$$X(t, \cdot) \circ X(-t, x) = X(t, X(-t, x)) = X(t - t, x) = X(0, x) = x \tag{2.3}$$

holds true for all $t \in \mathbb{R}$ and $x \in \Omega$. It follows that

$$X^{-1}(t, \cdot) := X(-t, \cdot) \tag{2.4}$$

is the inverse function of $X(t, \cdot)$, hence $X(t, \cdot)$ is a diffeomorphism in Ω. Using $u(x) = 0$ for $x \in \partial\Omega$, we obtain

$$X(t, \Omega) = \Omega \tag{2.5}$$

for all $t \in \mathbb{R}$.

Now consider LIOUVILLE's differential equation (see e. g. [4, p. 31–33] for a complete proof)

$$\partial_t \det \nabla X(t, x) = \det \nabla X(t, x) \left(\nabla_X \cdot u(X(t, x)) \right)$$

concerning the JACOBIAN determinant $\det \nabla X(t, x)$. Here $\nabla_X \cdot u(X)$ denotes the divergence of $X \mapsto u(X)$ with respect to its argument X.

If, in addition, $\nabla \cdot u = 0$ in Ω, then the above differential equation yields that $\det \nabla X(t,x)$ does not depend on time, hence

$$\det \nabla X(t,x) = \det \nabla X(0,x) = \det \nabla x = 1 \tag{2.6}$$

holds true for all $t \in \mathbb{R}$ and $x \in \Omega$.

This important property of the mappings $X : \Omega \to \Omega$ is called the *conservation of measure*.

Using that $X(t,\cdot) : \Omega \to \Omega$ is a diffeomorphism, this implies the identitiy

$$\|f(X(t,\cdot))\|_{0,p} = \|f(\cdot)\|_{0,p} \tag{2.7}$$

for all $t \in \mathbb{R}$ and $f \in L^p(\Omega)$, $1 \le p \le \infty$.

For $1 \le p < \infty$ this follows by substitution (compare [3, p. 211]) from

$$\begin{aligned}
\|f(X(t,\cdot))\|_{0,p}^p &= \int_\Omega |f(X(t,x))|^p \, \mathrm{d}x \overset{(2.5)}{=} \int_{X(t,\Omega)} |f(X(t,x))|^p \, \mathrm{d}x \\
&= \int_\Omega |f(X(t,X(-t,x)))|^p \cdot |\det \nabla X(-t,x)| \, \mathrm{d}x \\
&\overset{(2.3),(2.6)}{=} \int_\Omega |f(x)|^p \, \mathrm{d}x = \|f\|_{0,p}^p \, .
\end{aligned}$$

For $p = \infty$, we obtain the property immediately by $X(t,\Omega) = \Omega$.

This implies, in particular,

$$\begin{aligned}
\langle f \circ X(t,\cdot), g \circ X(t,\cdot) \rangle &= \langle f, g \rangle \, , \\
\langle f, g \circ X(t,\cdot) \rangle &= \left\langle f \circ X^{-1}(t,\cdot), g \right\rangle
\end{aligned} \tag{2.8}$$

for $f, g \in L^2(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$.

3 Time discretisation of the Navier-Stokes equations

In this section, we consider the nonstationary nonlinear NAVIER-STOKES equations

$$\begin{aligned} \partial_t v - \nu\Delta v + \nabla p + v \cdot \nabla v &= f && \text{in } (0,T) \times \Omega, \\ \nabla \cdot v &= 0 && \text{in } (0,T) \times \Omega, \\ v_{|\partial\Omega} &= 0 && \text{in } [0,T), \\ v_{|t=0} &= v_0 && \text{in } \Omega, \end{aligned} \tag{N_0}$$

where $0 < T < \infty$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. This system describes the motion of a viscous, incompressible, nonstationary fluid: The vector $v = v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ denotes the velocity and $p = p(t,x)$ some pressure function, while f, v_0 and the kinematic viscosity constant $\nu > 0$ are given data.

Using a time discretisation with stepsize $\varepsilon > 0$, we transfer the NAVIER-STOKES equations (N_0) into a finite number N of steady boundary value problems (N_ε^k) $(k = 0, \dots, N-1)$, the solution of which approximates the velocity v and the pressure p at the grid point $t_{k+1} = (k+1)\varepsilon$. Here in addition, the convective term $v(t_{k+1}) \cdot \nabla v(t_{k+1})$ will be linearized and treated with the method described in Chapter 2.

To explain our approach, let $0 < T < \infty$, $2 \le N \in \mathbb{N}$, and define a stepsize $\varepsilon := \frac{T}{N} > 0$. Setting $t_k := k\varepsilon$ $(k = 0, \dots, N)$, this establishes a time grid with equidistant grid points in $[0,T]$.

Restricting the NAVIER-STOKES equations (N_0) to the time $t = t_{k+1}$, we find

$$\partial_t v(t_{k+1}) - \nu\Delta v(t_{k+1}) + \nabla p(t_{k+1}) + v(t_{k+1}) \cdot \nabla v(t_{k+1}) = f(t_{k+1}),$$

and these equations will be modified as follows:

We approximate the time derivative $\partial_t v(t_{k+1})$ by a backwards difference quotient:

$$\partial_t v(t_{k+1}) \approx \frac{v(t_{k+1}) - v(t_k)}{\varepsilon}.$$

Using a time delay, we approximate the nonlinear term $v(t_{k+1}) \cdot \nabla v(t_{k+1})$ by a linearization:

$$v(t_{k+1}) \cdot \nabla v(t_{k+1}) \approx v(t_k) \cdot \nabla v(t_{k+1}).$$

Finally, we replace $f(t_{k+1})$ by the average

$$f(t_{k+1}) \approx \frac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} f(\tau)\, \mathrm{d}\tau =: f^{k+1}.$$

Inserting these modifications into (N_0), for every $k = 0, \dots, N-1$ we obtain a steady boundary value problem of the following type:

$$\begin{aligned} \frac{v^{k+1}-v^k}{\varepsilon} - \nu\Delta v^{k+1} + \nabla p^{k+1} + v^k \cdot \nabla v^{k+1} &= f^{k+1} && \text{in } \Omega, \\ \nabla \cdot v^{k+1} &= 0 && \text{in } \Omega, \\ v^{k+1}_{|\partial\Omega} &= 0. \end{aligned} \qquad (\widetilde{N}^k_\varepsilon)$$

Here v^k is a given function satisfying $\nabla \cdot v^k = 0$ in Ω and $v^k_{|\partial\Omega} = 0$.

To further simplify the linearized convective term $v^k \cdot \nabla v^{k+1}$, we take into account its physical deduction: The term results from the total derivative of the velocity field $v(t,x)$, hence so-called total or LAGRANGIAN difference quotients could be used for approximation. In the following, these quotients are introduced.

Definition 3.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Let $u \in C^1(\overline{\Omega})$ be divergence-free in Ω with vanishing values on $\partial\Omega$, and, for $\varepsilon \in \mathbb{R}$, let $X(\varepsilon,\cdot) : \Omega \to \Omega$ denote the mapping* (2.2) *constructed from the initial value problem*

$$\begin{aligned} \dot{x}(t) &= u(x(t)), \\ x(0) &= x_0 \in \Omega. \end{aligned}$$

Then for every function $v : \Omega \to \mathbb{R}^3$ the quotients

$$\begin{aligned} L^+_\varepsilon v(x) &:= \frac{1}{\varepsilon}\{v(X(\varepsilon,x)) - v(x)\}, \\ L^-_\varepsilon v(x) &:= \frac{1}{\varepsilon}\{v(x) - v(X(-\varepsilon,x))\} \end{aligned}$$

*are well defined in $x \in \Omega$ and called a forward and a backward total (*LAGRANGIAN*) difference quotient, respectively. Averaging both quotients leads to the central total (*LAGRANGIAN*) difference quotient*

$$L_\varepsilon v(x) := \frac{1}{2\varepsilon}\{v(X(\varepsilon,x)) - v(X(-\varepsilon,x))\}.$$

Remark 3.2 If, in addition, $v \in C^1(\overline{\Omega})$, then all the above defined difference quotients converge to $u \cdot \nabla v$ as $\varepsilon \to 0$. For example, for the forward quotient, using a mean value theorem and the fundamental theorem of calculus, we find

$$\begin{aligned} L^+_\varepsilon v(x) &= \frac{1}{\varepsilon}\{v(X(\varepsilon,x)) - v(x)\} \\ &= \frac{1}{\varepsilon}\{v(X(\varepsilon,x)) - v(X(0,x))\} \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \partial_\tau X(\tau,x) \cdot \nabla_X v(X(\tau,x))\, \mathrm{d}\tau \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon u(X(\tau,x)) \cdot \nabla_X v(X(\tau,x))\, \mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\int_0^\varepsilon (u\cdot\nabla v)\circ X(\tau,x)\;\mathrm{d}\tau \\
&= (u\cdot\nabla v)\circ X(\tau_0,x) \qquad \text{for some } \tau_0 \text{ with } 0\le|\tau_0|\le\varepsilon \\
&\overset{\varepsilon\to 0}{\longrightarrow} (u\cdot\nabla v)\circ X(0,x) = u(x)\cdot\nabla v(x).
\end{aligned}
$$

It is well known (see [21, p. 109]) that for functions $u\in\mathcal{H}^1(\Omega)$ and $v,w\in H_0^1(\Omega)$ the relations

$$
\begin{aligned}
\langle u\cdot\nabla v, w\rangle &= -\langle u\cdot\nabla w, v\rangle\,,\\
\langle u\cdot\nabla v, v\rangle &= 0
\end{aligned}
$$

hold true. Hopf (see [11]) used these important relations to prove the global in time existence of weak solutions of the Navier-Stokes equations (N_0). The following analogue of these relations can be proved for the central total (Lagrangian) difference quotient:

Lemma 3.3 *Let the assumptions of Definition 3.1 be satisfied, and let $v,w\in L^2(\Omega)$. Then for the central total (*Lagrangian*) difference quotient the following relations hold true:*

$$\langle L_\varepsilon v, w\rangle = -\langle v, L_\varepsilon w\rangle\,, \tag{3.1a}$$

$$\langle L_\varepsilon v, v\rangle = 0. \tag{3.1b}$$

Proof: Using the conservation of measure (2.8), for the mapping $X := X(\varepsilon,\cdot)$ we obtain

$$
\begin{aligned}
\langle L_\varepsilon v, w\rangle &= \frac{1}{2\varepsilon}\left(\langle v\circ X, w\rangle - \left\langle v\circ X^{-1}, w\right\rangle\right)\\
&\overset{(2.8)}{=} \frac{1}{2\varepsilon}\left(\left\langle v\circ X\circ X^{-1}, w\circ X^{-1}\right\rangle - \left\langle v\circ X^{-1}\circ X, w\circ X\right\rangle\right)\\
&= -\left\langle v, \frac{1}{2\varepsilon}\left(w\circ X - w\circ X^{-1}\right)\right\rangle\\
&= -\langle v, L_\varepsilon w\rangle\,,
\end{aligned}
$$

which is equivalent to the first relation (3.1a). The second relation (3.1b) follows immediately setting $w=v$. □

Remark 3.4 From Lemma 3.3 it follows that sufficiently regular solutions of a system regularized by central total differences satisfy the energy equality (see Proposition 5.4). As seen from the proof above, this important equation does not hold true if a one-sided total difference quotient is used. Thus, the central total (Lagrangian) difference quotient should be preferred for approximation of the convective term.

In the following, we shall use the above results to approximate the linearized convective term $v^k \cdot \nabla v^{k+1}$ in $(\widetilde{N}_\varepsilon^k)$ as follows:

$$\begin{aligned} v^k(x) \cdot \nabla v^{k+1}(x) &\approx \frac{1}{2\varepsilon}\left\{ v^{k+1}(X_k(\varepsilon, x)) - v^{k+1}(X_k^{-1}(\varepsilon, x)) \right\} \\ &=: L_\varepsilon^k v^{k+1}(x). \end{aligned}$$

Here the mappings X_k, X_k^{-1} have to be constructed from the solution $t \mapsto x(t) =: X_k(t, x_0)$ of the initial value problem

$$\begin{aligned} \dot{x}(t) &= v^k(x(t)), \\ x(0) &= x_0. \end{aligned}$$

For $k := 0, \ldots, N-1$ this leads to the following boundary value problem:

$$\begin{aligned} v^{k+1} - \varepsilon\nu\Delta v^{k+1} + \varepsilon L_\varepsilon^k v^{k+1} + \varepsilon\nabla p^{k+1} &= \varepsilon f^{k+1} + v^k \qquad &&\text{in } \Omega, \\ \nabla \cdot v^{k+1} &= 0 &&\text{in } \Omega, \\ v^{k+1}_{|\partial\Omega} &= 0. \end{aligned} \tag{N_ε^k}$$

Assuming v^k to be an approximation of the solution $t \mapsto v(t)$ of (N_0) at time $t = t_k$, this system suggests v^{k+1} and p^{k+1} to be approximations of $v(t_{k+1})$ and $p(t_{k+1})$.

In the next chapter we will show under which assumptions on the data v^k and f, the solutions v^{k+1}, p^{k+1} of (N_ε^k) can be constructed.

4 A boundary value problem of Navier-Stokes-type

In this chapter, we consider the boundary value problem (N_ε^k), which was derived in Chapter 3, for fixed $\varepsilon > 0$ and fixed $k \in \{0, \dots, N-1\}$. We prove existence and uniqueness of a weak solution and derive some regularity statements.

For simplicity, we replace the solutions v^{k+1} by v and p^{k+1} by p, and denote the given functions by $u := v^k$ and $g := f^{k+1}$. Thus, for a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$, we obtain the system

$$\begin{aligned} v - \varepsilon\nu\Delta v + \varepsilon L_\varepsilon^u v + \varepsilon\nabla p &= \varepsilon g + u && \text{in } \Omega, \\ \nabla \cdot v &= 0 && \text{in } \Omega, \\ v_{|\partial\Omega} &= 0, \end{aligned} \qquad (N_\varepsilon^u)$$

where $\varepsilon > 0$ and $L_\varepsilon^u v$ is defined by

$$L_\varepsilon^u v(x) := \frac{1}{2\varepsilon}\left\{v(X(\varepsilon, x)) - v(X^{-1}(\varepsilon, x))\right\}$$

with mappings X, X^{-1} constructed from the solution $t \mapsto x(t) =: X(t, x_0)$ of the initial value problem

$$\begin{aligned} \dot{x}(t) &= u(x(t)), \\ x(0) &= x_0. \end{aligned} \tag{4.1}$$

Here $u \in C^1(\overline{\Omega})$ is a given function, which is divergence-free in Ω with vanishing values on the boundary $\partial\Omega$.

In order to show the existence of a weak solution v of (N_ε^u), we assume $\partial\Omega \in C^2$ and use a Galerkin approximation based on the eigenfunctions e_i, $i \in \mathbb{N}$, of the Stokes operator $-P\Delta$ (compare (1.16)).

Applying the Helmholtz projection

$$P : L^2(\Omega) \to \mathcal{H}^0(\Omega)$$

(compare (1.15)) on the first equation of (N_ε^u), since u, v are divergence-free functions, we obtain the projected system

$$v - \varepsilon\nu P\Delta v + \varepsilon P L_\varepsilon^u v = \varepsilon P g + u. \tag{4.2}$$

Let us remind on the properties of the Stokes operator (compare pages 22–24): It is a positive and self-adjoint operator with eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

The corresponding sequence $\{e_i\}_i$ of eigenfunctions $e_i \in C_{0,\sigma}^\infty(\Omega)$ of $-P\Delta$ defines a complete orthonormal system in $\mathcal{H}^0(\Omega)$ satisfying

$$v \in \mathcal{H}^0(\Omega) \Rightarrow v = \sum_{i=1}^{\infty} \langle v, e_i \rangle \, e_i, \tag{4.3a}$$

$$-P\Delta e_i = \lambda_i e_i \quad \text{in } \Omega \quad (i \in \mathbb{N}), \tag{4.3b}$$

$$\langle e_i, e_j \rangle = \delta_{ij} \quad (i, j \in \mathbb{N}), \tag{4.3c}$$

where $\delta_{ij} := \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$.

Now, for $n \in \mathbb{N}$ let $V_n := \operatorname{span}\{e_1, \dots, e_n\} \subset C^\infty_{0,\sigma}(\Omega)^n$ and define a Galerkin *ansatz function* $v^n \in V_n$ (compare [8, pp. 110f.]) in the form

$$v^n : \begin{cases} \Omega \to \mathbb{R}^3 \\ x \mapsto v^n(x) := \sum\limits_{j=1}^n c_{jn} e_j(x), \quad c_{jn} \in \mathbb{R}. \end{cases} \tag{4.4}$$

To determine the unknown coefficients c_{jn} $(j = 1, \dots, n)$ in (4.4), we replace v by v^n in (4.2), multiply the resulting equation by e_i $(i = 1, \dots, n)$, and integrate over Ω. Thus, from (4.2) for $i = 1, \dots, n$ we obtain the equation

$$\langle v^n, e_i \rangle - \varepsilon\nu \langle P\Delta v^n, e_i \rangle + \varepsilon \langle PL^u_\varepsilon v^n, e_i \rangle = \langle \varepsilon Pg + u, e_i \rangle =: F^n_i, \tag{4.5}$$

where $L^u_\varepsilon v^n$ is defined by

$$L^u_\varepsilon v^n(x) := \frac{1}{2\varepsilon} \left\{ v^n(X(\varepsilon, x)) - v^n(X^{-1}(\varepsilon, x)) \right\}$$

with $\varepsilon > 0$ and mappings X, X^{-1} constructed from the initial value problem (4.1).

Using the above mentioned properties (4.3a)–(4.3c) of the eigenfunctions $e_i \in C^\infty_{0,\sigma}(\Omega)$, $i \in \mathbb{N}$, and setting $X := X(\varepsilon)$ we find

$$\begin{aligned}
\langle v^n, e_i \rangle &= \sum_{j=1}^n c_{jn} \langle e_j, e_i \rangle \overset{(4.3c)}{=} \sum_{j=1}^n c_{jn} \delta_{ij} = c_{in}, \\
-\varepsilon\nu \langle P\Delta v^n, e_i \rangle &= \varepsilon\nu \langle v^n, -P\Delta e_i \rangle \overset{(4.3b)}{=} \varepsilon\nu\lambda_i \langle v^n, e_i \rangle = \varepsilon\nu\lambda_i c_{in}, \\
\varepsilon \langle PL^u_\varepsilon v^n, e_i \rangle &= \varepsilon \langle L^u_\varepsilon v^n, Pe_i \rangle = \varepsilon \langle L^u_\varepsilon v^n, e_i \rangle \overset{(3.1a)}{=} -\varepsilon \langle v^n, L^u_\varepsilon e_i \rangle \\
&= \langle v^n, -\varepsilon L^u_\varepsilon e_i \rangle = \left\langle v^n, -\frac{1}{2} \left\{ e_i \circ X - e_i \circ X^{-1} \right\} \right\rangle \\
&= \sum_{j=1}^n c_{jn} \left\langle e_j, \left(\frac{1}{2} e_i \circ X^{-1} - \frac{1}{2} e_i \circ X \right) \right\rangle \\
&\overset{(2.8)}{=} \sum_{j=1}^n \left(\frac{1}{2} \langle e_j \circ X, e_i \rangle - \frac{1}{2} \langle e_i \circ X, e_j \rangle \right) c_{jn}.
\end{aligned}$$

So for $i = 1, \dots, n$, from (4.5) it follows

$$(1 + \varepsilon\nu\lambda_i) c_{in} + \sum_{j=1}^n \left(\frac{1}{2} \langle e_j \circ X, e_i \rangle - \frac{1}{2} \langle e_i \circ X, e_j \rangle \right) c_{jn} = F^n_i.$$

This is a linear algebraic system of the type

$$(D+A)c_n = F^n, \tag{4.6}$$

where $D = (d_{ij})$ is an $n\times n$ diagonal matrix with diagonal elements $d_{ii} = 1 + \varepsilon\nu\lambda_i > 1$ $(i = 1, \dots, n)$, where $A = (a_{ij})$ is an $n\times n$ skew-symmetric $(a_{ji} = -a_{ij})$ matrix with elements $a_{ij} = \frac{1}{2}\langle e_j \circ X, e_i\rangle - \frac{1}{2}\langle e_i \circ X, e_j\rangle$, and where $F^n := (F_1^n, \dots, F_n^n)^T$ is the given right hand side. Due to the special structure of the matrices D and A, there is a unique solution $c_n = (c_{1n}, \dots, c_{nn})^T$ of the system (4.6). This follows by contradiction: Let us assume that there exists a second solution $\widetilde{c}_n \neq c_n \in \mathbb{R}^n$ of (4.6). Then $x := c_n - \widetilde{c}_n \in \mathbb{R}^n$ satisfies $x \neq 0$ and $(D + A)x = 0$. Since A is skew-symmetric we find

$$x^T Ax = \sum_{i,j=1}^{n} x_i a_{ij} x_j = \sum_{i,j=1}^{n} -x_j a_{ji} x_i = -x^T Ax,$$

hence $x^T Ax = 0$. It follows

$$0 = (D + A)x = x^T(D + A)x = x^T Dx + x^T Ax = x^T Dx = \sum_{i=1}^{n} d_{ii} x_i^2,$$

and since $d_{ii} > 0$ for $i = 1, \dots, n$ we obtain $x = 0$, which contradicts our assumption. Thus (4.6) is uniquely solvable.

Definition 4.1 (Galerkin approximation) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^2$. For $n \in \mathbb{N}$, the function*

$$v^n : \begin{cases} \Omega \to \mathbb{R}^3 \\ x \mapsto v^n(x) := \sum\limits_{j=1}^{n} c_{jn} e_j(x) \end{cases}$$

is called a GALERKIN *approximation of order n. Here the coefficients c_{jn} $(j = 1, \dots, n)$ represent the unique solution of the linear algebraic system* (4.6)*, and the functions $e_j \in C_{0,\sigma}^\infty(\Omega)$, $j = 1, \dots, n$ are the eigenfunctions of the* STOKES *operator $-P\Delta$.*

In the following we shall establish some a-priori estimates for the GALERKIN approximations.

Lemma 4.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^2$, let $\varepsilon > 0$, $g \in L^2(\Omega)$, and let $u \in C^1(\overline{\Omega})$ be divergence-free in Ω with $u = 0$ on $\partial\Omega$.*

Then the GALERKIN *approximation v^n from Definition 4.1 exists for each $n \in \mathbb{N}$, and the sequence $\{v^n\}_n$ is bounded in $H^2(\Omega)$. In particular, the following a-priori estimates*

$$\|v^n\|^2 + \|v^n - u\|^2 + \varepsilon\nu \|\nabla v^n\|^2 \leq \|u\|^2 + \frac{\varepsilon c_p^2}{\nu} \|g\|^2, \tag{4.7a}$$

$$\begin{aligned}\|\nabla v^n\|^2 + \|\nabla v^n - \nabla u\|^2 + \varepsilon\nu \|P\Delta v^n\|^2 &\le \|\nabla u\|^2 + \frac{2\varepsilon}{\nu}\|g\|^2 + \frac{2}{\varepsilon\nu}\|v^n\|^2 \\ &\le \|\nabla u\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2}\right)\|g\|^2 + \frac{2}{\varepsilon\nu}\|u\|^2 ,\end{aligned} \tag{4.7b}$$

$$\begin{aligned}\|\nabla v^n\|^2 + \|\nabla v^n - \nabla u\|^2 + \frac{1}{\varepsilon\nu}\|v^n - u\|^2 &\le \|\nabla u\|^2 + \frac{2\varepsilon}{\nu}\|g\|^2 + \frac{2}{\varepsilon\nu}\|v^n\|^2 \\ &\le \|\nabla u\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2}\right)\|g\|^2 + \frac{2}{\varepsilon\nu}\|u\|^2 ,\end{aligned} \tag{4.7c}$$

hold true, where c_p is the Poincaré *constant and the right hand sides do not depend on n.*

Proof: The difference quotients

$$L_\varepsilon^u v^n(x) := \frac{1}{2\varepsilon}\left\{v^n \circ X(\varepsilon, x) - v^n \circ X^{-1}(\varepsilon, x)\right\},$$

with mappings X, X^{-1} resulting from the solutions of (4.1) are well defined, and X, X^{-1} are measure conserving mappings, since $u \in C^1(\overline{\Omega})$, $\nabla \cdot u = 0$ in Ω, and $u = 0$ on $\partial\Omega$. Thus, the algebraic system (4.6) is uniquely solvable and the Galerkin approximation v^n is well defined for each $n \in \mathbb{N}$.

Now let $n \in \mathbb{N}$ be fixed. To prove (4.7a), we multiply the equation (4.5) by c_{in} and afterwards sum up for $i = 1, \ldots, n$. Setting $X := X(\varepsilon)$, this gives us

$$\begin{aligned}\underbrace{\langle v^n, v^n\rangle - \langle u, v^n\rangle}_{=:a_1} \underbrace{- \varepsilon\nu \langle P\Delta v^n, v^n\rangle}_{=:a_2} &= \underbrace{-\frac{1}{2}\left\langle P\left\{v^n \circ X - v^n \circ X^{-1}\right\}, v^n\right\rangle}_{=:a_3} \\ &\quad + \underbrace{\varepsilon\langle Pg, v^n\rangle}_{=:a_4} .\end{aligned} \tag{4.8}$$

For a_1 we have

$$\begin{aligned} a_1 &= \langle v^n, v^n\rangle - \langle u, v^n\rangle \\ &= \frac{1}{2}\left(\langle v^n, v^n\rangle - 2\langle v^n, u\rangle + \langle u, u\rangle + \langle v^n, v^n\rangle - \langle u, u\rangle\right) \\ &= \frac{1}{2}\left(\langle v^n - u, v^n - u\rangle + \langle v^n, v^n\rangle - \langle u, u\rangle\right) \\ &= \frac{1}{2}\left(\|v^n - u\|^2 + \|v^n\|^2 - \|u\|^2\right).\end{aligned}$$

Since $v^n \in C^\infty_{0,\sigma}(\Omega)$ and $\partial\Omega \in C^1$, for a_2 we find

$$\begin{aligned} a_2 &= -\varepsilon\nu \langle P\Delta v^n, v^n\rangle = -\varepsilon\nu \langle \Delta v^n, Pv^n\rangle \\ &= -\varepsilon\nu \langle \Delta v^n, v^n\rangle \overset{(1.11)}{=} \varepsilon\nu \langle \nabla v^n, \nabla v^n\rangle = \varepsilon\nu \|\nabla v^n\|^2 . \end{aligned}$$

For a_3 we use (3.1b) from Lemma 3.3 and obtain

$$a_3 = -\varepsilon \langle PL^u_\varepsilon v^n, v^n\rangle = -\varepsilon \langle L^u_\varepsilon v^n, Pv^n\rangle = -\varepsilon \langle L^u_\varepsilon v^n, v^n\rangle \overset{(3.1b)}{=} 0.$$

For a_4, using Hölder, Young and the Poincaré inequality, we find

$$\begin{aligned} a_4 &= \varepsilon \langle g, Pv^n\rangle = \varepsilon \langle g, v^n\rangle = \varepsilon \int_\Omega g\cdot v^n \,\mathrm{d}x \\ &\le \varepsilon \int_\Omega |g\cdot v^n| \,\mathrm{d}x \overset{(1.5)}{\le} \varepsilon \|g\| \,\|v^n\| \overset{(1.2)}{\le} \varepsilon \|g\| \,c_p \|\nabla v^n\| \\ &= \frac{\varepsilon c_p}{\sqrt{\varepsilon\nu}} \|g\| \sqrt{\varepsilon\nu}\, \|\nabla v^n\| \overset{(1.10)}{\le} \frac{\varepsilon c_p^2}{2\nu} \|g\|^2 + \frac{\varepsilon\nu}{2} \|\nabla v^n\|^2 . \end{aligned}$$

Now we can apply these estimates on (4.8) and obtain

$$\frac12 \|v^n\|^2 + \frac12 \|v^n - u\|^2 + \varepsilon\nu \|\nabla v^n\|^2 \le \frac12 \|u\|^2 + \frac{\varepsilon c_p^2}{2\nu} \|g\|^2 + \frac{\varepsilon\nu}{2} \|\nabla v^n\|^2 .$$

Eliminating $\frac{\varepsilon\nu}{2} \|\nabla v^n\|$ on the right hand side and multiplying by 2, this leads to

$$\|v^n\|^2 + \|v^n - u\|^2 + \varepsilon\nu \|\nabla v^n\|^2 \le \|u\|^2 + \frac{\varepsilon c_p^2}{\nu} \|g\|^2 .$$

To prove (4.7b), we multiply the equation (4.5) by $\lambda_i c_{in}$, where λ_i denotes the i^{th} eigenvalue of the Stokes operator. Thus, using the relation $-P\Delta e_i = \lambda_i e_i$ from (4.3b), summing up for $i = 1, \dots, n$, and setting $X := X(\varepsilon)$, we obtain

$$\begin{aligned} &\underbrace{\langle v^n, -P\Delta v^n\rangle - \langle u, -P\Delta v^n\rangle}_{=:b_1} \underbrace{- \varepsilon\nu \langle P\Delta v^n, -P\Delta v^n\rangle}_{=:b_2} \\ &\qquad = \underbrace{-\frac12 \left\langle P\left\{v^n\circ X - v^n \circ X^{-1}\right\}, -P\Delta v^n\right\rangle}_{=:b_3} + \underbrace{\varepsilon \langle Pg, -P\Delta v^n\rangle}_{=:b_4} . \end{aligned}$$

Here, similar to the estimates of a_i above, we have

$$\begin{aligned} b_1 &= \langle v^n, -\Delta v^n\rangle - \langle u, -\Delta v^n\rangle \overset{(1.11)}{=} \langle \nabla v^n, \nabla v^n\rangle - \langle \nabla u, \nabla v^n\rangle \\ &= \frac12 \left(\|\nabla v^n\|^2 + \|\nabla v^n - \nabla u\|^2 - \|\nabla u\|^2 \right), \end{aligned}$$

$$b_2 = \varepsilon\nu \|P\Delta v^n\|^2 ,$$

$$\begin{aligned}
b_3 &= \frac{1}{2}\left\langle v^n \circ X - v^n \circ X^{-1}, P\Delta v^n \right\rangle \overset{(1.5)}{\leq} \frac{1}{2}\left\| v^n \circ X - v^n \circ X^{-1} \right\| \left\| P\Delta v^n \right\| \\
&\leq \frac{1}{2}\left(\| v^n \circ X \| + \left\| v^n \circ X^{-1} \right\| \right) \| P\Delta v^n \| \overset{(2.7)}{=} \| v^n \| \, \| P\Delta v^n \| \\
&= \sqrt{\frac{2}{\varepsilon\nu}} \| v^n \| \sqrt{\frac{\varepsilon\nu}{2}} \| P\Delta v^n \| \overset{(1.10)}{\leq} \frac{1}{\varepsilon\nu} \| v^n \|^2 + \frac{\varepsilon\nu}{4} \| P\Delta v^n \|^2 ,
\end{aligned}$$

$$b_4 = \varepsilon \left\langle g, -P\Delta v^n \right\rangle \overset{(1.5)}{\leq} \varepsilon \sqrt{\frac{2}{\varepsilon\nu}} \| g \| \sqrt{\frac{\varepsilon\nu}{2}} \| P\Delta v^n \| \overset{(1.10)}{\leq} \frac{\varepsilon}{\nu} \| g \|^2 + \frac{\varepsilon\nu}{4} \| P\Delta v^n \|^2 .$$

These estimates lead to

$$\begin{aligned}
&\frac{1}{2} \| \nabla v^n \|^2 + \frac{1}{2} \| \nabla v^n - \nabla u \|^2 + \varepsilon\nu \| P\Delta v^n \|^2 \\
&\qquad \leq \frac{1}{2} \| \nabla u \|^2 + \frac{\varepsilon}{\nu} \| g \|^2 + \frac{\varepsilon\nu}{4} \| P\Delta v^n \|^2 + \frac{1}{\varepsilon\nu} \| v^n \|^2 + \frac{\varepsilon\nu}{4} \| P\Delta v^n \|^2 .
\end{aligned}$$

Eliminating $\frac{\varepsilon\nu}{2} \| P\Delta v^n \|^2$ of the right hand side and multiplying by 2, we obtain

$$\begin{aligned}
\| \nabla v^n \|^2 + \| \nabla v^n - \nabla u \|^2 + \varepsilon\nu \| P\Delta v^n \|^2 &\leq \| \nabla u \|^2 + \frac{2\varepsilon}{\nu} \| g \|^2 + \frac{2}{\varepsilon\nu} \| v^n \|^2 \\
&\overset{(4.7a)}{\leq} \| \nabla u \|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2} \right) \| g \|^2 + \frac{2}{\varepsilon\nu} \| u \|^2 .
\end{aligned}$$

Finally, to prove (4.7c), again setting $X := X(\varepsilon)$, we consider

$$\begin{aligned}
&\underbrace{\langle v^n, v^n - u \rangle - \langle u, v^n - u \rangle}_{=:c_1} \underbrace{- \varepsilon\nu \langle P\Delta v^n, v^n - u \rangle}_{=:c_2} \\
&\qquad = \underbrace{-\frac{1}{2} \left\langle P \left\{ v^n \circ X - v^n \circ X^{-1} \right\}, v^n - u \right\rangle}_{=:c_3} + \underbrace{\varepsilon \langle Pg, v^n - u \rangle}_{=:c_4} .
\end{aligned}$$

Here we have

$$c_1 = \langle v^n - u, v^n - u \rangle = \| v^n - u \|^2 ,$$

$$\begin{aligned}
c_2 &= -\varepsilon\nu \langle \Delta v^n, v^n - u \rangle \overset{(1.11)}{=} \varepsilon\nu \langle \nabla v^n, \nabla v^n - \nabla u \rangle \\
&= \varepsilon\nu \langle \nabla v^n - \nabla u, \nabla v^n \rangle = \frac{\varepsilon\nu}{2} \left(\| \nabla v^n \|^2 + \| \nabla v^n - \nabla u \|^2 - \| \nabla u \|^2 \right) ,
\end{aligned}$$

$$\begin{aligned}
c_3 &= -\frac{1}{2} \left\langle v^n \circ X - v^n \circ X^{-1}, v^n - u \right\rangle \overset{(1.5)}{\leq} \frac{1}{2} \left\| v^n \circ X - v^n \circ X^{-1} \right\| \| v^n - u \| \\
&\leq \frac{1}{2} \left(\| v^n \circ X \| + \left\| v^n \circ X^{-1} \right\| \right) \| v^n - u \| \overset{(2.7)}{=} \| v^n \| \, \| v^n - u \|
\end{aligned}$$

$$= \sqrt{2}\,\|v^n\|\,\frac{1}{\sqrt{2}}\,\|v^n - u\| \overset{(1.10)}{\leq} \|v^n\|^2 + \frac{1}{4}\,\|v^n - u\|^2\,,$$

$$c_4 = \varepsilon\,\langle g, v^n - u\rangle \overset{(1.5)}{\leq} \varepsilon\sqrt{2}\,\|g\|\,\frac{1}{\sqrt{2}}\,\|v^n - u\| \overset{(1.10)}{\leq} \varepsilon^2\,\|g\|^2 + \frac{1}{4}\,\|v^n - u\|^2\,.$$

These estimates lead to

$$\begin{aligned}&\frac{\varepsilon\nu}{2}\,\|\nabla v^n\|^2 + \frac{\varepsilon\nu}{2}\,\|\nabla v^n - \nabla u\|^2 + \|v^n - u\|^2\\ &\qquad\leq \frac{\varepsilon\nu}{2}\,\|\nabla u\|^2 + \varepsilon^2\,\|g\|^2 + \frac{1}{4}\,\|v^n - u\|^2 + \|v^n\|^2 + \frac{1}{4}\,\|v^n - u\|^2\,.\end{aligned}$$

Eliminating $\frac{1}{2}\,\|v^n - u\|^2$ of the right hand side and multiplying by $\frac{2}{\varepsilon\nu}$, we obtain

$$\begin{aligned}\|\nabla v^n\|^2 + \|\nabla v^n - \nabla u\|^2 + \frac{1}{\varepsilon\nu}\,\|v^n - u\|^2 &\leq \|\nabla u\|^2 + \frac{2\varepsilon}{\nu}\,\|g\|^2 + \frac{2}{\varepsilon\nu}\,\|v^n\|^2\\ &\overset{(4.7a)}{\leq} \|\nabla u\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2}\right)\|g\|^2 + \frac{2}{\varepsilon\nu}\,\|u\|^2\,.\end{aligned}$$

These estimates immediately imply the boundedness of $\{v^n\}_n$ in $H^2(\Omega)$. □

Now we define a *weak solution* for the discretized NAVIER-STOKES-equations (N_ε^u) (compare e. g. SHINBROT [19, p. 160]).

Definition 4.3 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^2$, let $\varepsilon > 0$, $g \in L^2(\Omega)$, and let $u \in C^1(\overline{\Omega})$ be divergence-free in Ω with $u = 0$ on $\partial\Omega$. A function $v \in \mathcal{H}^1(\Omega)$ is called a weak solution of (N_ε^u), if*

$$\langle v, \varphi\rangle + \varepsilon\nu\,\langle\nabla v, \nabla\varphi\rangle + \varepsilon\,\langle L_\varepsilon^u v, \varphi\rangle = \varepsilon\,\langle g, \varphi\rangle + \langle u, \varphi\rangle \tag{4.9}$$

holds true for all test functions $\varphi \in C_{0,\sigma}^\infty(\Omega)$.

Using the estimates of the GALERKIN approximations v^n derived in Lemma 4.2, we are able to prove existence and uniqueness of a weak solution of (N_ε^u).

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^2$, let $\varepsilon > 0$ and $g \in L^2(\Omega)$, and let $u \in C^1(\overline{\Omega})$ be divergence-free in Ω with $u = 0$ on $\partial\Omega$.*
Then there exists a uniquely determined weak solution $v \in \mathcal{H}^1(\Omega) \cap H^2(\Omega)$ of (N_ε^u).

Proof: By Lemma 4.2, the GALERKIN approximation v^n exists for every $n \in \mathbb{N}$. Moreover, using the estimate (4.7b), we find

$$\|P\Delta v^n\| \leq c,$$

where c is independent of $n \in \mathbb{N}$.

With help of CATTABRIGA's estimate

$$\|u\|_{2,2} \leq c_c \|P\Delta u\|$$

for the solution of the steady STOKES system (compare Proposition 1.21), this implies that the sequence $\{v^n\}_n$ of the GALERKIN approximations $v^n \in C^\infty_{0,\sigma}(\Omega)$ is bounded in $H^2(\Omega)$. Thus there exists a subsequence which converges weakly in $H^2(\Omega)$ and – due to the compactness of the imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ for bounded Ω – strongly in $H^1(\Omega)$ (compare Propositions 1.15 and 1.20). We denote this subsequence again by $\{v^n\}_n$, and its limit function by v, where $v \in \mathcal{H}^1(\Omega) \cap H^2(\Omega)$ since $v^n \in \mathcal{H}^1(\Omega) \cap H^2(\Omega)$, and $\mathcal{H}^1(\Omega)$ is a closed subspace of $H^1(\Omega)$.

Thus, for every $i \in \mathbb{N}$, in (4.5) we can pass to the limit $n \to \infty$, obtaining

$$\langle v, e_i \rangle - \varepsilon\nu \langle P\Delta v, e_i \rangle + \varepsilon \langle PL^u_\varepsilon v, e_i \rangle = \varepsilon \langle Pg, e_i \rangle + \langle u, e_i \rangle , \tag{4.10}$$

where, in particular, setting $X := X(\varepsilon)$ we use

$$\begin{aligned} \varepsilon \langle PL^u_\varepsilon (v^n - v), e_i \rangle &= \left\langle \frac{1}{2} \left\{ (v^n - v) \circ X - (v^n - v) \circ X^{-1} \right\}, e_i \right\rangle \\ &\overset{(1.5)}{\leq} \frac{1}{2} \left(\|(v^n - v) \circ X\| + \left\|(v^n - v) \circ X^{-1}\right\| \right) \|e_i\| \\ &\overset{(2.7)}{=} \|v^n - v\| \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

The corresponding sequence $\{e_i\}_i$ of eigenfunctions of the STOKES operator represents a complete orthonormal system in $\mathcal{H}^0(\Omega)$. Thus, for each test function $\varphi \in C^\infty_{0,\sigma}(\Omega) \subset \mathcal{H}^0(\Omega)$ there exists a sequence $\{\mu_i\}_i$ with $\mu_i \in \mathbb{R}$ for $i \in \mathbb{N}$ and

$$\varphi = \sum_{i=1}^\infty \mu_i e_i.$$

Using GREEN's formula (1.11) on (4.10), multiplying by μ_i and summing up over i, this implies

$$\langle v, \varphi \rangle + \varepsilon\nu \langle \nabla v, \nabla\varphi \rangle + \varepsilon \langle PL^u_\varepsilon v, \varphi \rangle = \varepsilon \langle Pg, \varphi \rangle + \langle u, \varphi \rangle$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega) \subset \mathcal{H}^1(\Omega)$, i. e. $v \in \mathcal{H}^1(\Omega) \cap H^2(\Omega)$ is a weak solution of (N^u_ε) in the sense of Definition 4.3.

To prove the uniqueness, let $\widetilde{v} \in \mathcal{H}^1(\Omega) \cap H^2(\Omega)$ be a second weak solution of (N^u_ε), and denote by $w := v - \widetilde{v}$ the difference of these two solutions. It follows

$$w - \varepsilon\nu P\Delta w + \varepsilon PL^u_\varepsilon w = 0 \qquad \text{in } \mathcal{H}^0(\Omega),$$

and scalar multiplication in $L^2(\Omega)$ with w implies

$$\langle w, w \rangle - \varepsilon\nu \langle P\Delta w, w \rangle + \varepsilon \langle PL^u_\varepsilon w, w \rangle \overset{(1.11)}{=} \|w\|^2 + \varepsilon\nu \|\nabla w\|^2 = 0,$$

since $\langle PL^u_\varepsilon w, w \rangle \overset{(3.1b)}{=} 0$. Thus it follows $\|w\| = 0$, hence $w(x) = 0$ in $x \in \Omega$, which implies the asserted uniqueness. □

Remark 4.5 Consider any other accumulation point $\widetilde{v}$ obtained by extracting some different subsequence in the proof of Theorem 4.4. Then $\widetilde{v}$ is a weak solution of (N_ε^u), and – due to the uniquenes of the weak solution v derived in Theorem 4.4 – we obtain $v = \widetilde{v}$. Thus, there exists only one accumulation point v, and the whole sequence of Galerkin approximations $\{v^n\}_n$ converges to v in the corresponding norms.

To derive some a-priori estimates for the weak solution $v \in H^2 \cap \mathcal{H}^1(\Omega)$ of (N_ε^u), we consider the decompositions

$$\begin{aligned} v &= v - v^n + v^n, \\ \nabla v &= \nabla v - \nabla v^n + \nabla v^n. \end{aligned}$$

Here – due to the strong convergence $v^n \stackrel{n\to\infty}{\longrightarrow} v$ in $H^1(\Omega)$ – we can use

$$\begin{aligned} &\|v - v^n\| \stackrel{n\to\infty}{\longrightarrow} 0, \\ &\|\nabla v - \nabla v^n\| = \|\nabla(v - v^n)\| \stackrel{(1.1)}{\leq} \|v - v^n\|_{1,2} \stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus we can extend the a-priori estimates (4.7a) and (4.7c) of v^n from Lemma 4.2 to v. This leads to

Corollary 4.6 *Let the assumptions of Theorem 4.4 be satisfied. Then the weak solution v of (N_ε^u) satisfies*

$$\|v\|^2 + \|v - u\|^2 + \varepsilon\nu \|\nabla v\|^2 \leq \|u\|^2 + \frac{\varepsilon c_p^2}{\nu} \|g\|^2, \tag{4.11a}$$

$$\|\nabla v\|^2 + \|\nabla v - \nabla u\|^2 + \frac{1}{\varepsilon\nu} \|v - u\|^2 \leq \|\nabla u\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2}\right) \|g\|^2 + \frac{2}{\varepsilon\nu} \|u\|^2, \tag{4.11b}$$

where c_p is the Poincaré *constant.*

In the following, we will prove the equivalent of estimate (4.7b) in Lemma 4.2:

Proposition 4.7 *Let the assumptions of Theorem 4.4 be satisfied. Then the weak solution v of (N_ε^u) satisfies*

$$\|\nabla v\|^2 + \|\nabla v - \nabla u\|^2 + \varepsilon\nu \|P\Delta v\|^2 \leq \|\nabla u\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2}\right) \|g\|^2 + \frac{2}{\varepsilon\nu} \|u\|^2, \tag{4.11c}$$

where c_p is the Poincaré *constant.*

Proof: The weak solution v of (N_ε^u), which we derived in Theorem 4.4, satisfies

$$v - u - \varepsilon\nu P\Delta v + \varepsilon P L_\varepsilon^u v = \varepsilon P g \qquad \text{in } \mathcal{H}^0(\Omega),$$

and scalar multiplication in $L^2(\Omega)$ with $-P\Delta v \in \mathcal{H}^0(\Omega)$ implies

$$\underbrace{\langle v, -P\Delta v\rangle - \langle u, -P\Delta v\rangle}_{=:a_1} \underbrace{- \varepsilon\nu \langle P\Delta v, -P\Delta v\rangle}_{=:a_2} = \underbrace{-\varepsilon \langle PL_\varepsilon^u v, -P\Delta v\rangle}_{=:a_3} + \underbrace{\varepsilon \langle Pg, -P\Delta v\rangle}_{=:a_4}.$$

Now we can proceed as in the proof of estimate (4.7b) in Lemma 4.2. We obtain

$$\begin{aligned} a_1 &\overset{(1.11)}{=} \frac{1}{2}\left(\|\nabla v\|^2 + \|\nabla v - \nabla u\|^2 - \|\nabla u\|^2\right), \\ a_2 &= \varepsilon\nu \|P\Delta v\|^2, \end{aligned}$$

and using the conservation of measure (2.7), Hölder (1.5) and the Poincaré inequality (1.2), we have

$$\begin{aligned} a_3 &\leq \frac{1}{\varepsilon\nu}\|v\|^2 + \frac{\varepsilon\nu}{4}\|P\Delta v\|^2, \\ a_4 &\leq \frac{\varepsilon}{\nu}\|g\|^2 + \frac{\varepsilon\nu}{4}\|P\Delta v\|^2. \end{aligned}$$

These estimates lead to

$$\begin{aligned} \|\nabla v\|^2 + \|\nabla v - \nabla u\|^2 + \varepsilon\nu\|P\Delta v\|^2 &\leq \|\nabla u\|^2 + \frac{2\varepsilon}{\nu}\|g\|^2 + \frac{2}{\varepsilon\nu}\|v\|^2 \\ &\overset{(4.11a)}{\leq} \|\nabla u\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu^2}\right)\|g\|^2 + \frac{2}{\varepsilon\nu}\|u\|^2. \end{aligned}$$

□

In the next chapter, we define a non-steady velocity field $v^\varepsilon : [-\varepsilon, T] \to \mathbb{R}^3$ piecewise constant in time, using the steady solution of the boundary value problem (N_ε^u). For this purpose we need the following regularity of the weak solution. Here parts of the proof are motivated by Asanalieva, Heutling & Varnhorn [5, pp. 345–346]):

Proposition 4.8 *Let the assumptions of Theorem 4.4 be satisfied. In addition, let* $\partial\Omega \in C^3$ *and* $g \in H^1(\Omega)$. *Then, the weak solution* v *of* (N_ε^u) *satisfies*

$$v \in H^3(\Omega) \cap \mathcal{H}^1(\Omega).$$

Proof: We consider the system (4.2) in the form

$$-P\Delta v = \frac{1}{\varepsilon\nu}\Big(\underbrace{(u - v)}_{:=a_1} \underbrace{-\frac{1}{2}P\left\{v \circ X - v \circ X^{-1}\right\}}_{:=a_2} + \underbrace{\varepsilon Pg}_{:=a_3}\Big), \tag{4.12}$$

where $X := X(\varepsilon, x)$.

Using Cattabriga's estimate

$$\|v\|_{3,2} \leq c_c \|P\Delta v\|_{1,2}$$

(compare Proposition 1.21), we obtain $v \in H^3(\Omega)$ if the right hand side in (4.12) is in $H^1(\Omega)$.

By Theorem 4.4 we know $v \in H^2(\Omega)$. Thus, together with the assumptions of this proposition, we obtain immediately that $a_1 = (u - v) \in H^1(\Omega)$ and $a_3 = \varepsilon P g \in H^1(\Omega)$.

To prove $a_2 \in H^1(\Omega)$ we use the conservation of measure. For $X := X(\varepsilon, x)$, and, analogously, for $X^{-1} := X^{-1}(\varepsilon, x)$, we have

$$\|P(v \circ X)\| = \|v \circ X\| \overset{(2.7)}{=} \|v\|,$$

hence $v \circ X \in L^2(\Omega)$.

For the gradient $\|\nabla(v \circ X)\|$ we obtain

$$\begin{aligned}\|\nabla(v \circ X)\| &= \|((\nabla_X v) \circ X)(\nabla X)\| \overset{(1.8)}{\leq} c\,\|(\nabla_X v) \circ X\|\;\|\nabla X\|_{0,\infty} \\ &\overset{(2.7)}{\leq} c\,\|\nabla v\|\;\|\nabla X\|_{0,\infty}.\end{aligned}$$

To estimate $\|\nabla X\|_{0,\infty}$ we observe

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla X(t,x)\|_{0,\infty} &\leq \|\nabla \partial_t X(t,x)\|_{0,\infty} = \|\nabla(u \circ X(t,x))\|_{0,\infty} \\ &= \|((\nabla_X u) \circ X)(\nabla X)\|_{0,\infty} \overset{(1.8)}{\leq} c\,\|(\nabla_X u) \circ X\|_{0,\infty}\,\|\nabla X\|_{0,\infty} \\ &\overset{(2.7)}{=} c\,\|\nabla u\|_{0,\infty}\,\|\nabla X\|_{0,\infty} \overset{(*)}{\leq} c_1\,\|\nabla X\|_{0,\infty},\end{aligned}$$

where in $(*)$ we use $u \in C^1(\overline{\Omega})$, which implies $\nabla u \in C(\overline{\Omega})$, thus ∇u is bounded by its maximum $\max\limits_{x \in \overline{\Omega}} |\nabla u(x)|$.

Setting $\varphi(t) := \|\nabla X(t,x)\|_{0,\infty}$, this yields the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \leq c_1 \varphi(t),$$

hence

$$\varphi(t) \leq \varphi(0) + c_1 \int_0^t \varphi(\tau)\;\mathrm{d}\tau,$$

where

$$\varphi(0) = \|\nabla X(0,x)\|_{0,\infty} = \|\nabla x\|_{0,\infty} = 1.$$

Using the Lemma of GRONWALL (see Proposition 1.9) we obtain $\varphi(t) \leq e^{tc_1}$, i. e.

$$\|\nabla X(\varepsilon, x)\|_{0,\infty} \leq e^{\varepsilon c_1} < \infty.$$

Thus, by $\|v \circ X\|_{1,2}^2 \overset{(1.1)}{=} \|v \circ X\|^2 + \|\nabla(v \circ X)\|^2$, we have $a_2 \in H^1(\Omega)$, which proves the assertion. □

Corollary 4.9 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, let $\varepsilon > 0$ and $g \in H^1(\Omega)$. Then for any function*

$$u \in H^3(\Omega) \cap \mathcal{H}^1(\Omega),$$

the weak solution v of (N_ε^u) exists, is uniquely determined and satisfies

$$v \in H^3(\Omega) \cap \mathcal{H}^1(\Omega).$$

In particular, all results of this chapter for the weak solution v hold true.

Proof: Using Lemma 1.1, the regularity $u \in H^3(\Omega)$ implies

$$u \in C^1(\overline{\Omega}),$$

and $\nabla \cdot u = 0$ in Ω with $u = 0$ on $\partial\Omega$ follows from $u \in \mathcal{H}^1(\Omega)$.

Hence all assumptions of Theorem 4.4 and all additional assumptions of Proposition 4.8 are satisfied, which proves the assertion. $\square$

5 An approximate Navier-Stokes solution

In this chapter we use the weak solution of the boundary value problem (N_ε^u) from Chapter 4 to define a uniquely determined weak solution v^{k+1} of the discretized Navier-Stokes equations (N_ε^k), $k = 0, \dots, N-1$, formulated at the end of Chapter 3.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and let $0 < T < \infty$. For $2 \le N \in \mathbb{N}$, as in Section 3 we define a time grid in $[0,T]$ with $N+1$ equidistant grid points $t_k := k\varepsilon$ $(k = 0, \dots, N)$ of stepsize $\varepsilon := \frac{T}{N} > 0$.

If now $v^k := u$ is given for some $k \in \{0, \dots, N-1\}$, then $v^{k+1} := v$ can be constructed according to Corollary 4.9. More precisely, defining $L_\varepsilon^k v^{k+1}$ by

$$L_\varepsilon^k v^{k+1}(x) := \frac{1}{2\varepsilon} \left\{ v^{k+1}(X_k(\varepsilon, x)) - v^{k+1}(X_k^{-1}(\varepsilon, x)) \right\}$$

with mappings X_k, X_k^{-1} constructed from the solution $t \mapsto x(t) =: X_k(t, x_0)$ of the initial value problem

$$\begin{aligned} \dot{x}(t) &= v^k(x(t)), \\ x(0) &= x_0, \end{aligned} \tag{5.1}$$

we have

Proposition 5.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and let $0 < T < \infty$. For $2 \le N \in \mathbb{N}$ let $\varepsilon := \frac{T}{N} > 0$ and set $t_k := k\varepsilon$ $(k = 0, \dots, N)$. Let $f \in L^2(0, T; H^1(\Omega))$ with*

$$f^{k+1} := \frac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} f(\tau) \, \mathrm{d}\tau, \qquad k = 0, \dots, N-1, \tag{5.2}$$

and let $v^0 := v_0 \in C^1(\overline{\Omega})$ be divergence-free in Ω with $v_0 = 0$ on $\partial\Omega$.

Then the system

$$\begin{aligned} v^{k+1} - \varepsilon\nu\Delta v^{k+1} + \varepsilon L_\varepsilon^k v^{k+1} + \varepsilon\nabla p^{k+1} &= \varepsilon f^{k+1} + v^k \qquad && \text{in } \Omega, \\ \nabla \cdot v^{k+1} &= 0 && \text{in } \Omega, \\ v^{k+1}_{|\partial\Omega} &= 0, && \end{aligned} \tag{N_ε^k}$$

is, successively for $k = 0, \dots, N-1$, uniquely solvable and the weak solutions $v^1, v^2, \dots, v^N$ with the corresponding uniquely determined pressure gradients $\nabla p^1, \nabla p^2, \dots, \nabla p^N$ satisfy

$$v^j \in H^3(\Omega) \cap \mathcal{H}^1(\Omega), \quad \nabla p^j \in H^1(\Omega), \qquad j = 1, \dots, N.$$

Proof: Let $k \in \{0, \dots, N-1\}$. Using $f \in L^2(0, T; H^1(\Omega))$ we have $f \in L^1(0, T; H^1(\Omega))$ and it follows $f^{k+1} \in H^1(\Omega)$ since

$$\left\| f^{k+1} \right\|_{1,2} = \left\| \frac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} f(\tau) \, \mathrm{d}\tau \right\|_{1,2} \le \frac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} \| f(\tau) \|_{1,2} \, \mathrm{d}\tau < \infty.$$

Now let $k = 0$. By Theorem 4.4 and Proposition 4.8, there exists a uniquely determined weak solution v^1 of (N_ε^0) satisfying

$$v^1 \in H^3(\Omega) \cap \mathcal{H}^1(\Omega).$$

Then, successively for $k = 1, \dots, N-1$, using Corollary 4.9, we obtain v^{k+1} as uniquely determined weak solution of (N_ε^k) with

$$v^{k+1} \in H^3(\Omega) \cap \mathcal{H}^1(\Omega).$$

The statement concerning the pressure gradients follows from the HELMHOLTZ decomposition (1.14) by using the Projection Theorem (compare [7, pp. 31f.]). □

Corollary 5.2 *Let the assumptions of Proposition 5.1 be satisfied and let v^{k+1} denote the weak solution of the system (N_ε^k), $k = 0, \dots, N-1$. Then it holds*

$$\begin{aligned} &\left\| v^{k+1} \right\|^2 + \left\| v^{k+1} - v^k \right\|^2 + \varepsilon\nu \left\| \nabla v^{k+1} \right\|^2 \\ &\qquad \le \left\| v^k \right\|^2 + \frac{\varepsilon c_p^2}{\nu} \left\| f^{k+1} \right\|^2, \end{aligned} \tag{5.3a}$$

$$\begin{aligned} &\left\| \nabla v^{k+1} \right\|^2 + \left\| \nabla v^{k+1} - \nabla v^k \right\|^2 + \frac{1}{\varepsilon\nu} \left\| v^{k+1} - v^k \right\|^2 \\ &\qquad \le \left\| \nabla v^k \right\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu} \right) \left\| f^{k+1} \right\|^2 + \frac{2}{\varepsilon\nu} \left\| v^k \right\|^2, \end{aligned} \tag{5.3b}$$

$$\begin{aligned} &\left\| \nabla v^{k+1} \right\|^2 + \left\| \nabla v^{k+1} - \nabla v^k \right\|^2 + \varepsilon\nu \left\| P\Delta v^{k+1} \right\|^2 \\ &\qquad \le \left\| \nabla v^k \right\|^2 + \left(\frac{2\varepsilon}{\nu} + \frac{2c_p^2}{\nu} \right) \left\| f^{k+1} \right\|^2 + \frac{2}{\varepsilon\nu} \left\| v^k \right\|^2, \end{aligned} \tag{5.3c}$$

where c_p is the POINCARÉ *constant.*

Proof: These estimates follow immediately from Corollary 4.6 and Proposition 4.7 for $v := v^{k+1}$, $u := v^k$, and $g := f^{k+1}$. □

Using the steady solutions v^{k+1} $(k = 0, \dots, N-1)$ constructed above we define a corresponding step function in $[0, T]$:

Definition 5.3 *Let the assumptions of Proposition 5.1 be satisfied and let v^{k+1} denote the solution of the system (N_ε^k), $k = 0, \dots, N-1$. Then we define a non-steady step function $v^\varepsilon : [-\varepsilon, T] \to \mathbb{R}^3$ by*

$$v^\varepsilon(t) := \begin{cases} v_0 & , t \in [-\varepsilon, 0] \\ v^{k+1} & , t \in (t_k, t_{k+1}], \qquad k = 0, \dots, N-1. \end{cases}$$

We shall see that the step function v^ε, $\varepsilon > 0$, in a certain sense represents an approximate solution of the NAVIER-STOKES equations. In the following, we prove some important properties of this function.

Proposition 5.4 *Let the assumptions of Proposition 5.1 be satisfied. Then the function v^ε from Definition 5.3 satisfies*

$$v^\varepsilon \in L^\infty\left(0, T; H^3(\Omega) \cap \mathcal{H}^1(\Omega)\right).$$

The energy equality

$$\begin{aligned}\|v^\varepsilon(t_k)\|^2 + \frac{1}{\varepsilon}\int_0^{t_k} \|v^\varepsilon(\tau) - v^\varepsilon(\tau-\varepsilon)\|^2 \ \mathrm{d}\tau + 2\nu \int_0^{t_k} \|\nabla v^\varepsilon(\tau)\|^2 \ \mathrm{d}\tau \\ = \|v_0\|^2 + 2\int_0^{t_k} \langle f(\tau), v^\varepsilon(\tau)\rangle \ \mathrm{d}\tau\end{aligned} \tag{5.4}$$

holds true for all grid points $t_k = k\varepsilon$ ($k = 0, \ldots, N$).

Proof: The energy equality is trivial at time $t_0 = 0$. Now consider the grid points t_{k+1} for $k = 0, \ldots, N-1$. Then – by construction of v^ε (compare Proposition 5.1 and Definition 5.3) – we know $v^\varepsilon(t_{k+1}) = v^{k+1}$ is a weak solution of the boundary value problem (N_ε^k), satisfying

$$v^{k+1} - \varepsilon\nu\Delta v^{k+1} + \varepsilon L_\varepsilon^k v^{k+1} = \varepsilon f^{k+1} + v^k \qquad \text{in } \mathcal{H}^0(\Omega).$$

Scalar multiplication with v^{k+1} in $L^2(\Omega)$ implies

$$\underbrace{\left\langle v^{k+1}, v^{k+1}\right\rangle - \left\langle v^k, v^{k+1}\right\rangle}_{=:a_1} \underbrace{- \varepsilon\nu\left\langle \Delta v^{k+1}, v^{k+1}\right\rangle}_{=:a_2} + \underbrace{\varepsilon\left\langle L_\varepsilon^k v^{k+1}, v^{k+1}\right\rangle}_{=:a_3} = \underbrace{\varepsilon\left\langle f^{k+1}, v^{k+1}\right\rangle}_{=:a_4}.$$

For the left hand side of this equation – similar to the proof of the first equation (4.7a) in Lemma 4.2 – we obtain

$$\begin{aligned} a_1 &= \frac{1}{2}\left(\left\|v^{k+1}\right\|^2 - \left\|v^k\right\|^2 + \left\|v^{k+1} - v^k\right\|^2\right), \\ a_2 &\overset{(1.11)}{=} \varepsilon\nu\left\|\nabla v^{k+1}\right\|^2, \\ a_3 &\overset{(3.1b)}{=} 0. \end{aligned}$$

Using (5.2) and FUBINI, for a_4 we have

$$\begin{aligned} a_4 &= \varepsilon\left\langle f^{k+1}, v^{k+1}\right\rangle = \int_\Omega \int_{t_k}^{t_{k+1}} f(\tau)\ \mathrm{d}\tau \cdot v^{k+1}\ \mathrm{d}x \\ &= \int_{t_k}^{t_{k+1}} \int_\Omega f(\tau)\cdot v^{k+1}\ \mathrm{d}x\ \mathrm{d}\tau = \int_{t_k}^{t_{k+1}} \left\langle f(\tau), v^{k+1}\right\rangle\ \mathrm{d}\tau. \end{aligned}$$

This leads to

$$\frac{1}{2}\left(\left\|v^{k+1}\right\|^2-\left\|v^k\right\|^2\right)+\frac{1}{2}\left\|v^{k+1}-v^k\right\|^2+\varepsilon\nu\left\|\nabla v^{k+1}\right\|^2$$
$$=\int_{t_k}^{t_{k+1}}\left\langle f(\tau),v^{k+1}\right\rangle\ \mathrm{d}\tau$$

for $k=0,\ldots,N-1$. Setting $j=k$ and summing up for $j=0,\ldots,k$, we obtain

$$\sum_{j=0}^{k}\left(\left\|v^{j+1}\right\|^2-\left\|v^j\right\|^2\right)+\sum_{j=0}^{k}\left\|v^{j+1}-v^j\right\|^2+2\varepsilon\nu\sum_{j=0}^{k}\left\|\nabla v^{j+1}\right\|^2$$
$$=2\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\left\langle f(\tau),v^{j+1}\right\rangle\ \mathrm{d}\tau.$$

Hence, using Definition 5.3 regarding the step function v^ε, it holds

$$\|v^\varepsilon(t_{k+1})\|^2+\frac{1}{\varepsilon}\int_0^{t_{k+1}}\|v^\varepsilon(\tau)-v^\varepsilon(\tau-\varepsilon)\|^2\ \mathrm{d}\tau+2\nu\int_0^{t_{k+1}}\|\nabla v^\varepsilon(\tau)\|^2\ \mathrm{d}\tau$$
$$=\|v_0\|^2+2\int_0^{t_{k+1}}\langle f(\tau),v^\varepsilon(\tau)\rangle\ \mathrm{d}\tau$$

for all $k=0,\ldots,N-1$. This implies the asserted energy equality.

To prove the regularity $v^\varepsilon\in L^\infty\left(0,T;H^3(\Omega)\cap\mathcal{H}^1(\Omega)\right)$, let $t\in(0,T]$, be fixed. Then we have $v^\varepsilon(t)=v^{k+1}$ for some $k\in\{0,\ldots,N-1\}$ as weak solution of (N_ε^k), satisfying $v^k\in H^3(\Omega)\cap\mathcal{H}^1(\Omega)$ (compare Proposition 5.1). Setting $c:=\max\limits_{0\le k\le N-1}\left\|v^{k+1}\right\|_{3,2}$ we obtain

$$\operatorname*{ess\,sup}_{t\in[0,T]}\|v^\varepsilon(t)\|_{3,2}=\max_{t\in(0,T]}\|v^\varepsilon(t)\|_{3,2}\le c,$$

which proves the assertion. □

Proposition 5.5 *Let the assumptions of Proposition 5.1 be satisfied. Then the function v^ε from Definition 5.3 satisfies the estimate*

$$\|v^\varepsilon(t)\|^2+\frac{1}{\varepsilon}\int_0^{t}\|v^\varepsilon(\tau)-v^\varepsilon(\tau-\varepsilon)\|^2\ \mathrm{d}\tau+\nu\int_0^{t}\|\nabla v^\varepsilon(\tau)\|^2\ \mathrm{d}\tau$$
$$\le\|v_0\|^2+\frac{c_p^2}{\nu}\int_0^{t_{k+1}}\|f(\tau)\|^2\ \mathrm{d}\tau \tag{5.5}$$

for all $t\in[0,T]$. Here t_{k+1} is the smallest grid point with $t\le t_{k+1}$.

Proof: Before we prove the assertion, we consider $\left\|f^{j+1}\right\|$ for some $j\in\{0,\ldots,N-1\}$. We find

$$\left\|f^{j+1}\right\|^2\ =\ \left\|\frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}}f(\tau)\ \mathrm{d}\tau\right\|^2$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon^2}\int_\Omega \left| \int_{t_j}^{t_{j+1}} f(\tau)\, \mathrm{d}\tau \right|^2 \mathrm{d}x \\
&\le \frac{1}{\varepsilon^2}\int_\Omega \left(\int_{t_j}^{t_{j+1}} 1 \cdot |f(\tau)|\, \mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\overset{(1.4)}{\le} \frac{1}{\varepsilon^2}\int_\Omega \left(\left(\int_{t_j}^{t_{j+1}} |1|^2\, \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_{t_j}^{t_{j+1}} |f(\tau)|^2\, \mathrm{d}\tau \right)^{\frac{1}{2}} \right)^2 \mathrm{d}x \\
&= \frac{1}{\varepsilon^2}\int_\Omega \varepsilon \int_{t_j}^{t_{j+1}} |f(\tau)|^2\, \mathrm{d}\tau\, \mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} \int_\Omega |f(\tau)|^2\, \mathrm{d}x\, \mathrm{d}\tau \\
&= \frac{1}{\varepsilon}\int_{t_j}^{t_{j+1}} \|f(\tau)\|^2\, \mathrm{d}\tau. \qquad (5.6)
\end{aligned}
$$

For $t = 0$, the assertion is trivial. For $t \in (0, T]$ we have $v^\varepsilon(t) = v^{k+1}$ for some $k \in \{0, \ldots, N-1\}$ as weak solution of (N_ε^k) satisfying (5.3a). Setting $j = k$ and summing up for $j = 0, \ldots, k$ we obtain

$$
\begin{aligned}
&\sum_{j=0}^{k} \left(\|v^{j+1}\|^2 - \|v^j\|^2 \right) + \sum_{j=0}^{k} \|v^{j+1} - v^j\|^2 + \varepsilon\nu \sum_{j=0}^{k} \|\nabla v^{j+1}\|^2 \\
&\quad \overset{(5.3a)}{\le} \frac{\varepsilon c_p^2}{\nu} \sum_{j=0}^{k} \|f^{j+1}\|^2 \\
&\quad \overset{(5.6)}{\le} \frac{c_p^2}{\nu} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \|f(\tau)\|^2\, \mathrm{d}\tau,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\left(\|v^\varepsilon(t)\|^2 - \|v_0\|^2 \right) + \frac{1}{\varepsilon}\int_0^{t_{k+1}} \|v^\varepsilon(\tau) - v^\varepsilon(\tau - \varepsilon)\|^2\, \mathrm{d}\tau + \nu \int_0^{t_{k+1}} \|\nabla v^\varepsilon(\tau)\|^2\, \mathrm{d}\tau \\
&\qquad \le \frac{c_p^2}{\nu} \int_0^{t_{k+1}} \|f(\tau)\|^2\, \mathrm{d}\tau.
\end{aligned}
$$

Since $t \le t_{k+1}$ and all integrands on the left hand side are non-negative, this implies the inequality (5.5). $\square$

In Chapter 6, we will derive some limit function v of v^ε as $\varepsilon \to 0$. In order to show that v is a weak solution of (N_0), we proceed as in Shinbrot [19, pp. 164–173] and firstly prove that v^ε satisfies equation (5.7) of Lemma 5.7. For this, we need to estimate the difference between the time derivative and its difference quotient:

Lemma 5.6 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Let $0 < T < \infty$, $\varphi \in C^1([0,T]; C^\infty_{0,\sigma}(\Omega))$, and let $\varepsilon > 0$. Then for each $t \in [0,T)$ it holds*

$$\left\| \dot{\varphi}(t) - \frac{\varphi(t+\varepsilon) - \varphi(t)}{\varepsilon} \right\| = o(1) \qquad \text{as } \varepsilon \to 0.$$

Proof: Let $t \in [0,T)$. Setting

$$r_t(\varepsilon) := \frac{(\varepsilon \dot{\varphi}(t) - \varphi(t+\varepsilon)) + \varphi(t)}{\varepsilon}$$

and applying the rule of de L'Hospital we find

$$\lim_{\varepsilon \to 0} r_t(\varepsilon) = \lim_{\varepsilon \to 0} \frac{\dot{\varphi}(t) - \dot{\varphi}(t+\varepsilon)}{1} = 0.$$

□

To precisely formulate the next lemma, for $t \in (t_k, t_{k+1}]$, $k = 0, \dots, N-1$, we set

$$\tilde{f}(t) := f^{k+1} := \frac{1}{\varepsilon} \int_{t_k}^{t_{k+1}} f(\tau) \, \mathrm{d}\tau$$

and define

$$\begin{aligned} L_\varepsilon(t)\varphi(t,x) &:= L^k_\varepsilon \varphi(t,x) \\ &:= \frac{1}{2\varepsilon} \left\{ \varphi(t) \circ X_k(\varepsilon, x) - \varphi(t) \circ X_k^{-1}(\varepsilon, x) \right\}. \end{aligned}$$

Here the mappings X_k, X_k^{-1} are constructed from the solution $\tau \mapsto x(\tau) =: X_k(\tau, x_0)$ of the initial value problem (5.1):

$$\begin{aligned} \dot{x}(\tau) &= v^k(x(\tau)), \\ x(0) &= x_0. \end{aligned}$$

Lemma 5.7 *Let the assumptions of Proposition 5.1 be satisfied. Then the function v^ε from Definition 5.3 satisfies*

$$\begin{aligned} -\int_0^T \langle v^\varepsilon(t), \dot{\varphi}(t) \rangle \; \mathrm{d}t + \nu \int_0^T \langle \nabla v^\varepsilon(t), \nabla \varphi(t) \rangle \; \mathrm{d}t - \int_0^T \langle v^\varepsilon(t), L_\varepsilon(t)\varphi(t) \rangle \; \mathrm{d}t \\ = \langle v_0, \varphi(0) \rangle + \int_0^T \left\langle \tilde{f}(t), \varphi(t) \right\rangle \; \mathrm{d}t + o(1) \end{aligned} \tag{5.7}$$

for all $\varphi \in C^1_0([0,T); C^\infty_{0,\sigma}(\Omega))$.

Proof: For fixed $t \in (0,T]$ we have $v^\varepsilon(t) = v^{k+1}$ for some $k \in \{0, \dots, N-1\}$ as weak solution of (N^k_ε). Hence it holds

$$\langle v^\varepsilon(t), \varphi(t) \rangle + \varepsilon\nu \langle \nabla v^\varepsilon(t), \nabla \varphi(t) \rangle + \varepsilon \langle L_\varepsilon(t) v^\varepsilon(t), \varphi(t) \rangle = \varepsilon \left\langle \tilde{f}(t), \varphi(t) \right\rangle + \langle v^\varepsilon(t-\varepsilon), \varphi(t) \rangle$$

for each $\varphi \in C_0^1([0,T); C_{0,\sigma}^\infty(\Omega))$. Integrating over t, we find

$$\begin{aligned}\int_0^T \langle v^\varepsilon(t), \varphi(t)\rangle \; \mathrm{d}t + \varepsilon\nu \int_0^T \langle \nabla v^\varepsilon(t), \nabla\varphi(t)\rangle \; \mathrm{d}t + \varepsilon \int_0^T \langle L_\varepsilon(t) v^\varepsilon(t), \varphi(t)\rangle \; \mathrm{d}t \\ = \varepsilon \int_0^T \left\langle \tilde{f}(t), \varphi(t)\right\rangle \; \mathrm{d}t + \int_0^T \langle v^\varepsilon(t-\varepsilon), \varphi(t)\rangle \; \mathrm{d}t.\end{aligned} \tag{5.8}$$

Since, for $t \in (t_k, t_{k+1}]$

$$\begin{aligned}\varepsilon \langle L_\varepsilon(t) v^\varepsilon(t), \varphi(t)\rangle &= \varepsilon \left\langle L_\varepsilon^k v^\varepsilon(t), \varphi(t)\right\rangle \\ &= \frac{1}{2}\left\langle v^\varepsilon(t)\circ X_k(\varepsilon,\cdot) - v^\varepsilon(t)\circ X_k^{-1}(\varepsilon,\cdot), \varphi(t)\right\rangle \\ &= \frac{1}{2}\langle v^\varepsilon(t)\circ X_k(\varepsilon,\cdot), \varphi(t)\rangle - \frac{1}{2}\left\langle v^\varepsilon(t)\circ X_k^{-1}(\varepsilon,\cdot), \varphi(t)\right\rangle \\ &\overset{(2.8)}{=} \frac{1}{2}\left\langle v^\varepsilon(t), \varphi(t)\circ X_k^{-1}(\varepsilon,\cdot)\right\rangle - \frac{1}{2}\langle v^\varepsilon(t), \varphi(t)\circ X_k(\varepsilon,\cdot)\rangle \\ &= -\frac{1}{2}\left\langle v^\varepsilon(t), \varphi(t)\circ X_k(\varepsilon,\cdot) - \varphi(t)\circ X_k^{-1}(\varepsilon,\cdot)\right\rangle \\ &= -\varepsilon\left\langle v^\varepsilon(t), L_\varepsilon^k\varphi(t)\right\rangle \\ &= -\varepsilon\langle v^\varepsilon(t), L_\varepsilon(t)\varphi(t)\rangle,\end{aligned} \tag{5.9}$$

we obtain from (5.8)

$$\begin{aligned}\int_0^T \left\langle \frac{v^\varepsilon(t) - v^\varepsilon(t-\varepsilon)}{\varepsilon}, \varphi(t)\right\rangle \; \mathrm{d}t + \nu \int_0^T \langle \nabla v^\varepsilon(t), \nabla\varphi(t)\rangle \; \mathrm{d}t - \int_0^T \langle v^\varepsilon(t), L_\varepsilon(t)\varphi(t)\rangle \; \mathrm{d}t \\ = \int_0^T \left\langle \tilde{f}(t), \varphi(t)\right\rangle \; \mathrm{d}t.\end{aligned}$$

Thus everything is proved if only

$$\int_0^T \left\langle \frac{v^\varepsilon(t) - v^\varepsilon(t-\varepsilon)}{\varepsilon}, \varphi(t)\right\rangle \; \mathrm{d}t = -\int_0^T \langle v^\varepsilon(t), \dot{\varphi}(t)\rangle \; \mathrm{d}t - \langle v_0, \varphi(0)\rangle - o(1). \tag{5.10}$$

To prove (5.10), in a first step we pass the difference quotient of v^ε on the test function φ. We find

$$\begin{aligned}&\int_0^T \left\langle \frac{v^\varepsilon(t) - v^\varepsilon(t-\varepsilon)}{\varepsilon}, \varphi(t)\right\rangle \; \mathrm{d}t \\ &\quad = \frac{1}{\varepsilon}\int_0^T \langle v^\varepsilon(t), \varphi(t)\rangle \; \mathrm{d}t - \frac{1}{\varepsilon}\int_0^T \langle v^\varepsilon(t-\varepsilon), \varphi(t)\rangle \; \mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \left\langle v^\varepsilon(t), \frac{\varphi(t)-\varphi(t+\varepsilon)}{\varepsilon} \right\rangle \,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^T \langle v^\varepsilon(t), \varphi(t+\varepsilon)\rangle \,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon}\int_0^T \langle v^\varepsilon(t-\varepsilon), \varphi(t)\rangle \,\mathrm{d}t \\
&\overset{(*)}{=} -\int_0^T \left\langle v^\varepsilon(t), \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\rangle \,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^{T-\varepsilon} \langle v^\varepsilon(t), \varphi(t+\varepsilon)\rangle \,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon}\int_{-\varepsilon}^{T-\varepsilon} \langle v^\varepsilon(t), \varphi(t+\varepsilon)\rangle \,\mathrm{d}t \\
&= -\int_0^T \left\langle v^\varepsilon(t), \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\rangle \,\mathrm{d}t - \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} \langle v^\varepsilon(t), \varphi(t+\varepsilon)\rangle \,\mathrm{d}t, \qquad (5.11)
\end{aligned}
$$

where in $(*)$ we use $\varphi(t) = 0$ for $t \geq T$, since φ has compact support in $[0, T)$.

In a second step we add a zero to the term (5.11) and obtain

$$
\begin{aligned}
&\int_0^T \left\langle \frac{v^\varepsilon(t) - v^\varepsilon(t-\varepsilon)}{\varepsilon}, \varphi(t) \right\rangle \,\mathrm{d}t \\
&\overset{(5.11)}{=} -\int_0^T \left\langle v^\varepsilon(t), \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\rangle \,\mathrm{d}t - \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} \langle v^\varepsilon(t), \varphi(t+\varepsilon)\rangle \,\mathrm{d}t \\
&\quad - \int_0^T \langle v^\varepsilon(t), \dot\varphi(t)\rangle \,\mathrm{d}t - \langle v_0, \varphi(0)\rangle \\
&\quad + \int_0^T \langle v^\varepsilon(t), \dot\varphi(t)\rangle \,\mathrm{d}t + \langle v_0, \varphi(0)\rangle \\
&= -\int_0^T \langle v^\varepsilon(t), \dot\varphi(t)\rangle \,\mathrm{d}t - \langle v_0, \varphi(0)\rangle \\
&\quad + \underbrace{\int_0^T \langle v^\varepsilon(t), \dot\varphi(t)\rangle \,\mathrm{d}t - \int_0^T \left\langle v^\varepsilon(t), \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\rangle \,\mathrm{d}t}_{=:a_1(\varepsilon)} \\
&\quad + \underbrace{\langle v_0, \varphi(0)\rangle - \frac{1}{\varepsilon}\int_{-\varepsilon}^{0} \langle v^\varepsilon(t), \varphi(t+\varepsilon)\rangle \,\mathrm{d}t}_{=:a_2(\varepsilon)}.
\end{aligned}
$$

Thus it remains to prove $a_1(\varepsilon) = o(1)$ and $a_2(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Concerning $a_1(\varepsilon)$ we find

$$
\begin{aligned}
|a_1(\varepsilon)| &= \left| \int_0^T \langle v^\varepsilon(t), \dot\varphi(t)\rangle \,\mathrm{d}t - \int_0^T \left\langle v^\varepsilon(t), \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\rangle \,\mathrm{d}t \right| \\
&= \left| \int_0^T \left\langle v^\varepsilon(t), \dot\varphi(t) - \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\rangle \,\mathrm{d}t \right| \\
&\overset{(1.5)}{\leq} \int_0^T \|v^\varepsilon(t)\| \left\| \dot\varphi(t) - \frac{\varphi(t+\varepsilon)-\varphi(t)}{\varepsilon} \right\| \,\mathrm{d}t
\end{aligned}
$$

$$\stackrel{(**)}{\leq} \int_0^T \|v^\varepsilon(t)\| \ \mathrm{d}t \ \left\| \dot{\varphi}(\tilde{t}) - \frac{\varphi(\tilde{t}+\varepsilon) - \varphi(\tilde{t})}{\varepsilon} \right\|$$
$$\stackrel{5.6}{=} o(1) \int_0^T \|v^\varepsilon(t)\| \ \mathrm{d}t \quad \text{as } \varepsilon \to 0,$$

where in $(**)$ we use $\varphi \in C_0^1([0,T); C_{0,\sigma}^\infty(\Omega))$ and choose $\tilde{t} \in [0,T)$ such that

$$\left\| \dot{\varphi}(\tilde{t}) - \frac{\varphi(\tilde{t}+\varepsilon) - \varphi(\tilde{t})}{\varepsilon} \right\| = \max_{t \in [0,T]} \left\| \dot{\varphi}(t) - \frac{\varphi(t+\varepsilon) - \varphi(t)}{\varepsilon} \right\|.$$

Since

$$\int_0^T \|v^\varepsilon(t)\| \ \mathrm{d}t \stackrel{(5.5)}{\leq} T \left(\|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^T \|f(\tau)\|^2 \ \mathrm{d}\tau \right)^{\frac{1}{2}} < \infty,$$

we have $a_1(\varepsilon) = o(1)$ as $\varepsilon \to 0$. Using $v^\varepsilon(t) = v_0$ in $[-\varepsilon, 0]$, for $a_2(\varepsilon)$ we obtain

$$\begin{aligned}
|a_2(\varepsilon)| &= \left| \langle v_0, \varphi(0) \rangle - \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \langle v^\varepsilon(t), \varphi(t+\varepsilon) \rangle \ \mathrm{d}t \right| \\
&= \left| -\frac{1}{\varepsilon} \int_{-\varepsilon}^0 \langle v_0, \varphi(t+\varepsilon) - \varphi(0) \rangle \ \mathrm{d}t \right| \\
&= \left| \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left\langle v_0, \int_0^{t+\varepsilon} \dot{\varphi}(\tau) \ \mathrm{d}\tau \right\rangle \ \mathrm{d}t \right| \\
&= \left| \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \int_0^{t+\varepsilon} \langle v_0, \dot{\varphi}(\tau) \rangle \ \mathrm{d}\tau \ \mathrm{d}t \right| \\
&\stackrel{(1.5)}{\leq} \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \int_0^{t+\varepsilon} \|v_0\| \ \|\dot{\varphi}(\tau)\| \ \mathrm{d}\tau \ \mathrm{d}t \\
&\leq \frac{1}{\varepsilon} \|v_0\| \max_{\tau \in [0,\varepsilon]} \|\dot{\varphi}(\tau)\| \int_{-\varepsilon}^0 \int_0^{t+\varepsilon} \mathrm{d}\tau \ \mathrm{d}t \\
&\leq \frac{\varepsilon}{2} \|v_0\| \max_{\tau \in [0,T]} \|\dot{\varphi}(\tau)\|,
\end{aligned}$$

which even implies $a_2(\varepsilon) = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. Thus everything is proved. □

6 A weak solution of the Navier-Stokes equations

Throughout this chapter, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and let $0 < T < \infty$. For $2 \leq N \in \mathbb{N}$ we consider an equidistant time grid in $[0, T]$ with stepsize $\varepsilon := \frac{T}{N} > 0$ and grid points $t_k := k\varepsilon$, $k = 0, \ldots, N$.

Now let $f \in L^2(0, T; H^1(\Omega))$, and let $v^0 := v_0 \in C^1(\overline{\Omega})$ be divergence-free in Ω with $v_0 = 0$ on $\partial\Omega$. By v^{k+1}, $k = 0, \ldots, N-1$, we denote the weak solution of (N_ε^k) (compare Proposition 5.1). In this chapter, we consider the step function

$$v^\varepsilon(t) := \begin{cases} v_0 & , t \in [-\varepsilon, 0] \\ v^{k+1} & , t \in (t_k, t_{k+1}], \qquad k = 0, \ldots, N-1 \end{cases}$$

from Definition 5.3 and investigate the limit as $\varepsilon \to 0$.

To do so, we first prove that for $\varepsilon := \varepsilon_N := \frac{T}{N} \to 0$, i.e. $N \to \infty$, there exists an accumulation point v of $\{v^{\varepsilon_N}\}_N$ that satisfies an energy inequality. In Section 6.2 we shall prove that v is a weak solution of (N_0).

6.1 Convergence properties of $\{v^{\varepsilon_N}\}_N$

Before we investigate the limit behaviour of the sequence $\{v^{\varepsilon_N}\}_N$ as $N \to \infty$, in the next lemma we prove the weak equicontinuity of this sequence. This important property is needed to prove the weak convergence of a subsequence of $\{v^{\varepsilon_N}(t)\}_N$ in $\mathcal{H}^0(\Omega)$ for all $t \in [0, T]$.

Lemma 6.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and let $0 < T < \infty$. For fixed $2 \leq N \in \mathbb{N}$ we set $\varepsilon := \frac{T}{N} > 0$ and $t_k := k\varepsilon$ ($k = 0, \ldots, N$). In addition, let $f \in L^2(0, T; H^1(\Omega))$ and let $v^0 := v_0 \in C^1(\overline{\Omega})$ be divergence-free in Ω with $v_0 = 0$ on $\partial\Omega$.*

Then, for each $\varphi \in C_{0,\sigma}^\infty(\Omega)$, the step function v^ε from Definition 5.3 satisfies

$$|\langle v^\varepsilon(t) - v^\varepsilon(s), \varphi\rangle| \leq c\left(|t-s| + \left|\int_s^t \|f(\tau)\|^2 \, \mathrm{d}\tau\right|\right) + o(1) \quad \text{as } \varepsilon \to 0 \tag{6.1}$$

for all $s, t \in [0, T]$, where c does not depend on s, t, ε, and where $o(1)$ does not depend on s, t.

Proof: Let $\varphi \in C_{0,\sigma}^\infty(\Omega)$ and let v^{k+1} denote the weak solution of (N_ε^k) for $k = 0, \ldots, N-1$. Using (4.9) for $v := v^{k+1}$, $u := v^k$, $g := f^{k+1} := \frac{1}{\varepsilon}\int_{t_k}^{t_{k+1}} f(\tau) \, \mathrm{d}\tau$ and

$$\varepsilon\nu \left\langle \nabla v^{k+1}, \nabla\varphi \right\rangle \overset{(1.11)}{=} -\varepsilon\nu \left\langle v^{k+1}, \Delta\varphi \right\rangle,$$

$$\varepsilon \left\langle L_\varepsilon^k v^{k+1}, \varphi \right\rangle \overset{(5.9)}{=} -\varepsilon \left\langle v^{k+1}, L_\varepsilon^k \varphi \right\rangle,$$

we find

$$\left\langle v^{k+1} - v^k, \varphi \right\rangle = \varepsilon\nu \left\langle v^{k+1}, \Delta\varphi \right\rangle + \varepsilon \left\langle v^{k+1}, L_\varepsilon^k \varphi \right\rangle + \varepsilon \left\langle f^{k+1}, \varphi \right\rangle \tag{6.2}$$

for $k = 0, \ldots, N-1$.

Now let $s, t \in (0, T]$. Then there are numbers $k_s, k_t \in \{0, \ldots, N-1\}$ such that $v^\varepsilon(s) = v^{k_s+1}$ and $v^\varepsilon(t) = v^{k_t+1}$. For $s = t = 0$ as well as for $k_s = k_t$ the left side of (6.1) is 0, thus the assertion is trivial. Now let $s, t \in [0, T]$ with $k_s < k_t$ (where for the case $s = 0$ we set $k_s := -1$). Then we have

$$\begin{aligned}
\langle v^\varepsilon(t) - v^\varepsilon(s), \varphi \rangle &= \left\langle v^{k_t+1} - v^{k_s+1}, \varphi \right\rangle \\
&= \sum_{k=k_s+1}^{k_t} \left\langle v^{k+1} - v^k, \varphi \right\rangle \\
&\overset{(6.2)}{=} \sum_{k=k_s+1}^{k_t} \left(\varepsilon\nu \left\langle v^{k+1}, \Delta\varphi \right\rangle + \varepsilon \left\langle v^{k+1}, L_\varepsilon^k \varphi \right\rangle + \varepsilon \left\langle f^{k+1}, \varphi \right\rangle \right).
\end{aligned}$$

This implies

$$\begin{aligned}
|\langle v^\varepsilon(t) - v^\varepsilon(s), \varphi \rangle| &\overset{(1.5)}{\leq} \varepsilon \Big(\underbrace{\nu (k_t - k_s) \|\Delta\varphi\| \max_{i \in \{k_s+1, \ldots, k_t\}} \left\| v^{i+1} \right\|}_{=:a_1} \\
&\quad + \underbrace{\max_{i \in \{k_s+1, \ldots, k_t\}} \left\| v^{i+1} \right\| \sum_{k=k_s+1}^{k_t} \left\| L_\varepsilon^k \varphi \right\|}_{=:a_2} + \underbrace{\|\varphi\| \sum_{k=k_s+1}^{k_t} \left\| f^{k+1} \right\|}_{=:a_3} \Big).
\end{aligned}$$

Due to $t_{k_t} < t$ and $s - \varepsilon \leq t_{k_s}$, hence $-t_{k_s} \leq -s + \varepsilon$, we have

$$\varepsilon(k_t - k_s) = t_{k_t} - t_{k_s} < t - s + \varepsilon, \tag{6.3}$$

and to estimate $\max_{i \in \{k_s+1, \ldots, k_t\}} \left\| v^{i+1} \right\|$ independently of k_s, k_t, ε, we use

$$\begin{aligned}
\max_{i \in \{k_s+1, \ldots, k_t\}} \left\| v^{i+1} \right\| &\leq \max_{i \in \{0, \ldots, N-1\}} \left\| v^{i+1} \right\| \\
&\overset{(5.5)}{\leq} \left(\|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^T \|f(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&=: M.
\end{aligned}$$

Hence, for εa_1 we find

$$\begin{aligned}
\varepsilon a_1 &\overset{(6.3)}{<} \nu M \|\Delta\varphi\| (t - s + \varepsilon) \\
&=: \nu M \|\Delta\varphi\| \, |t - s| + \mathcal{O}(\varepsilon) \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

To estimate εa_2, we first consider

$$L^k_\varepsilon \varphi(x) := \frac{1}{2\varepsilon}\left\{\varphi \circ X_k(\varepsilon, x) - \varphi \circ X_k^{-1}(\varepsilon, x)\right\},$$

with mappings X_k, X_k^{-1} constructed from the solution $t \mapsto x(t) =: X_k(t, x_0)$ of the initial value problem (5.1):

$$\begin{aligned} \dot{x}(t) &= v^k(x(t)), \\ x(0) &= x_0. \end{aligned}$$

For $\varphi \circ X_k(\varepsilon, x) - \varphi(x)$ we can use the fundamental theorem of calculus and obtain

$$\begin{aligned} \varphi \circ X_k(\varepsilon, x) - \varphi(x) &= \varphi \circ X_k(\varepsilon, x) - \varphi \circ X_k(0, x) \\ &= \int_0^\varepsilon \partial_\tau \varphi(X_k(\tau, x))\ \mathrm{d}\tau \\ &= \int_0^\varepsilon \partial_\tau X_k(\tau, x) \cdot \nabla_X \varphi(X_k(\tau, x))\ \mathrm{d}\tau \\ &= \int_0^\varepsilon v^k \circ X_k(\tau, x) \cdot \nabla_X \varphi(X_k(\tau, x))\ \mathrm{d}\tau \\ &= \int_0^\varepsilon [v^k \cdot \nabla_X \varphi] \circ X_k(\tau, x)\ \mathrm{d}\tau. \end{aligned} \tag{6.4}$$

Analogously, we have

$$\varphi(x) - \varphi \circ X_k^{-1}(\varepsilon, x) = \int_0^\varepsilon [v^k \cdot \nabla_X \varphi] \circ X_k^{-1}(\tau, x)\ \mathrm{d}\tau.$$

Thus, we can use the conservation of measure to estimate $\left\|L^k_\varepsilon \varphi\right\|$ and obtain

$$\begin{aligned} \left\|L^k_\varepsilon \varphi\right\| &= \frac{1}{2\varepsilon}\left\|\varphi \circ X_k(\varepsilon, \cdot) - \varphi \circ X_k^{-1}(\varepsilon, \cdot)\right\| \\ &= \frac{1}{2\varepsilon}\left\|\Big(\varphi \circ X_k(\varepsilon, \cdot) - \varphi(\cdot)\Big) + \Big(\varphi(\cdot) - \varphi \circ X_k^{-1}(\varepsilon, \cdot)\Big)\right\| \\ &\leq \frac{1}{2\varepsilon}\left\|\varphi \circ X_k(\varepsilon, \cdot) - \varphi(\cdot)\right\| + \frac{1}{2\varepsilon}\left\|\varphi(\cdot) - \varphi \circ X_k^{-1}(\varepsilon, \cdot)\right\| \\ &\overset{(6.4)}{=} \frac{1}{2\varepsilon}\left\|\int_0^\varepsilon [v^k \cdot \nabla_X \varphi] \circ X_k(\tau, \cdot)\ \mathrm{d}\tau\right\| + \frac{1}{2\varepsilon}\left\|\int_0^\varepsilon [v^k \cdot \nabla_X \varphi] \circ X_k^{-1}(\tau, \cdot)\ \mathrm{d}\tau\right\| \\ &\overset{(*)}{\leq} \frac{1}{2\sqrt{\varepsilon}}\left(\int_0^\varepsilon \left\|[v^k \cdot \nabla_X \varphi] \circ X_k(\tau, \cdot)\right\|^2\ \mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\quad + \frac{1}{2\sqrt{\varepsilon}}\left(\int_0^\varepsilon \left\|[v^k \cdot \nabla_X \varphi] \circ X_k^{-1}(\tau, \cdot)\right\|^2\ \mathrm{d}\tau\right)^{\frac{1}{2}} \\ &\overset{(2.7)}{=} \frac{1}{\sqrt{\varepsilon}}\left(\int_0^\varepsilon \left\|v^k \cdot \nabla \varphi\right\|^2\ \mathrm{d}\tau\right)^{\frac{1}{2}} \overset{(1.7)}{\leq} c\left\|v^k\right\| \left\|\nabla \varphi\right\|_{0,\infty} \leq cM \left\|\nabla \varphi\right\|_{0,\infty}, \end{aligned} \tag{6.5}$$

where $(*)$ is proved similar to (5.6). This yields an estimate of $\left\|L_\varepsilon^k\varphi\right\|$ independent of k and ε, hence we obtain

$$\begin{aligned}
\varepsilon a_2 &\leq cM^2 \|\nabla\varphi\|_{0,\infty}\,\varepsilon(k_t - k_s) \\
&\overset{(6.3)}{<} cM^2 \|\nabla\varphi\|_{0,\infty}\,(t - s + \varepsilon) \\
&=: cM^2 \|\nabla\varphi\|_{0,\infty}\,|t - s| + \mathcal{O}(\varepsilon) \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

For εa_3 we use $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$, which implies

$$\begin{aligned}
\varepsilon a_3 &= \varepsilon \|\varphi\| \sum_{k=k_s+1}^{k_t} \left\|f^{k+1}\right\| = \frac{\varepsilon}{2}\|\varphi\| \sum_{k=k_s+1}^{k_t} 2\cdot 1\cdot \left\|f^{k+1}\right\| \\
&\leq \frac{\varepsilon}{2}\|\varphi\| \sum_{k=k_s+1}^{k_t}\left(1 + \left\|f^{k+1}\right\|^2\right) \\
&= \frac{\varepsilon}{2}\|\varphi\|(k_t - k_s) + \frac{\varepsilon}{2}\|\varphi\| \sum_{k=k_s+1}^{k_t}\left\|f^{k+1}\right\|^2 \\
&\overset{(*)}{\leq} \frac{1}{2}\|\varphi\|(t - s + \varepsilon) + \frac{1}{2}\|\varphi\| \sum_{k=k_s+1}^{k_t}\int_{t_k}^{t_{k+1}} \|f(\tau)\|^2 \,\mathrm{d}\tau,
\end{aligned}$$

where in $(*)$ we use (5.6) and (6.3). Hence, setting $f(\tau) = 0$ for $\tau > T$ and using $s \leq t_{k_s+1}$ and $t_{k_t+1} < t + \varepsilon$, we obtain

$$\begin{aligned}
\varepsilon a_3 &\leq \frac{1}{2}\|\varphi\|\left(t - s + \varepsilon + \int_s^{t+\varepsilon}\|f(\tau)\|^2\,\mathrm{d}\tau\right) \\
&\leq \frac{1}{2}\|\varphi\|\left(t - s + \varepsilon + \int_s^t \|f(\tau)\|^2\,\mathrm{d}\tau + \operatorname*{ess\,sup}_{\sigma\in[0,T]}\int_\sigma^{\sigma+\varepsilon}\|f(\tau)\|^2\,\mathrm{d}\tau\right) \\
&=: \frac{1}{2}\|\varphi\|\left(|t - s| + \int_s^t \|f(\tau)\|^2\,\mathrm{d}\tau\right) + \mathcal{O}(\varepsilon) + o(1) \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

For $k_s > k_t$ everything works analogously, thus the assertion is proved. □

The following theorem states an important result about the convergence behaviour of $\{v^{\varepsilon_N}\}_N$ as $N \to \infty$. Some parts of the proof are motivated by Shinbrot [19, pp. 169–173].

Theorem 6.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and let $0 < T < \infty$. For fixed $2 \leq N \in \mathbb{N}$ we set $\varepsilon := \varepsilon_N := \frac{T}{N} > 0$ and $t_k := k\varepsilon$ ($k = 0, \ldots, N$). In addition, let $f \in L^2(0, T; H^1(\Omega))$, let $v_0 \in C^1(\overline{\Omega})$ be divergence-free in Ω with $v_0 = 0$ on $\partial\Omega$, and let v^ε be the step function from Definition 5.3.*

Then, there exists a subsequence $\{v^{\varepsilon_{N_k}}\}_k$ of $\{v^{\varepsilon_N}\}_N$ and a function $v \in L^2\left(0, T; \mathcal{H}^1(\Omega)\right)$ with the following properties:

$$v^{\varepsilon_{N_k}}(t) \overset{k\to\infty}{\rightharpoonup} v(t) \qquad \text{in } \mathcal{H}^0(\Omega) \textit{ for all } t \in [0,T], \tag{6.6a}$$

$$v^{\varepsilon_{N_k}} \overset{k\to\infty}{\longrightarrow} v \qquad \text{in } L^2(0,T;\mathcal{H}^0(\Omega)), \tag{6.6b}$$

$$v^{\varepsilon_{N_k}} \overset{k\to\infty}{\rightharpoonup} v \qquad \text{in } L^2(0,T;\mathcal{H}^1(\Omega)). \tag{6.6c}$$

Proof: To prove (6.6a), let $t \in [0,T]$ be fixed. By Propositions 5.4 and 5.5 we have $v^\varepsilon(t) \in H^3(\Omega) \cap \mathcal{H}^1(\Omega)$ for all $t \in (0,T]$ with

$$\|v^\varepsilon(t)\|^2 \le \|v_0\|^2 + \frac{c_p^2}{\nu}\int_0^T \|f(\tau)\|^2 \, \mathrm{d}\tau$$

for all $\varepsilon = \varepsilon_N > 0$. Hence $\{v^{\varepsilon_N}(t)\}_N$ is bounded in $\mathcal{H}^0(\Omega)$. This implies the existence of a subsequence $\{v^{\varepsilon_{N_k}}(t)\}_k$ and of a function $v(t) \in \mathcal{H}^0(\Omega)$ satisfying

$$v^{\varepsilon_{N_k}}(t) \overset{k\to\infty}{\rightharpoonup} v(t) \qquad \text{in } \mathcal{H}^0(\Omega)$$

(compare Proposition 1.15). Of course, the subsequence $\{N_k\}_k$ of $\{N\}_N$ depends on t, i. e. we have $N_k = N_k(t)$.

Next we construct a subsequence $\{N_k\}_k$ of $\{N\}_N$ independent of t, i. e. satisfying $v^{\varepsilon_{N_k}}(t) \overset{k\to\infty}{\rightharpoonup} v(t)$ in $\mathcal{H}^0(\Omega)$ for all $t \in [0,T]$. This construction is carried out in two steps.

In a first step we consider $t \in [0,T] \cap \mathbb{Q} =: \{t_1, t_2, \dots\}$, i. e. t is contained in a countable set of rational numbers. In this case, the above procedure for $t = t_1$ yields a subsequence $\{N_k(t_1)\}_k$ of $\{N\}_N$ satisfying $v^{\varepsilon_{N_k(t_1)}}(t_1) \overset{k\to\infty}{\rightharpoonup} v(t_1)$ in $\mathcal{H}^0(\Omega)$.

Since $\{v^{\varepsilon_{N_k(t_1)}}(t_2)\}_k$ is bounded in $\mathcal{H}^0(\Omega)$, there exists a subsequence $\{N_k(t_2)\}_k$ of $\{N_k(t_1)\}_k$ satisfying $v^{\varepsilon_{N_k(t_2)}}(t_1) \overset{k\to\infty}{\rightharpoonup} v(t_1)$ in $\mathcal{H}^0(\Omega)$ as well as $v^{\varepsilon_{N_k(t_2)}}(t_2) \overset{k\to\infty}{\rightharpoonup} v(t_2)$ in $\mathcal{H}^0(\Omega)$.

Repeating this procedure for all rational points $t_1, t_2, t_3, \dots$, the usual diagonal argument leads to a subsequence $\{N_k\}_k$ of $\{N\}_N$ such that

$$v^{\varepsilon_{N_k}}(t) \overset{k\to\infty}{\rightharpoonup} v(t) \qquad \text{in } \mathcal{H}^0(\Omega)$$

for all $t \in [0,T] \cap \mathbb{Q}$, where $v(t) \in \mathcal{H}^0(\Omega)$.

For simplicity, throughout this proof we denote this subsequence $\{N_k\}_k$ again by $\{N\}_N$.

Now, in the second step, let $t \in [0,T]$ be non-rational and let $\tilde{t} \in [0,T] \cap \mathbb{Q}$. To prove the weak continuity of $v^{\varepsilon_N}(t)$ in $\mathcal{H}^0(\Omega)$, we use the decomposition

$$
\begin{aligned}
|\langle v^{\varepsilon_N}(t),\varphi\rangle - \langle v^{\varepsilon_M}(t),\varphi\rangle| \quad \le \quad & \underbrace{\left|\langle v^{\varepsilon_N}(t),\varphi\rangle - \left\langle v^{\varepsilon_N}(\tilde t),\varphi\right\rangle\right|}_{=:a_1} \\
& + \underbrace{\left|\left\langle v^{\varepsilon_N}(\tilde t),\varphi\right\rangle - \left\langle v^{\varepsilon_M}(\tilde t),\varphi\right\rangle\right|}_{=:a_2} \\
& + \underbrace{\left|\left\langle v^{\varepsilon_M}(\tilde t),\varphi\right\rangle - \langle v^{\varepsilon_M}(t),\varphi\rangle\right|}_{=:a_3}.
\end{aligned}
$$

Here it suffices to consider $\varphi \in C^\infty_{0,\sigma}(\Omega)$ since this set is dense in $\mathcal{H}^0(\Omega)$.

Using Lemma 6.1, for a_1 we find

$$
a_1 \le c\left(|t-\tilde t| + \left|\int_{\tilde t}^t \|f(\tau)\|^2 \ \mathrm{d}\tau\right|\right) + o(1) \qquad \text{as } N\to\infty,
$$

where the constant c does not depend on $t, \tilde t, N$, hence the term a_1 can be made as small as desired by choosing N large enough and $|t-\tilde t|$ small enough. Analogously we can make a_3 as small as desired by choosing M large enough and $|t-\tilde t|$ small enough.

The remaining term a_2 is getting small for sufficiently large N and M since $\tilde t \in \mathbb{Q}$.

It follows that $\{\langle v^{\varepsilon_N}(t),\varphi\rangle\}_N$ is a Chauchy sequence also for $t\in[0,T]\backslash\mathbb{Q}$, hence we obtain

$$
v^{\varepsilon_N}(t) \overset{N\to\infty}{\rightharpoonup} v(t) \qquad \text{in } \mathcal{H}^0(\Omega) \text{ for all } t\in[0,T]
$$

for some function $v : [0,T] \to \mathcal{H}^0(\Omega)$.

To prove (6.6c) we use the subsequence $\{\varepsilon_N\}_N$ from above, providing weak convergence $v^{\varepsilon_N}(t) \overset{N\to\infty}{\rightharpoonup} v(t)$ in $\mathcal{H}^0(\Omega)$ for all $t\in[0,T]$. By Proposition 5.5, setting $t=T$ in (5.5), we have

$$
\int_0^T \|\nabla v^{\varepsilon_N}(t)\|^2 \ \mathrm{d}t \overset{(5.5)}{\le} \frac{1}{\nu}\left(\|v_0\|^2 + \frac{c_p^2}{\nu}\int_0^T \|f(\tau)\|^2 \ \mathrm{d}\tau\right), \tag{6.7}
$$

where the right hand side does not depend on ε, hence the sequence $\{v^{\varepsilon_N}\}_N$ is bounded in $L^2(0,T;\mathcal{H}^1(\Omega))$. Thus there exists a subsequence – here denoted by $\{v^{\varepsilon_N}\}_N$ again – satisfying

$$
v^{\varepsilon_N} \overset{N\to\infty}{\rightharpoonup} v \quad \text{ in } L^2(0,T;\mathcal{H}^1(\Omega)).
$$

Of course, here $v\in L^2(0,T;\mathcal{H}^1(\Omega))$ is the same limit function as for the weak convergence pointwise in $\mathcal{H}^0(\Omega)$.

To prove (6.6b), again we start with the subsequence $\{\varepsilon_N\}_N$ constructed above and satisfying (6.6a) and (6.6c). Using Proposition 1.18 and Proposition 5.5, for the limit function $v\in L^2(0,T;\mathcal{H}^1(\Omega))$ we obtain

$$
\|v(t)\|^2 + \nu\int_0^t \|\nabla v(\tau)\|^2 \ \mathrm{d}\tau \overset{(1.13c)}{\le} \liminf_{N\to\infty} \|v^{\varepsilon_N}(t)\|^2 + \nu \liminf_{N\to\infty}\int_0^t \|\nabla v^{\varepsilon_N}(\tau)\|^2 \ \mathrm{d}\tau
$$

$$\begin{aligned}
&= \liminf_{N\to\infty} \left(\|v^{\varepsilon_N}(t)\|^2 + \nu \int_0^t \|\nabla v^{\varepsilon_N}(\tau)\|^2 \, \mathrm{d}\tau \right) \\
&\overset{(5.5)}{\leq} \liminf_{N\to\infty} \left(\|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^{t_{j(N)}} \|f(\tau)\|^2 \, \mathrm{d}\tau \right) \\
&= \|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^t \|f(\tau)\|^2 \, \mathrm{d}\tau, \qquad (6.8)
\end{aligned}$$

where, for all $N \in \mathbb{N}$, $t_{j(N)}$ denotes the smallest grid point with $t \leq t_{j(N)}$, hence $t_{j(N)} \overset{N\to\infty}{\longrightarrow} t$.

In addition, using the Poincaré inequality from Proposition 1.3 on (6.7), the weak convergence of $\{v^{\varepsilon_N}\}_N$ in $L^2(0,T;\mathcal{H}^1(\Omega))$ immediately implies the weak convergence in $L^2(0,T;\mathcal{H}^0(\Omega))$ for some subsequence, again denoted by $\{v^{\varepsilon_N}\}_N$.

To prove the strong convergence in $L^2(0,T;\mathcal{H}^0(\Omega))$, we need to show

$$\int_0^T \|v^{\varepsilon_N}(t) - v(t)\|^2 \, \mathrm{d}t \overset{N\to\infty}{\longrightarrow} 0.$$

Let $\{w_j\}_j$ with $w_j \in C^\infty_{0,\sigma}(\Omega)$ be the sequence of functions from Lemma 1.10. Since $v^{\varepsilon_N}(t) - v(t) \in \mathcal{H}^1(\Omega)$ for almost all $t \in [0,T]$, the lemma states that for all $\delta > 0$ we find an $M = M_\delta \in \mathbb{N}$ with

$$\|v^{\varepsilon_N}(t) - v(t)\|^2 \leq \delta \, \|\nabla(v^{\varepsilon_N}(t) - v(t))\|^2 + \sum_{j=1}^{M_\delta} |\langle v^{\varepsilon_N}(t) - v(t), \omega_j\rangle|^2$$

for almost all $t \in [0,T]$.

Integrating over $[0,T]$, we obtain

$$\begin{aligned}
&\int_0^T \|v^{\varepsilon_N}(t) - v(t)\|^2 \, \mathrm{d}t \\
&\qquad \leq \underbrace{\delta \int_0^T \|\nabla v^{\varepsilon_N}(t) - \nabla v(t)\|^2 \, \mathrm{d}t}_{=:b_1} + \underbrace{\sum_{j=1}^{M_\delta} \int_0^T |\langle v^{\varepsilon_N}(t) - v(t), \omega_j\rangle|^2 \, \mathrm{d}t}_{=:b_2}.
\end{aligned}$$

Using $2ab \leq a^2 + b^2$, we can estimate the term b_1 by

$$\begin{aligned}
b_1 &\leq \delta \int_0^T \|\nabla v^{\varepsilon_N}(t)\|^2 \, \mathrm{d}t + \delta \int_0^T \|\nabla v(t)\|^2 \, \mathrm{d}t + \delta \int_0^T 2\,\|\nabla v^{\varepsilon_N}(t)\| \, \|\nabla v(t)\| \, \mathrm{d}t \\
&\leq 2\delta \int_0^T \|\nabla v^{\varepsilon_N}(t)\|^2 \, \mathrm{d}t + 2\delta \int_0^T \|\nabla v(t)\|^2 \, \mathrm{d}t \\
&\overset{(*)}{\leq} \frac{4\delta}{\nu} \left(\|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^T \|f(\tau)\|^2 \, \mathrm{d}\tau \right),
\end{aligned}$$

where in $(*)$ we use (5.5) and (6.8). Here we can choose δ as small as desired.

Once δ is fixed, also $M = M_\delta \in \mathbb{N}$ is fixed, and, due to the weak convergence $v^{\varepsilon_N} \overset{N\to\infty}{\rightharpoonup} v$ in $L^2(0,T;\mathcal{H}^0(\Omega))$, we can make b_2 as small as desired by choosing N sufficiently large. This leads to

$$v^{\varepsilon_N} \overset{N\to\infty}{\longrightarrow} v \quad \text{in } L^2(0,T;\mathcal{H}^0(\Omega)),$$

and the proof is finished. □

Corollary 6.3 *Let the assumptions of Theorem 6.2 be satisfied. Then the limit function v of $\{v^{\varepsilon_N}\}_N$ from Theorem 6.2 satisfies*

$$\|v(t)\|^2 + \nu \int_0^t \|\nabla v(\tau)\|^2 \ \mathrm{d}\tau \le \|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^t \|f(\tau)\|^2 \ \mathrm{d}\tau \tag{6.9}$$

for all $t \in [0,T]$.

Proof: The above estimate was shown in (6.8) in the proof of Theorem 6.2. □

In the following Proposition we transfer the energy equality of v^ε to v and obtain the energy inequality for v.

Proposition 6.4 *Let the assumptions of Theorem 6.2 be satisfied. Then the limit function v of $\{v^{\varepsilon_N}\}_N$ from Theorem 6.2 satisfies*

$$v \in L^\infty\left(0,T;\mathcal{H}^0(\Omega)\right) \cap L^2\left(0,T;\mathcal{H}^1(\Omega)\right).$$

In addition, the energy inequality

$$\|v(t)\|^2 + 2\nu \int_0^t \|\nabla v(\tau)\|^2 \ \mathrm{d}\tau \le \|v_0\|^2 + 2\int_0^t \langle f(\tau), v(\tau)\rangle \ \mathrm{d}\tau \tag{6.10}$$

holds true for all $t \in [0,T]$.

Proof: From Theorem 6.2 we have $v \in L^2(0,T;\mathcal{H}^1(\Omega))$ and by

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\| &\overset{(6.9)}{\le} \operatorname*{ess\,sup}_{t\in[0,T]} \left(\|v_0\|^2 + \frac{c_p^2}{\nu}\int_0^T \|f(\tau)\|^2 \ \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&= \left(\|v_0\|^2 + \frac{c_p^2}{\nu}\int_0^T \|f(\tau)\|^2 \ \mathrm{d}\tau \right)^{\frac{1}{2}} < \infty
\end{aligned}$$

we find $v \in L^\infty(0,T;\mathcal{H}^0(\Omega)) \cap L^2(0,T;\mathcal{H}^1(\Omega))$.

To prove the energy inequality, for $N \in \mathbb{N}$, $t \in [0,T]$ let $t_{j(N)}$ denote the smallest grid point satisfying $t \leq t_{j(N)}$. Then – using $v^{\varepsilon_N}(t) = v^{\varepsilon_N}(t_{j(N)})$ and the energy equality of v^{ε_N} from Proposition 5.4 – we obtain

$$\begin{aligned}
\|v(t)\|^2 + 2\nu \int_0^t \|\nabla v(\tau)\|^2 \, \mathrm{d}\tau &\overset{(1.13c)}{\leq} \liminf_{N\to\infty} \left(\|v^{\varepsilon_N}(t)\|^2 + 2\nu \int_0^t \|\nabla v^{\varepsilon_n}(\tau)\|^2 \, \mathrm{d}\tau \right) \\
&\leq \liminf_{N\to\infty} \left(\left\|v^{\varepsilon_N}(t_{j(N)})\right\|^2 + 2\nu \int_0^{t_{j(N)}} \|\nabla v^{\varepsilon_N}(\tau)\|^2 \, \mathrm{d}\tau \right) \\
&\overset{(5.4)}{\leq} \liminf_{N\to\infty} \left(\|v_0\|^2 + 2 \int_0^{t_{j(N)}} \langle f(\tau), v^{\varepsilon_N}(\tau) \rangle \, \mathrm{d}\tau \right) \\
&= \liminf_{N\to\infty} \Bigg(\|v_0\|^2 + 2 \int_0^t \langle f(\tau), v^{\varepsilon_N}(\tau) - v(\tau) \rangle \, \mathrm{d}\tau \\
&\quad + 2 \int_0^t \langle f(\tau), v(\tau) \rangle \, \mathrm{d}\tau + 2 \int_t^{t_{j(N)}} \langle f(\tau), v^{\varepsilon_N}(\tau) \rangle \, \mathrm{d}\tau \Bigg) \\
&= \|v_0\|^2 + 2 \int_0^t \langle f(\tau), v(\tau) \rangle \, \mathrm{d}\tau \\
&\quad + 2 \underbrace{\liminf_{N\to\infty} \int_0^t \langle f(\tau), v^{\varepsilon_N}(\tau) - v(\tau) \rangle \, \mathrm{d}\tau}_{=:a_1} \\
&\quad + 2 \underbrace{\liminf_{N\to\infty} \int_t^{t_{j(N)}} \langle f(\tau), v^{\varepsilon_N}(\tau) \rangle \, \mathrm{d}\tau}_{=:a_2} .
\end{aligned}$$

Thus the assertion is proved, if only $a_1 = a_2 = 0$.

Using the convergence $v^{\varepsilon_N} \overset{N\to\infty}{\longrightarrow} v$ in $L^2(0,T;\mathcal{H}^0(\Omega))$ from Theorem 6.2, we obtain the estimate

$$\begin{aligned}
|a_1| &= \liminf_{N\to\infty} \left| \int_0^t \langle f(\tau), v^{\varepsilon_N}(\tau) - v(\tau) \rangle \, \mathrm{d}\tau \right| \\
&\overset{(1.5)}{\leq} \liminf_{N\to\infty} \int_0^t \|f(\tau)\| \, \|v^{\varepsilon_N}(\tau) - v(\tau)\| \, \mathrm{d}\tau \\
&\leq \liminf_{N\to\infty} \int_0^T \|f(\tau)\| \, \|v^{\varepsilon_N}(\tau) - v(\tau)\| \, \mathrm{d}\tau \\
&\overset{(1.4)}{\leq} \liminf_{N\to\infty} \left(\int_0^T \|f(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_0^T \|v^{\varepsilon_N}(\tau) - v(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\overset{(6.6b)}{=} 0.
\end{aligned}$$

Using Proposition 5.5, for a_2 we find

$$\left| \int_t^{t_{j(N)}} \langle f(\tau), v^{\varepsilon_N}(\tau) \rangle \, \mathrm{d}\tau \right| \overset{(1.5)}{\leq} \int_t^{t_{j(N)}} \|f(\tau)\| \, \|v^{\varepsilon_N}(\tau)\| \, \mathrm{d}\tau$$

$$\overset{(1.4)}{\leq} \left(\int_t^{t_{j(N)}} \|f(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_t^{t_{j(N)}} \|v^{\varepsilon_N}(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}}$$
$$\overset{(5.5)}{\leq} c\sqrt{t_{j(N)} - t} \left(\int_t^{t_{j(N)}} \|f(\tau)\| \, \mathrm{d}\tau \right)^{\frac{1}{2}},$$

where c is independent of t and ε. Hence, by $t_{j(N)} \overset{N\to\infty}{\longrightarrow} t$, we have

$$a_2 = \liminf_{N\to\infty} \int_t^{t_{j(N)}} \langle f(\tau), v^{\varepsilon_N}(\tau)\rangle \, \mathrm{d}\tau = 0$$

and the energy inequality is proved. □

6.2 Existence of a weak solution of (N_0)

In this section, we prove that the function v from Theorem 6.2 is a weak solution of the Navier-Stokes equations

$$\begin{aligned} \partial_t v - \nu\Delta v + \nabla p + v\cdot\nabla v &= f && \text{in } (0,T)\times\Omega, \\ \nabla\cdot v &= 0 && \text{in } (0,T)\times\Omega, \\ v_{|\partial\Omega} &= 0 && \text{in } [0,T), \\ v_{|t=0} &= v_0 && \text{in } \Omega. \end{aligned} \tag{N_0}$$

Let us first define the notion of a weak solution in the sense of Leray-Hopf (compare Hopf [11, p. 220] and Leray [13], [14])

Definition 6.5 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, let $f \in L^2(0,T;L^2(\Omega))$ and let $v_0 \in \mathcal{H}^0(\Omega)$. A function*

$$v \in L^\infty(0,T;\mathcal{H}^0(\Omega)) \cap L^2(0,T;\mathcal{H}^1(\Omega)) \tag{6.11a}$$

is called a weak solution of (N_0), if it satisfies

$$\int_0^T \Big(-\langle v,\dot\varphi\rangle + \nu\langle\nabla v,\nabla\varphi\rangle - \langle v\cdot\nabla\varphi, v\rangle \Big)\,\mathrm{d}t = \langle v_0,\varphi(0)\rangle + \int_0^T \langle f,\varphi\rangle \,\mathrm{d}t \tag{6.11b}$$

for all test functions $\varphi \in C_0^1([0,T);C_{0,\sigma}^\infty(\Omega))$, and if the energy inequality

$$\|v(t)\|^2 + 2\nu\int_0^t \|\nabla v(\tau)\|^2 \,\mathrm{d}\tau \leq \|v_0\|^2 + 2\int_0^t \langle f(\tau), v(\tau)\rangle \,\mathrm{d}\tau \tag{6.11c}$$

holds true for all $t \in [0,T]$.

Now our main result is stated as follows:

Theorem 6.6 *Every accumulation point v of the sequence $\{v^{\varepsilon_N}\}_N$ constructed as in Theorem 6.2 is a weak solution of the* NAVIER-STOKES *equations* (N_0), *i. e. it satisfies* (6.11a), (6.11b) *and* (6.11c).

Proof: Let the assumptions of Theorem 6.2 be satisfied and let $\{v^{\varepsilon_N}\}_N$ denote any subsequence satisfying

$$v^{\varepsilon_N}(t) \overset{N\to\infty}{\rightharpoonup} v(t) \qquad \text{in } \mathcal{H}^0(\Omega) \text{ for all } t \in [0,T], \tag{6.12a}$$

$$v^{\varepsilon_N} \overset{N\to\infty}{\longrightarrow} v \qquad \text{in } L^2(0,T;\mathcal{H}^0(\Omega)), \tag{6.12b}$$

$$v^{\varepsilon_N} \overset{N\to\infty}{\rightharpoonup} v \qquad \text{in } L^2(0,T;\mathcal{H}^1(\Omega)). \tag{6.12c}$$

By Proposition 6.4, we know that v satisfies

$$v \in L^\infty\left(0,T;\mathcal{H}^0(\Omega)\right) \cap L^2\left(0,T;\mathcal{H}^1(\Omega)\right)$$

and the energy inequality

$$\|v(t)\|^2 + 2\nu \int_0^t \|\nabla v(\tau)\|^2 \,\mathrm{d}\tau \le \|v_0\|^2 + 2\int_0^t \langle f(\tau), v(\tau)\rangle \,\mathrm{d}\tau$$

for all $t \in [0,T]$. Thus (6.11a) and (6.11c) are proved.

To prove (6.11b), we consider (5.7) from Lemma 5.7:

$$\begin{aligned}
&\underbrace{-\int_0^T \langle v^{\varepsilon_N}(t), \dot\varphi(t)\rangle \,\mathrm{d}t - \langle v_0, \varphi(0)\rangle}_{=:a_1^N} + \underbrace{\nu\int_0^T \langle \nabla v^{\varepsilon_N}(t), \nabla\varphi(t)\rangle \,\mathrm{d}t}_{=:a_2^N} \\
&\underbrace{-\int_0^T \langle L_{\varepsilon_N}(t)\varphi(t), v^{\varepsilon_N}(t)\rangle \,\mathrm{d}t}_{=:a_3^N} \\
= \;&\underbrace{\int_0^T \left\langle \tilde f(t), \varphi(t)\right\rangle \,\mathrm{d}t}_{=:a_4^N} + o(1) \quad \text{as } N\to\infty,
\end{aligned} \tag{5.7}$$

where for $t \in (t_k, t_{k+1}]$ we have

$$\tilde f(t) := f^{k+1} := \frac{1}{\varepsilon_N}\int_{t_k}^{t_{k+1}} f(\tau)\,\mathrm{d}\tau$$

and

$$\begin{aligned}
L_{\varepsilon_N}(t)\varphi(t,x) &:= L_{\varepsilon_N}^k \varphi(t,x) \\
&:= \frac{1}{2\varepsilon}\left\{\varphi(t)\circ X_k(\varepsilon_N,x) - \varphi(t)\circ X_k^{-1}(\varepsilon_N,x)\right\},
\end{aligned}$$

with mappings X_k, X_k^{-1} constructed from the solution $\tau \mapsto x(\tau) =: X_k(\tau, x_0)$ of the initial value problem (5.1):

$$\begin{aligned} \dot{x}(\tau) &= v^k(x(\tau)), \\ x(0) &= x_0. \end{aligned}$$

Now let $\varphi \in C_0^1([0,T); C_{0,\sigma}^\infty(\Omega))$. To prove that v is a weak solution of (N_0) we shall show that v satisfies

$$\underbrace{-\int_0^T \langle v(t), \dot{\varphi}(t)\rangle \,\mathrm{d}t - \langle v_0, \varphi(0)\rangle}_{=:b_1} + \underbrace{\nu \int_0^T \langle \nabla v(t), \nabla\varphi(t)\rangle \,\mathrm{d}t}_{=:b_2} - \underbrace{\int_0^T \langle v(t)\cdot\nabla\varphi(t), v(t)\rangle \,\mathrm{d}t}_{=:b_3}$$
$$= \underbrace{\int_0^T \langle f(t), \varphi(t)\rangle \,\mathrm{d}t}_{=:b_4}$$

by proving

$$a_i^N \overset{N\to\infty}{\longrightarrow} b_i, \qquad i = 1, \dots, 4.$$

Due to the weak convergence of $\{v^{\varepsilon_N}\}_N$ in $L^2(0,T;\mathcal{H}^0(\Omega))$ and in $L^2(0,T;\mathcal{H}^1(\Omega))$ we obtain

$$\begin{aligned} \lim_{N\to\infty} \int_0^T \langle v^{\varepsilon_N}(t), \dot{\varphi}(t)\rangle \,\mathrm{d}t &\overset{(6.12b)}{=} \int_0^T \langle v(t), \dot{\varphi}(t)\rangle \,\mathrm{d}t, \\ \lim_{N\to\infty} \int_0^T \langle \nabla v^{\varepsilon_N}(t), \nabla\varphi(t)\rangle \,\mathrm{d}t &\overset{(6.12c)}{=} \int_0^T \langle \nabla v(t), \nabla\varphi(t)\rangle \,\mathrm{d}t, \end{aligned}$$

i. e. $a_1^N \overset{N\to\infty}{\longrightarrow} b_1$ and $a_2^N \overset{N\to\infty}{\longrightarrow} b_2$ are proved.

To show $a_3^N \overset{N\to\infty}{\longrightarrow} b_3$ means to prove

$$\lim_{N\to\infty} \int_0^T \langle L_{\varepsilon_N}(t)\varphi(t), v^{\varepsilon_N}(t)\rangle \,\mathrm{d}t = \int_0^T \langle v(t)\cdot\nabla\varphi(t), v(t)\rangle \,\mathrm{d}t.$$

This is the most crucial point, and therefore it will be shown separately in the next Theorem 6.7.

The last statement

$$\lim_{N\to\infty} a_4^N = b_4$$

can be proved as follows: Let $M := M(N) := \frac{T}{\varepsilon_N}$ be the number of positive grid points in $[0,T]$ for the stepsize ε_N. Then we find

$$\int_0^T \left\langle \tilde{f}(t), \varphi(t)\right\rangle \,\mathrm{d}t \;=\; \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \left\langle f^{k+1}, \varphi(t)\right\rangle \,\mathrm{d}t$$

$$
\begin{aligned}
&= \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \left\langle \frac{1}{\varepsilon_N} \int_{t_k}^{t_{k+1}} f(\tau)\ \mathrm{d}\tau, \varphi(t) \right\rangle\ \mathrm{d}t \\
&= \frac{1}{\varepsilon_N} \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \langle f(\tau), \varphi(t) \rangle\ \mathrm{d}\tau\ \mathrm{d}t \\
&= \underbrace{\frac{1}{\varepsilon_N} \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \langle f(\tau), \varphi(t) - \varphi(\tau) \rangle\ \mathrm{d}\tau\ \mathrm{d}t}_{=:c_1^N} \\
&\quad + \underbrace{\frac{1}{\varepsilon_N} \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \langle f(\tau), \varphi(\tau) \rangle\ \mathrm{d}\tau\ \mathrm{d}t}_{=:c_2^N}.
\end{aligned}
$$

We can estimate c_1^N by

$$
\begin{aligned}
c_1^N &\overset{(1.5)}{\leq} \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \|f(\tau)\| \left\| \frac{\varphi(t) - \varphi(\tau)}{\varepsilon_N} \right\|\ \mathrm{d}\tau\ \mathrm{d}t \\
&\overset{(*)}{\leq} \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \|f(\tau)\|\ \|\dot{\varphi}(\sigma_k)\|\ \mathrm{d}\tau\ \mathrm{d}t \\
&\leq \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \|f(\tau)\| \max_{\sigma \in [0,T]} \|\dot{\varphi}(\sigma)\|\ \mathrm{d}\tau\ \mathrm{d}t \\
&=: c\varepsilon_N \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \|f(\tau)\|\ \mathrm{d}\tau \\
&= c\varepsilon_N \int_0^T \|f(\tau)\|\ \mathrm{d}\tau \\
&\overset{(1.4)}{\leq} c\varepsilon_N \sqrt{T} \left(\int_0^T \|f(\tau)\|^2\ \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\overset{N\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

where for the estimate $(*)$ we use a mean value theorem.

Concerning c_2^N we find

$$
\begin{aligned}
c_2^N &= \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \langle f(\tau), \varphi(\tau) \rangle\ \mathrm{d}\tau \\
&= \int_0^T \langle f(\tau), \varphi(\tau) \rangle\ \mathrm{d}\tau \\
&= b_4,
\end{aligned}
$$

hence

$$\lim_{N\to\infty} \int_0^T \left\langle \tilde{f}(t), \varphi(t) \right\rangle \, \mathrm{d}t = \int_0^T \langle f(t), \varphi(t) \rangle \, \mathrm{d}t,$$

i. e. $\lim\limits_{N\to\infty} a_4^N = b_4$ is proved. $\square$

We still need to show the statement $a_3^N \overset{N\to\infty}{\longrightarrow} b_3$. This is the most challenging part. It means that the weak limit procedure to the nonlinear term is possible if only the sequence of approximate solutions is strongly convergent in $L^2(0,T;\mathcal{H}^0(\Omega))$ and weakly convergent in $L^2(0,T;\mathcal{H}^1(\Omega))$. To precisely formulate this statement, let us recall: For $t \in (t_k, t_{k+1}]$, $k = 0, \dots, M-1$, where $M := \frac{T}{\varepsilon_N}$ denotes the number of positive grid points, we have

$$\begin{aligned} L_{\varepsilon_N}(t)\varphi(t,x) &:= L^k_{\varepsilon_N}\varphi(t,x) \\ &:= \frac{1}{2\varepsilon_N} \left\{ \varphi(t) \circ X_k(\varepsilon_N, x) - \varphi(t) \circ X_k^{-1}(\varepsilon_N, x) \right\}. \end{aligned} \tag{6.13}$$

Here the mappings X_k, X_k^{-1} are constructed from the solution $\tau \mapsto x(\tau) =: X_k(\tau, x_0)$ of the initial value problem (5.1):

$$\begin{aligned} \dot{x}(\tau) &= v^k(x(\tau)), \\ x(0) &= x_0. \end{aligned}$$

Some parts of the following proof correspond to the investigation of Asanalieva, Heutling & Varnhorn [5, pp. 349–353].

Theorem 6.7 *For every accumulation point v of a sequence $\{v^{\varepsilon_N}\}_N$ constructed as in Theorem 6.2 it holds*

$$\lim_{N\to\infty} \int_0^T \langle L_{\varepsilon_N}(t)\varphi(t), v^{\varepsilon_N}(t) \rangle \, \mathrm{d}t = \int_0^T \langle v(t) \cdot \nabla\varphi(t), v(t) \rangle \, \mathrm{d}t \tag{6.14}$$

for all $\varphi \in C_0^1([0,T); C_{0,\sigma}^\infty(\Omega))$.

Proof: Let the assumptions of Theorem 6.2 be satisfied and let $\{v^{\varepsilon_N}\}_N$ denote any subsequence satisfying (6.12a)–(6.12c). To prove (6.14), we show

$$\lim_{N\to\infty} \int_0^T \underbrace{\Big(\langle L_{\varepsilon_N}(t)\varphi(t), v^{\varepsilon_N}(t) \rangle - \langle v(t) \cdot \nabla\varphi(t), v(t) \rangle \Big)}_{=:I^N(t)} \, \mathrm{d}t = 0.$$

For $t \in (t_k, t_{k+1}]$, $k = 0, \dots, N-1$, setting $X_k := X_k(\varepsilon_N)$, the Integrand $I^N(t)$ is decomposed as follows:

$$
\begin{aligned}
I^N(t) &= \left\langle L^k_{\varepsilon_N}\varphi(t), v^{\varepsilon_N}(t)\right\rangle - \langle v(t)\cdot\nabla\varphi(t), v(t)\rangle \\
&= \left\langle L^k_{\varepsilon_N}\varphi(t), v^{\varepsilon_N}(t)\right\rangle \underbrace{- \left\langle L^k_{\varepsilon_N}\varphi(t), v(t)\right\rangle + \left\langle L^k_{\varepsilon_N}\varphi(t), v(t)\right\rangle}_{=0} - \langle v(t)\cdot\nabla\varphi(t), v(t)\rangle \\
&= \left\langle L^k_{\varepsilon_N}\varphi(t), v^{\varepsilon_N}(t) - v(t)\right\rangle + \left\langle L^k_{\varepsilon_N}\varphi(t) - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle \\
&= \left\langle L^k_{\varepsilon_N}\varphi(t), v^{\varepsilon_N}(t) - v(t)\right\rangle \\
&\quad + \left\langle \frac{1}{2\varepsilon_N}\left\{\varphi(t)\circ X_k - \varphi(t)\circ X_k^{-1}\right\} - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle \\
&= \left\langle L^k_{\varepsilon_N}\varphi(t), v^{\varepsilon_N}(t) - v(t)\right\rangle \\
&\quad + \left\langle \frac{1}{2\varepsilon_N}\Big\{\varphi(t)\circ X_k \underbrace{-\varphi(t) + \varphi(t)}_{=0} - \varphi(t)\circ X_k^{-1}\Big\} - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle \\
&= \underbrace{\left\langle L^k_{\varepsilon_N}\varphi(t), v^{\varepsilon_N}(t) - v(t)\right\rangle}_{=:a_1^N(t)} \\
&\quad + \underbrace{\frac{1}{2}\left\langle \frac{1}{\varepsilon_N}\Big(\varphi(t)\circ X_k - \varphi(t)\Big) - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle}_{=:a_2^N(t)} \\
&\quad + \underbrace{\frac{1}{2}\left\langle \frac{1}{\varepsilon_N}\Big(\varphi(t) - \varphi(t)\circ X_k^{-1}\Big) - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle}_{=:a_3^N(t)}.
\end{aligned}
$$

To estimate the integral over $a_1^N(t)$ we use the strong convergence $v^{\varepsilon_N} \overset{N\to\infty}{\longrightarrow} v$ in $L^2(0,T;\mathcal{H}^0(\Omega))$, see (6.12b), and the boundedness of $\left\|L^k_{\varepsilon_N}(t)\varphi(t)\right\|$, which can be derived from (6.5) with $\varphi \in C_0^1([0,T);C^\infty_{0,\sigma}(\Omega))$, which implies

$$
\|\nabla\varphi(t)\|_{0,\infty} \le \max_{\tau\in[0,T]} \|\nabla\varphi(\tau)\|_{0,\infty} \le c \tag{6.15}
$$

for some constant c independent of t. We obtain

$$
\begin{aligned}
\left|\int_0^T a_1^N(t)\ \mathrm{d}t\right| &= \left|\int_0^T \langle L_{\varepsilon_N}(t)\varphi(t), v^{\varepsilon_N}(t) - v(t)\rangle\ \mathrm{d}t\right| \\
&\overset{(1.5)}{\le} \int_0^T \|L_{\varepsilon_N}(t)\varphi(t)\|\ \|v^{\varepsilon_N}(t) - v(t)\|\ \mathrm{d}t \\
&\overset{(1.4)}{\le} \Big(\int_0^T \|L_{\varepsilon_N}(t)\varphi(t)\|^2\ \mathrm{d}t\Big)^{\frac{1}{2}} \Big(\int_0^T \|v^{\varepsilon_N}(t) - v(t)\|^2\ \mathrm{d}t\Big)^{\frac{1}{2}}
\end{aligned}
$$

$$\overset{(6.5)}{\leq} \; c\Big(\int_0^T \|v^{\varepsilon_N}(t) - v(t)\|^2 \;\mathrm{d}t\Big)^{\frac{1}{2}}$$
$$\overset{N\to\infty}{\longrightarrow} 0.$$

The integrals over $a_2^N(t)$ and $a_3^N(t)$ can be treated in exactly the same way. Therefore we restrict our investigations to the integral over $a_2^N(t)$, only.

We use the construction of the mappings X_k to replace the term $\varphi(t) \circ X_k(\varepsilon_N) - \varphi(t)$ (compare (6.4) in the proof of Lemma 6.1) and obtain

$$\begin{aligned}
a_2^N(t) &= \frac{1}{2}\left\langle \frac{1}{\varepsilon_N}\Big(\varphi(t)\circ X_k(\varepsilon_N) - \varphi(t)\Big) - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle \\
&\overset{(6.4)}{=} \frac{1}{2}\left\langle \frac{1}{\varepsilon_N}\int_0^{\varepsilon_N} [v^{\varepsilon_N}(t_k)\cdot\nabla_X\varphi(t)]\circ X_k(\tau)\;\mathrm{d}\tau - v(t)\cdot\nabla\varphi(t), v(t)\right\rangle \\
&= \frac{1}{2\varepsilon_N}\int_0^{\varepsilon_N} \langle [v^{\varepsilon_N}(t_k)\cdot\nabla_X\varphi(t)]\circ X_k(\tau) - v(t)\cdot\nabla\varphi(t), v(t)\rangle\;\mathrm{d}\tau \\
&\overset{(2.8)}{=} \frac{1}{2\varepsilon_N}\int_0^{\varepsilon_N} \Big(\big\langle v^{\varepsilon_N}(t_k)\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle \\
&\qquad \underbrace{-\big\langle v^{\varepsilon_N}(t)\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle + \big\langle v^{\varepsilon_N}(t)\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle}_{=0} \\
&\qquad \underbrace{-\big\langle v(t)\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle + \big\langle v(t)\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle}_{=0} \\
&\qquad - \langle v(t)\cdot\nabla\varphi(t), v(t)\rangle \Big)\;\mathrm{d}\tau \\
&= \underbrace{\frac{1}{2\varepsilon_N}\int_0^{\varepsilon_N} \big\langle (v^{\varepsilon_N}(t_k) - v^{\varepsilon_N}(t))\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle\;\mathrm{d}\tau}_{=:b_1^N(t)} \\
&\qquad + \underbrace{\frac{1}{2\varepsilon_N}\int_0^{\varepsilon_N} \big\langle (v^{\varepsilon_N}(t) - v(t))\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle\;\mathrm{d}\tau}_{=:b_2^N(t)} \\
&\qquad \underbrace{-\frac{1}{2\varepsilon_N}\int_0^{\varepsilon_N} \big\langle v(t)\cdot\nabla\varphi(t), v(t) - v(t)\circ X_k^{-1}(\tau)\big\rangle\;\mathrm{d}\tau\,.}_{=:b_3^N(t)}
\end{aligned}$$

To estimate $b_1^N(t)$ we use (6.15) on $\|\nabla\varphi(t)\|_{0,\infty}$ and the energy inequality from Proposition 6.4 on $\|v(t)\|$, hence

$$\left|b_1^N(t)\right| \quad = \quad \frac{1}{2\varepsilon_N}\left|\int_0^{\varepsilon_N} \big\langle (v^{\varepsilon_N}(t_k) - v^{\varepsilon_N}(t))\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\big\rangle\;\mathrm{d}\tau\right|$$

$$
\begin{aligned}
&\overset{(1.7)}{\leq} && \frac{c}{2\varepsilon_N}\int_0^{\varepsilon_N} \left\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\right\| \left\|\nabla\varphi(t)\right\|_{0,\infty} \left\|v(t)\circ X_k^{-1}(\tau)\right\| \,\mathrm{d}\tau \\
&\overset{(2.7)}{=} && \frac{c}{2\varepsilon_N}\int_0^{\varepsilon_N} \left\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\right\| \left\|\nabla\varphi(t)\right\|_{0,\infty} \left\|v(t)\right\| \,\mathrm{d}\tau \\
&= && \frac{c}{2}\left\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\right\| \left\|\nabla\varphi(t)\right\|_{0,\infty} \left\|v(t)\right\| \\
&\overset{(6.9),(6.15)}{\leq} && c_1\left\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\right\|.
\end{aligned}
$$

To estimate $\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\|$ we use

$$
\begin{aligned}
\|u-w\|^2 &= \langle u-w, u+w-2w\rangle \\
&= \langle u-w, u+w\rangle - 2\langle u-w, w\rangle \\
&= \|u\|^2 - \|w\|^2 - 2\langle u-w, w\rangle,
\end{aligned}
$$

which implies the decomposition

$$
\begin{aligned}
\left\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\right\|^2 &= \left\|v^{\varepsilon_N}(t_k)\right\|^2 - \left\|v^{\varepsilon_N}(t)\right\|^2 \\
&\quad - 2\left\langle v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t), v^{\varepsilon_N}(t)\underbrace{-v(t)+v(t)}_{=0}\right\rangle \\
&= \underbrace{\left\|v^{\varepsilon_N}(t_k)\right\|^2 - \left\|v^{\varepsilon_N}(t)\right\|^2}_{=:c_1^N(t)} \\
&\quad \underbrace{-2\left\langle v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t), v^{\varepsilon_N}(t)-v(t)\right\rangle}_{=:c_2^N(t)} \underbrace{-2\left\langle v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t), v(t)\right\rangle}_{=:c_3^N(t)}.
\end{aligned}
$$

Concerning $c_1^N(t)$ we can use Corollary 5.2, since $v^{\varepsilon_N}(t)=v^{\varepsilon_N}(t_{k+1})$ for $t\in(t_k,t_{k+1}]$. Setting $f(\tau)=0$ for $\tau>T$ we find

$$
\begin{aligned}
|c_1^N(t)| &= \left|\left\|v^{\varepsilon_N}(t_{k+1})\right\|^2 - \left\|v^{\varepsilon_N}(t_k)\right\|^2\right| \\
&\overset{(5.3a)}{\leq} \frac{\varepsilon_N c_p^2}{\nu}\left\|f^{k+1}\right\|^2 \\
&\overset{(5.6)}{\leq} \frac{c_p^2}{\nu}\int_{t_k}^{t_{k+1}} \|f(\tau)\|^2 \,\mathrm{d}\tau \\
&\leq \frac{c_p^2}{\nu}\operatorname*{ess\,sup}_{\sigma\in[0,T]}\int_\sigma^{\sigma+\varepsilon_N} \|f(\tau)\|^2 \,\mathrm{d}\tau,
\end{aligned}
$$

hence

$$\begin{aligned}
\lim_{N\to\infty}\left|\int_0^T c_1^N(t)\,\mathrm{d}t\right| &\le \frac{c_p^2}{\nu}\lim_{N\to\infty}\int_0^T \operatorname*{ess\,sup}_{\sigma\in[0,T]}\int_\sigma^{\sigma+\varepsilon_N}\|f(\tau)\|^2\,\mathrm{d}\tau\,\mathrm{d}t\\
&= \frac{c_p^2 T}{\nu}\lim_{N\to\infty}\operatorname*{ess\,sup}_{\sigma\in[0,T]}\int_\sigma^{\sigma+\varepsilon_N}\|f(\tau)\|^2\,\mathrm{d}\tau\\
&= 0.
\end{aligned}$$

To estimate $c_2^N(t)$ we use Proposition 5.5 and find

$$\begin{aligned}
|c_2^N(t)| &\overset{(1.5)}{\le} 2\left(\|v^{\varepsilon_N}(t_k)\| + \|v^{\varepsilon_N}(t)\|\right)\|v^{\varepsilon_N}(t)-v(t)\|\\
&\overset{(5.5)}{\le} c\,\|v^{\varepsilon_N}(t)-v(t)\|\,.
\end{aligned}$$

Due to the strong convergence (6.12b) of v^{ε_N} in $L^2(0,T;\mathcal{H}^0(\Omega))$ we have

$$\begin{aligned}
\lim_{N\to\infty}\left|\int_0^T c_2^N(t)\,\mathrm{d}t\right| &\le c\lim_{N\to\infty}\int_0^T 1\cdot\|v^{\varepsilon_N}(t)-v(t)\|\,\mathrm{d}t\\
&\overset{(1.4)}{\le} c_1\lim_{N\to\infty}\left(\int_0^T\|v^{\varepsilon_N}(t)-v(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\overset{(6.12b)}{=} 0.
\end{aligned}$$

To estimate $c_3^N(t)$ we approximate the function $v\in L^2(0,T;\mathcal{H}^0(\Omega))$ by a sequence $\{v^n\}_n$ of smooth functions

$$v^n(t,x) := \sum_{i=1}^n c_{in}(t)a_i(x) \qquad , n\in\mathbb{N}.$$

Here $c_{in}\in C_0^\infty((0,T))$ and $\{a_i\}_i$ with $a_i\in C_{0,\sigma}^\infty(\Omega)$ denotes a complete orthonormal system in $\mathcal{H}^0(\Omega)$.

Now we decompose

$$\begin{aligned}
c_3^N(t) &= 2\left\langle v^{\varepsilon_N}(t)-v^{\varepsilon_N}(t_k), v(t)\underbrace{-v^n(t)+v^n(t)}_{=0}\right\rangle\\
&= \underbrace{2\left\langle v^{\varepsilon_N}(t)-v^{\varepsilon_N}(t_k), v(t)-v^n(t)\right\rangle}_{=:d_1^N(t)} + \underbrace{2\left\langle v^{\varepsilon_N}(t)-v^{\varepsilon_N}(t_k), v^n(t)\right\rangle}_{=:d_2^N(t)}.
\end{aligned}$$

The term $d_1^N(t)$ can be estimated analogously to $c_2^N(t)$, hence we find

$$\lim_{N\to\infty}\left|\int_0^T d_1^N(t)\,\mathrm{d}t\right| \overset{(1.4)}{\leq} c_1\left(\int_0^T \|v(t)-v^n(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}},$$

and the integral on the right hand side can be made as small as desired by choosing $n\in\mathbb{N}$ large enough.

Since $v^n(t)\in C_{0,\sigma}^\infty(\Omega)$ we can use Lemma 6.1 for $d_2^N(t)$. Setting $f(\tau)=0$ for $\tau>T$ we find

$$\begin{aligned}
|d_2^N(t)| &= 2\left\langle v^{\varepsilon_N}(t)-v^{\varepsilon_N}(t_k), v^n(t)\right\rangle \\
&\overset{(6.1)}{\leq} c\left((t-t_k)+\int_{t_k}^t \|f(\tau)\|^2\,\mathrm{d}\tau\right)+o(1) \quad \text{as } N\to\infty \\
&\leq c\left(\varepsilon_N+\operatorname*{ess\,sup}_{\sigma\in[0,T]}\int_\sigma^{\sigma+\varepsilon_N}\|f(\tau)\|^2\,\mathrm{d}\tau\right)+o(1) \quad \text{as } N\to\infty \\
&= c\varepsilon_N+o(1) \quad \text{as } N\to\infty,
\end{aligned}$$

where $o(1)$ does not depend on t, hence

$$\lim_{N\to\infty}\left|\int_0^T d_2^N(t)\,\mathrm{d}t\right| \leq cT\lim_{N\to\infty}(\varepsilon_N+o(1))=0.$$

Collecting the above estimates we find

$$\begin{aligned}
\lim_{N\to\infty}\left|\int_0^T b_1^N(t)\,\mathrm{d}t\right| &\leq \lim_{N\to\infty}\int_0^T c_1\|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\|\,\mathrm{d}t \\
&\overset{(1.4)}{\leq} \lim_{N\to\infty} c_1\sqrt{T}\left(\int_0^T \|v^{\varepsilon_N}(t_k)-v^{\varepsilon_N}(t)\|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= \lim_{N\to\infty} c_1\sqrt{T}\left(\int_0^T c_1^N(t)+c_2^N(t)+c_3^N(t)\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= 0.
\end{aligned}$$

For $b_2^N(t)$ we use the conservation of measure to eliminate $X_k^{-1}(\tau)$, which implies

$$\begin{aligned}
|b_2^N(t)| &= \frac{1}{2\varepsilon_N}\int_0^{\varepsilon_N}\left\langle (v^{\varepsilon_N}(t)-v(t))\cdot\nabla\varphi(t), v(t)\circ X_k^{-1}(\tau)\right\rangle\,\mathrm{d}\tau \\
&\overset{(1.7)}{\leq} \frac{c}{2\varepsilon_N}\int_0^{\varepsilon_N}\|v^{\varepsilon_N}(t)-v(t)\|\,\|\nabla\varphi(t)\|_{0,\infty}\left\|v(t)\circ X_k^{-1}(\tau)\right\|\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.7)}{=} \frac{c}{2\varepsilon_N}\int_0^{\varepsilon_N} \|v^{\varepsilon_N}(t)-v(t)\| \; \|\nabla\varphi(t)\|_{0,\infty} \; \|v(t)\| \; \mathrm{d}\tau \\
&= \frac{c}{2}\|\nabla\varphi(t)\|_{0,\infty} \; \|v(t)\| \; \|v^{\varepsilon_N}(t)-v(t)\|.
\end{aligned}
$$

Now, as before, using the boundedness of $\|\nabla\varphi(t)\|_{0,\infty}$ from (6.15) together with Proposition 6.4, the convergence property (6.12b) implies

$$
\begin{aligned}
&\lim_{N\to\infty}\left|\int_0^T b_2^N(t)\;\mathrm{d}t\right| \\
&\leq \frac{c}{2}\lim_{N\to\infty}\int_0^T \Big(\|\nabla\varphi(t)\|_{0,\infty}\;\|v(t)\|\Big)\,\|v^{\varepsilon_N}(t)-v(t)\| \;\mathrm{d}t \\
&\overset{(1.4)}{\leq} \frac{c}{2}\lim_{N\to\infty}\Big(\int_0^T \|\nabla\varphi(t)\|_{0,\infty}^2\;\|v(t)\|^2\;\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^T \|v^{\varepsilon_N}(t)-v(t)\|^2\;\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\overset{(6.9),(6.15)}{\leq} c_1\lim_{N\to\infty}\Big(\int_0^T \|v^{\varepsilon_N}(t)-v(t)\|^2\;\mathrm{d}t\Big)^{\frac{1}{2}} \\
&\overset{(6.12b)}{=} 0.
\end{aligned}
$$

Finally, we estimate $b_3^N(t)$. Similar to (6.4) in the proof of Lemma 6.1 we find

$$
\begin{aligned}
v(t)-v(t)\circ X_k^{-1}(\tau) &= v(t)\circ X_k(0)-v(t)\circ X_k(-\tau) \\
&= \int_{-\tau}^0 [v^{\varepsilon_N}(t_k)\cdot\nabla_X v(t)]\circ X_k(\sigma)\;\mathrm{d}\sigma \qquad (6.16)
\end{aligned}
$$

for each $\tau\in[0,\varepsilon_N]$. Thus, for $b_3^N(t)$ we obtain

$$
\begin{aligned}
|b_3^N(t)| &= \frac{1}{2\varepsilon_N}\Big|\int_0^{\varepsilon_N}\big\langle v(t)\cdot\nabla\varphi(t), v(t)-v(t)\circ X_k^{-1}(\tau)\big\rangle\;\mathrm{d}\tau\Big| \\
&\overset{(6.16)}{=} \frac{1}{2\varepsilon_N}\Big|\int_0^{\varepsilon_N}\Big\langle v(t)\cdot\nabla\varphi(t), \int_{-\tau}^0 [v^{\varepsilon_N}(t_k)\cdot\nabla_X v(t)]\circ X_k(\sigma)\;\mathrm{d}\sigma\Big\rangle\;\mathrm{d}\tau\Big| \\
&= \frac{1}{2\varepsilon_N}\Big|\int_0^{\varepsilon_N}\int_{-\tau}^0 \langle v(t)\cdot\nabla\varphi(t), [v^{\varepsilon_N}(t_k)\cdot\nabla_X v(t)]\circ X_k(\sigma)\rangle\;\mathrm{d}\sigma\;\mathrm{d}\tau\Big| \\
&\overset{(1.7)}{\leq} \frac{c}{\varepsilon_N}\int_0^{\varepsilon_N}\int_{-\tau}^0 \|v(t)\|_{0,6}\;\|\nabla\varphi(t)\|_{0,\infty}\;\|v^{\varepsilon_N}(t_k)\circ X_k(\sigma)\|_{0,3}\;\|\nabla_X v(t)\circ X_k(\sigma)\|\;\mathrm{d}\sigma\;\mathrm{d}\tau \\
&\overset{(2.7)}{=} \frac{c}{\varepsilon_N}\int_0^{\varepsilon_N}\int_{-\tau}^0 \|v(t)\|_{0,6}\;\|\nabla\varphi(t)\|_{0,\infty}\;\|v^{\varepsilon_N}(t_k)\|_{0,3}\;\|\nabla v(t)\|\;\mathrm{d}\sigma\;\mathrm{d}\tau \\
&= c_1\varepsilon_N\|v(t)\|_{0,6}\;\|\nabla\varphi(t)\|_{0,\infty}\;\|v^{\varepsilon_N}(t_k)\|_{0,3}\;\|\nabla v(t)\| \\
&\overset{(6.15)}{\leq} c_2\varepsilon_N\|v(t)\|_{0,6}\;\|v^{\varepsilon_N}(t_k)\|_{0,3}\;\|\nabla v(t)\|.
\end{aligned}
$$

Using the SOBOLEV Imbedding Theorem 1.2 for $m = 1$, $p = 2$, we find

$$W^{1,2}(\Omega) \to L^q(\Omega) \quad \text{for } 1 \leq q \leq 6, \tag{6.17}$$

and, together with the POINCARÉ inequality 1.3, this implies

$$\begin{aligned} \|u\|_{0,p} &\overset{(6.17)}{\leq} c\,\|u\|_{1,2} \\ &\overset{(1.1)}{=} c\left(\|u\|^2 + \|\nabla u\|^2\right)^{\frac{1}{2}} \\ &\leq c\,(\|u\| + \|\nabla u\|) \\ &\overset{(1.2)}{\leq} c_1\,\|\nabla u\| \end{aligned} \tag{6.18}$$

for $u = v(t)$ and for $u = v^{\varepsilon_N}(t_k)$ and we obtain

$$|b_3^N(t)| \overset{(6.18)}{\leq} c\varepsilon_N\,\|\nabla v^{\varepsilon_N}(t_k)\|\,\|\nabla v(t)\|^2 .$$

For $k = 0$ we have

$$\|\nabla v^{\varepsilon_N}(0)\|^2 = \|\nabla v_0\|^2 \leq c,$$

since $v_0 \in C^1(\overline{\Omega})$. For $k = 1, \ldots, M-1$ with $M := \frac{T}{\varepsilon_N}$ we can use Proposition 5.5 and obtain

$$\begin{aligned} \varepsilon_N\,\|\nabla v^{\varepsilon_N}(t_k)\|^2 &= \int_{t_{k-1}}^{t_k} \|\nabla v^{\varepsilon_N}(\tau)\|^2 \,\mathrm{d}\tau \\ &\leq \int_0^{t_k} \|\nabla v^{\varepsilon_N}(\tau)\|^2 \,\mathrm{d}\tau \\ &\overset{(5.5)}{\leq} c, \end{aligned}$$

hence for all grid points it holds

$$\|\nabla v^{\varepsilon_N}(t_k)\| \leq c_1 \varepsilon_N^{-\frac{1}{2}}.$$

Thus it follows

$$\left|b_3^N(t)\right| \leq c\varepsilon_N^{\frac{1}{2}}\,\|\nabla v(t)\|^2 ,$$

and, integrating over t, we find

$$\begin{aligned} \lim_{N\to\infty} \left|\int_0^T b_3^N(t)\,\mathrm{d}t\right| &\leq \lim_{N\to\infty} c\varepsilon_N^{\frac{1}{2}} \int_0^T \|\nabla v(t)\|^2 \,\mathrm{d}t \\ &\overset{(6.9)}{\leq} \lim_{N\to\infty} c_1 \varepsilon_N^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

Thus we have

$$\lim_{N\to\infty}\left|\int_0^T a_2^N(t)\ \mathrm{d}t\right| = 0,$$

which proves the theorem. □

Finally, we prove some important properties of the weak solution v.

Proposition 6.8 *Let v be a weak solution of (N_0) constructed as in Theorem 6.2. Then v satisfies*

$$\begin{aligned}
&v : [0,T] \to \mathcal{H}^0(\Omega) \textit{ is weakly continuous,}\\
&\lim_{0<t\to 0} \|v(t) - v_0\| = 0.
\end{aligned}$$

Proof: Let the assumptions of Theorem 6.2 be satisfied, and let $\{v^{\varepsilon_N}\}_N$ denote any subsequence satisfying (6.12a)–(6.12c).

Let $\varphi \in C^\infty_{0,\sigma}(\Omega)$. Since, for $t \in (t_k, t_{k+1}]$, the function $v^{\varepsilon_N}(t) = v^{k+1}$ is a weak solution of $(N^k_{\varepsilon_N})$, we have

$$\begin{aligned}
\left\langle v^{k+1},\varphi\right\rangle - \left\langle v^k,\varphi\right\rangle &= \varepsilon_N\nu\left\langle v^{k+1},\Delta\varphi\right\rangle - \varepsilon_N\left\langle L^k_{\varepsilon_N}v^{k+1},\varphi\right\rangle + \varepsilon_N\left\langle f^{k+1},\varphi\right\rangle\\
&\overset{(3.1a)}{=} \varepsilon_N\nu\left\langle v^{k+1},\Delta\varphi\right\rangle + \varepsilon_N\left\langle v^{k+1},L^k_{\varepsilon_N}\varphi\right\rangle + \varepsilon_N\left\langle f^{k+1},\varphi\right\rangle.
\end{aligned}$$

Setting $j = k$ and summing up for $j = 0,\dots,k$, we obtain

$$\begin{aligned}
&\sum_{j=0}^{k}\left(\left\langle v^{j+1},\varphi\right\rangle - \left\langle v^j,\varphi\right\rangle\right)\\
&\qquad = \varepsilon_N\nu\sum_{j=0}^{k}\left\langle v^{j+1},\Delta\varphi\right\rangle + \varepsilon_N\sum_{j=0}^{k}\left\langle v^{j+1},L^j_{\varepsilon_N}\varphi\right\rangle + \varepsilon_N\sum_{j=0}^{k}\left\langle f^{j+1},\varphi\right\rangle,
\end{aligned}$$

and this implies for $t \in (t_k, t_{k+1}]$ the identity

$$\begin{aligned}
&\langle v^{\varepsilon_N}(t),\varphi\rangle - \langle v_0,\varphi\rangle\\
&= \nu\int_0^{t_{k+1}}\langle v^{\varepsilon_N}(\tau),\Delta\varphi\rangle\ \mathrm{d}\tau + \int_0^{t_{k+1}}\langle v^{\varepsilon_N}(\tau),L_{\varepsilon_N}(\tau)\varphi\rangle\ \mathrm{d}\tau + \int_0^{t_{k+1}}\left\langle \tilde f(\tau),\varphi\right\rangle\ \mathrm{d}\tau\\
&= \nu\int_0^{t}\langle v^{\varepsilon_N}(\tau),\Delta\varphi\rangle\ \mathrm{d}\tau + \int_0^{t}\langle v^{\varepsilon_N}(\tau),L_{\varepsilon_N}(\tau)\varphi\rangle\ \mathrm{d}\tau + \int_0^{t}\left\langle \tilde f(\tau),\varphi\right\rangle\ \mathrm{d}\tau\\
&\quad \underbrace{+\,\nu\int_t^{t_{k+1}}\langle v^{\varepsilon_N}(\tau),\Delta\varphi\rangle\ \mathrm{d}\tau}_{=:a_1^N(t)} + \underbrace{\int_t^{t_{k+1}}\left\langle v^{\varepsilon_N}(\tau),L^k_{\varepsilon_N}\varphi\right\rangle\ \mathrm{d}\tau}_{=:a_2^N(t)} + \underbrace{\int_t^{t_{k+1}}\left\langle f^{k+1},\varphi\right\rangle\ \mathrm{d}\tau}_{=:a_3^N(t)}.
\end{aligned}\tag{6.19}$$

Here for $\tau \in (t_j, t_{j+1}]$, $j = 0, \dots, k$, we set $L_{\varepsilon_N}(\tau) := L^j_{\varepsilon_N}$ (compare (6.13) on page 65) and $\tilde{f}(\tau) := f^{j+1} := \frac{1}{\varepsilon_N} \int_{t_j}^{t_{j+1}} f(t)\, \mathrm{d}t$.

Next we prove that the remainders $a_i^N(t)$ for $i = 1, 2, 3$ vanish as $N \to \infty$. Since $t \in (t_k, t_{k+1}]$ we find, due to Proposition 5.5 and the boundedness of $\left\| L^k_{\varepsilon_N} \varphi \right\|$,

$$
\begin{aligned}
|a_1^N(t)| &\overset{(1.5)}{\leq} \nu \int_{t_k}^{t_{k+1}} \|v^{\varepsilon_N}(\tau)\| \, \|\Delta\varphi\| \, \mathrm{d}\tau \\
&\overset{(5.5)}{\leq} c\varepsilon_N \overset{N\to\infty}{\longrightarrow} 0, \\
|a_2^N(t)| &\overset{(1.5)}{\leq} \int_{t_k}^{t_{k+1}} \|v^{\varepsilon_N}(\tau)\| \left\| L^k_{\varepsilon_N} \varphi \right\| \, \mathrm{d}\tau \\
&\overset{(5.5),(6.5)}{\leq} c\varepsilon_N \overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

For $a_3^N(t)$ we use the estimate (5.6) and find

$$
\begin{aligned}
|a_3^N(t)| &\overset{(1.5)}{\leq} \int_{t_k}^{t_{k+1}} \left\| f^{k+1} \right\| \|\varphi\| \, \mathrm{d}\tau \\
&= \varepsilon_N \|\varphi\| \left\| f^{k+1} \right\| \\
&\overset{(5.6)}{\leq} c\varepsilon_N \left(\frac{1}{\varepsilon_N} \int_{t_k}^{t_{k+1}} \|f(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\leq c\varepsilon_N^{\frac{1}{2}} \left(\int_0^T \|f(\tau)\|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\overset{N\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Using the weak convergence $v^{\varepsilon_N}(t) \overset{n\to\infty}{\rightharpoonup} v(t)$ in $\mathcal{H}^0(\Omega)$ for all $t \in [0, T]$, we have

$$
\langle v^{\varepsilon_N}(t), \varphi \rangle \overset{N\to\infty}{\longrightarrow} \langle v(t), \varphi \rangle
$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$, and, analogously to the proof of Theorem 6.6, we obtain

$$
\begin{aligned}
\nu \int_0^t \langle v^{\varepsilon_N}(\tau), \Delta\varphi \rangle \, \mathrm{d}\tau &\overset{(1.11)}{=} -\nu \int_0^t \langle \nabla v^{\varepsilon_N}(\tau), \nabla\varphi \rangle \, \mathrm{d}\tau \\
&\overset{N\to\infty}{\longrightarrow} -\nu \int_0^t \langle \nabla v(\tau), \nabla\varphi \rangle \, \mathrm{d}\tau,
\end{aligned}
$$

$$
\int_0^t \langle v^{\varepsilon_N}(\tau), L_{\varepsilon_N}(\tau)\varphi \rangle \, \mathrm{d}\tau \overset{N\to\infty}{\longrightarrow} \int_0^t \langle v(\tau), v(\tau) \cdot \nabla\varphi \rangle \, \mathrm{d}\tau,
$$

$$\int_0^t \left\langle \tilde{f}(\tau), \varphi \right\rangle \, \mathrm{d}\tau \overset{N\to\infty}{\longrightarrow} \int_0^t \langle f(\tau), \varphi \rangle \, \mathrm{d}\tau.$$

From (6.19) this implies

$$\begin{aligned} \langle v(t), \varphi \rangle &= \langle v_0, \varphi \rangle - \nu \int_0^t \langle \nabla v(\tau), \nabla \varphi \rangle \, \mathrm{d}\tau \\ &\quad + \int_0^t \langle v(\tau), v(\tau) \cdot \nabla \varphi \rangle \, \mathrm{d}\tau + \int_0^t \langle f(\tau), \varphi \rangle \, \mathrm{d}\tau \end{aligned} \tag{6.20}$$

for all $\varphi \in C_{0,\sigma}^\infty(\Omega)$.

To prove the weak continuity in $\mathcal{H}^0(\Omega)$, we need so show that

$$\lim_{[0,T]\ni\tilde{t}\to t} \left\langle v(t) - v(\tilde{t}), \varphi \right\rangle = 0$$

holds true for each $t \in [0, T]$ and each $\varphi \in \mathcal{H}^0(\Omega)$.

For $\varphi \in C_{0,\sigma}^\infty(\Omega)$ the assertion is trivial, since the right hand side of (6.20) is continuous with respect to t.

Now let $\varphi \in \mathcal{H}^0(\Omega)$. Since $C_{0,\sigma}^\infty(\Omega)$ is dense in $\mathcal{H}^0(\Omega)$, for each $h > 0$ we find some $\widetilde{\varphi} \in C_{0,\sigma}^\infty(\Omega)$ such that $\|\varphi - \widetilde{\varphi}\| \le h$. Now we consider the decomposition

$$\left\langle v(t) - v(\tilde{t}), \varphi \right\rangle = \underbrace{\left\langle v(t) - v(\tilde{t}), \varphi - \widetilde{\varphi} \right\rangle}_{=:b_1(t)} + \underbrace{\left\langle v(t) - v(\tilde{t}), \widetilde{\varphi} \right\rangle}_{=:b_2(t)},$$

where we can estimate b_1 by

$$\begin{aligned} |b_1(t)| &= \left| \left\langle v(t) - v(\tilde{t}), \varphi - \widetilde{\varphi} \right\rangle \right| \\ &\overset{(1.5)}{\le} \left\| v(t) - v(\tilde{t}) \right\| \, \|\varphi - \widetilde{\varphi}\| \\ &\le \left(\|v(t)\| + \left\| v(\tilde{t}) \right\| \right) \|\varphi - \widetilde{\varphi}\| \\ &\overset{(6.9)}{\le} ch. \end{aligned}$$

Thus we can make $b_1(t)$ as small as desired by choosing h small enough. Once h is chosen, $\widetilde{\varphi} \in C_{0,\sigma}^\infty(\Omega)$ can be determined. Since we already know that $\langle v(t), \widetilde{\varphi} \rangle$ is continuous, we can make $b_2(t)$ as small as desired by reducing the distance between t and $\tilde{t}$.

Finally we prove the strong continuity in $t = 0$. From (6.20) we immediately obtain

$$\langle v(0), \varphi \rangle = \langle v_0, \varphi \rangle$$

for $\varphi \in C_{0,\sigma}^\infty(\Omega)$, and with a usual density argument we obtain

$$v(0) = v_0.$$

Using (1.13d) from Proposition 1.18, it suffices to prove

$$\|v_0\|^2 \geq \limsup_{0<t\to 0} \|v(t)\|^2$$

to obtain the continuity in $t = 0$. Thus assume $\|v_0\|^2 < \limsup\limits_{0<t\to 0} \|v(t)\|^2$. Then there exists a sequence $\{t_n\}_n \overset{n\to\infty}{\longrightarrow} 0$ satisfying

$$\|v_0\|^2 + h \leq \|v(t_n)\|^2$$

for some $h > 0$ fixed and all $n \in \mathbb{N}$. Using

$$\|v(t_n)\|^2 \overset{(6.9)}{\leq} \|v_0\|^2 + \frac{c_p^2}{\nu} \int_0^{t_n} \|f(\tau)\|^2 \, \mathrm{d}\tau$$

from Corollary 6.3, this implies

$$\begin{aligned} h &\leq \frac{c_p^2}{\nu} \int_0^{t_n} \|f(\tau)\|^2 \, \mathrm{d}\tau \\ &\overset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

which contradicts the assumption, thus the assertion is proved. □

References

[1] Robert A. ADAMS and John J. F. FOURNIER. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier Ltd, Amsterdam, 2nd edition, 2003.

[2] Hans W. ALT. *Lineare Funktionalanalysis. Eine anwendungsorientierte Einführung*. Springer, Berlin–Heidelberg, 4th edition, 2002.

[3] Herbert AMANN and Joachim ESCHER. *Analysis III*. Grundstudium Mathematik. Birkhäuser, Basel–Boston–Berlin, 2001.

[4] Nazgul ASANALIEVA. *Zur Approximation der Gleichungen von Navier-Stokes*. Dissertation, Kassel University, Kassel, 2011.

[5] Nazgul ASANALIEVA, Carolin HEUTLING, and Werner VARNHORN. Time Delay and Lagrangian Approximation for Navier-Stokes Flow. *Analysis*, 35(4):213–229, 2015.

[6] Lamberto CATTABRIGA. Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rendiconti del Seminario Matematico della Università di Padova*, 31:308–340, 1961.

[7] Manfred DOBROWOLSKI. *Angewandte Funktionalanalysis. Funktionalanalysis, Sobolev-Räume und elliptische Differentialgleichungen*. Springer, Berlin–Heidelberg, 2nd edition, 2010.

[8] Etienne EMMRICH. *Gewöhnliche und Operator-Differentialgleichungen. Eine integrierte Einführung in Randwertprobleme und Evolutionsgleichungen für Studierende*. Vieweg, Wiesbaden, 2004.

[9] Lawrence C. EVANS. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2nd edition, 2010.

[10] John G. HEYWOOD. The Navier-Stokes Equations. On the Existence, Regularity and Decay of Solutions. *Indiana University Mathematics Journal*, 29(5):639–681, 1980.

[11] Eberhard HOPF. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Mathematische Nachrichten*, 4:213–231, 1950.

[12] Olga A. LADYZHENSKAYA. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon an Breach, New York, 1969.

[13] Jean LERAY. Etude de diverses équations intégrales non linéaires et de quelques problémes que pose l' hydrodynamique. *Journal de Mathématiques Pures et Appliquées*, 12:1–82, 1933.

[14] Jean LERAY. Sur le mouvement d' un liquide visqueux emplissant l' espace. *Acta Mathematica*, 63:193–248, 1934.

[15] Norman G. Meyers and James Serrin. $H = W$. *Proceedings of the National Academy of Sciences of the United States of America*, 51(6):1055–1056, 1964.

[16] Reimund Rautmann. A Criterion for Global Existence in Case of Ordinary Differential Equations. *Applicable Analysis*, 2:187–194, 1972.

[17] Michael Renardy and Robert C. Rogers. *An Introduction to Partial Differential Equations*, volume 13 of *Texts in Applied Mathematics*. Springer, New York, 2nd edition, 2003.

[18] Ben Schweizer. *Partielle Differentialgleichungen. Eine anwendungsorientierte Einführung*. Springer, Berlin–Heidelberg, 2013.

[19] Marvin Shinbrot. *Lectures on fluid mechanics*. Gordon and Breach, New York, 1973.

[20] Hermann Sohr. *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*. Birkhäuser, Boston–Basel–Berlin, 2001.

[21] Roger Temam. *Navier-Stokes equations. Theory and Numerical Analysis*. American Mathematical Society, Providence (Rhode Island), 2001.

[22] Werner Varnhorn. The Navier-Stokes equations with particle methods. In Petr Kaplický and Šárka Nečasová, editors, *Topics on partial differential equations*, volume 2 of *Jindřich Nečas Center for Mathematical Modeling. Lecture Notes*, pages 121–157. MATFYZPRESS Publishing House of the Faculty of Mathematics and Physics, Charles University in Prague, 2007.

[23] Werner Varnhorn. Time Delay and Material Differences for Non-stationary Navier-Stokes Flow. *Far East Journal of Applied Mathematics*, 72(1):49–59, 2012.

[24] Wolfgang Walter. *Gewöhnliche Differentialgleichungen. Eine Einführung*. Springer, Berlin–Heidelberg, 6th edition, 1996.

[25] Dirk Werner. *Funktionalanalysis*. Springer, Berlin–Heidelberg, 5th edition, 2005.

[26] Joseph Wloka. *Partielle Differentialgleichungen. Sobolevräume und Randwertaufgaben*. Mathematische Leitfäden. Teubner, Stuttgart, 1982.

[27] Florian Zanger. *Time Discretization of the SST-generalized Navier-Stokes Equations: Positive and Negative Results*. Dissertation, Kassel University, Kassel, 2014.

Declaration of academic honesty

I herewith give assurance that I completed this dissertation independently without prohibited assistance of third parties or aids other than those identified in this dissertation. All passages that are drawn from published or unpublished writings, either word-for-word or in paraphrase, have been clearly identified as such. Third parties were not involved in the drafting of the content of this dissertation; most specifically I did not employ the assistance of a dissertation advisor. No part of this thesis has been used in another doctoral or tenure process.