

# Fourier Analysis of DG Schemes for Advection-Diffusion

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This work compares the wave propagation properties of discontinuous Galerkin (DG) schemes for advection-diffusion problems in particular with respect to the discretization of diffusion terms. Extending previous investigations, the advection discretization now additionally varies between the choices of central or upwind fluxes. The results show that a previously recognized better performance of central schemes for well-resolved problems only hold for even polynomial degrees and that upwind-type discretizations also perform better on Gauss-Lobatto nodes.

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## 1 Introduction

In case of high order methods, small numerical dissipation competes with robustness and has to be carefully analyzed. Therefore, dispersion and diffusion properties have been investigated for major classes of high order schemes [1–4]. The eigenanalysis of numerical schemes is often based on the linear advection equation while viscous flux discretization is ignored.

For advection-diffusion problems, Watkins et al. [5] analyze the influence of different interface fluxes within nodal DG schemes obtained via flux reconstruction. More precisely, for a DG scheme of polynomial degree  $N = 2$  on classical Gauss nodes, differences of its wave propagation properties are studied in case of either upwind or central flux for advection as well as either LDG or BR1 scheme for diffusion. It is shown that the corresponding schemes with central flux discretization produce smaller errors for well-resolved solutions whereas one-sided flux discretizations produce smaller errors in the under-resolved case. In [6], the study by Watkins et al. [5] has been extended to higher order DG schemes with  $N > 2$  to detect odd-even phenomena. Furthermore, DG schemes on Gauss-Lobatto nodes are considered as well as a larger range of interface fluxes for diffusion terms. In particular, it is shown in [6] that the two possible choices of alternating LDG fluxes lead to significant differences in the error vs. wave number characteristics and that the higher accuracy of the BR1 flux for well-resolved solutions observed in [5] is restricted to the DG scheme on Gauss nodes and to an even polynomial degree  $N$ .

In this work, the various choices of numerical diffusion fluxes and DG nodal sets are combined with either central or upwind fluxes for advection whereas [6] only considers upwind advection discretization.

## 2 The DG scheme for advection-diffusion equations

We consider the linear advection-diffusion equation in one space dimension given by

$$U_t + aU_x = dU_{xx}, \quad (x, t) \in Q = \Omega \times (0, T), \quad \Omega = (x_\alpha, x_\beta), \tag{1}$$

with diffusion coefficient  $d > 0$  and advective velocity  $a > 0$ , supplemented by periodic initial conditions  $U(x, 0) = U_0(x)$  and periodic boundary conditions. The spatial domain  $\Omega$  is partitioned into a uniform grid consisting of cells  $I_j, j = 1, \dots, E$ , having equal cell length  $\Delta x$ . The basis and test functions used to define the DG scheme are taken from the finite element space  $V_h = \{v \in L^2(\Omega) \mid v|_{I_j} \in \mathcal{P}_N(I_j) \forall j = 1, \dots, E\}$ . At element boundaries, the left-hand side and right-hand side values of a piecewise continuous function  $v$  are denoted by  $v^-$  and  $v^+$ , respectively. Each cell  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$  is transformed into the reference cell  $I = (-1, 1)$  by the map  $\Lambda_j(\xi) = \frac{1+\xi}{2} x_{j+\frac{1}{2}} + \frac{1-\xi}{2} x_{j-\frac{1}{2}}$  for  $\xi \in I$ . Nodal values of the approximate solution at the quadrature points  $x_\nu = \Lambda_j(\xi_\nu)$  within a DG cell are collected into the solution vector  $\mathbf{u} = (u_1, \dots, u_{N+1})^T$ . Furthermore, the vector valued function  $\mathbf{L} = (L_1, \dots, L_{N+1})^T$  is composed of the Lagrange polynomials  $L_k(\xi)$  corresponding to the DG nodal set on  $I$ . Defining the differentiation matrix  $\mathbf{D}$  and mass matrix  $\mathbf{M}$  by their entries  $D_{jk} = L'_k(\xi_j)$  and  $M_{jk} = \delta_{jk}\omega_j$ , the resulting DG formulation applied to the degenerate system of first order PDEs  $U_t = Q_x, Q = U_x$  reads

$$\frac{\Delta x}{2} \mathbf{u}_t + a\mathbf{D} \mathbf{u} - d\mathbf{D} \mathbf{q} = \mathbf{M}^{-1} (a[(u - \hat{u}_{adv})\mathbf{L}]_{-1}^1 - d[(q - \hat{q})\mathbf{L}]_{-1}^1), \tag{2}$$

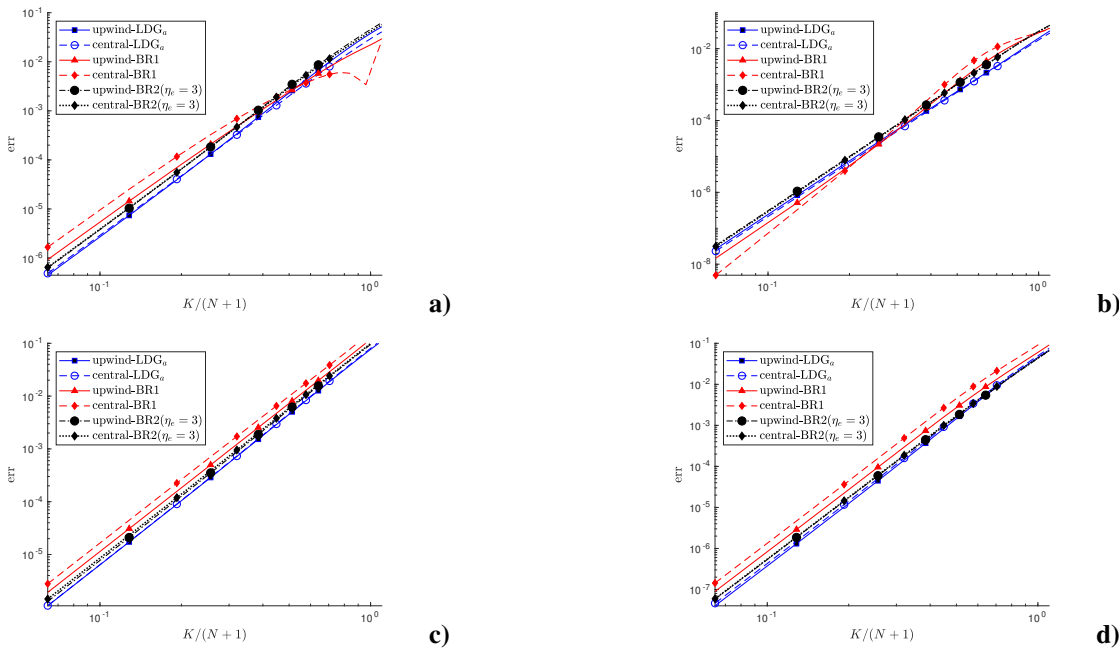
$$\frac{\Delta x}{2} \mathbf{q} - \mathbf{D} \mathbf{u} = -\mathbf{M}^{-1} [(u - \hat{u}_{div})\mathbf{L}]_{-1}^1, \tag{3}$$

where  $\hat{u}_{adv} = \frac{1}{2}(u^- + u^+) + \frac{\gamma}{2}(u^- - u^+)$  represents the choice of numerical advection flux of either upwind ( $\gamma = 1$ ) or central type ( $\gamma = 0$ ) and  $\hat{q}$  and  $\hat{u}_{div}$  denote the numerical diffusion fluxes LDG<sub>a</sub>, BR1 and BR2( $\eta_e = 3$ ) described in detail in [6].

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**Fig. 1:** Relative error  $\text{err}(t)$  vs. non-dimensional wave number comparing various numerical flux formulations. **a)** DG( $N = 3$ ) on Gauss nodes, **b)** DG( $N = 4$ ) on Gauss nodes, **c)** DG( $N = 3$ ) on Gauss-Lobatto nodes, **d)** DG( $N = 4$ ) on Gauss-Lobatto nodes.

### 3 Eigensolution analysis

The linear advection-diffusion equation (1) admits a traveling wave solution of the form  $u(x, t) = e^{i(kx - \omega t)}$  where  $k \in \mathbb{R}$  is the wave number and  $\omega$  denotes the frequency given by  $\omega = ak - idk^2$ . Inserting a corresponding numerical solution of the form  $\mathbf{u}_j(t) = \mathbf{c} e^{i(kj\Delta x - \tilde{\omega}t)}$  into the strong DG formulation (2), (3) yields

$$\left( (\mathbf{A}_0 + e^{-iK} \mathbf{A}_{-1}) + \frac{1}{Pe^*} (e^{-2iK} \mathbf{B}_{-2} + e^{-iK} \mathbf{B}_{-1} + \mathbf{B}_0 + e^{iK} \mathbf{B}_1 + e^{2iK} \mathbf{B}_2) \right) \mathbf{c} = i\Omega \mathbf{c}. \quad (4)$$

with real  $(N+1) \times (N+1)$  matrices  $\mathbf{A}_0, \mathbf{A}_{-1}, \mathbf{B}_k, k = 0, \pm 1, \pm 2$ , depending only on the chosen nodal set and the numerical fluxes which characterize the respective DG scheme. Hereby, the set of parameters is reduced by defining the non-dimensional wave number  $K = k\Delta x$ , the non-dimensional numerical frequency  $\Omega = \frac{\Delta x \tilde{\omega}}{a}$  and the grid Peclet number  $Pe^* = \frac{a\Delta x}{d}$ . Solving the eigenvalue problem (4) for any  $K \in [0, \pi(N+1)]$ , we obtain a set of  $N+1$  eigenvalues of equation (4) given by eigenvalues  $\Omega_p, p = 1, \dots, N+1$ , and corresponding normalized eigenvectors  $\mathbf{v}_p, p = 1, \dots, N+1$ . Now, an initial wave representing  $e^{ikx}$  is given by the initial nodal values  $u_\nu(0) = e^{i(j + \frac{\xi_\nu + 1}{2})K}$ ,  $\nu = 1, \dots, N+1$ , and can be represented as the linear combination  $\mathbf{u}(0) = \sum_{p=1}^{N+1} \beta_p \mathbf{v}_p e^{ijK}$ , where the coefficients  $\beta_p$  are obtained as the solution of  $\sum_{p=1}^{N+1} \beta_p \mathbf{v}_p = \alpha$ , with  $\alpha_\nu = e^{i\frac{\xi_\nu + 1}{2}K}$ . As in Watkins et al. [5], we now consider the error  $\text{err}(t)$  depending on the non-dimensional wave number,

$$\text{err}(t) := \frac{1}{\sqrt{N+1}} \left\| \sum_{p=1}^{N+1} \left( e^{\frac{a}{\Delta x} (-i(\Omega_p - K) + (Pe^*)^{-1}K^2)t} - 1 \right) \beta_p \mathbf{v}_p \right\|_{l_2}. \quad (5)$$

Fig. 1 depicts this error bound for  $Pe^* = 10, t = 0.1$  when applying the DG scheme on Gauss or Gauss-Legendre nodes using various numerical advection and diffusion fluxes of both one-sided and central type. For the considered polynomial degrees of  $N = 3, 4$ , the DG schemes using upwind advection flux and one-sided (LDG) diffusion fluxes perform best except for DG( $N = 3$ ) on Gauss nodes with a better performance of BR1 for particularly low wave numbers. In addition, the change between upwind or central advection discretization mostly affects the BR1 diffusion fluxes whereas for other choices of the DG diffusion discretization, the results are less influenced by the type of advection flux.

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