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# Efficient Solution of Distributed MIP in Control of Networked Systems

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Certain classes of optimization-based control problems stated for networked systems involving hybrid dynamics and logical constraints can be cast into Mixed-Integer Programming (MIP) problems. Since these belong to the complexity class NP-hard, the motivation arises to find approximations of the optimal solution by distributed solution efficiently. For the cases that the cost functional is linear or quadratic and the constraints are linear, this paper proposes an alternative to the standard centralized schemes, by employing dual decomposition into a set of local problems of moderate size which can be solved in parallel. Numerical examples demonstrate that the scheme can efficiently approximate the global solution.



## 1 Introduction and Problem Definition

For distributed systems consisting of $n$ coupled subsystems with index $i \in N = \{1, \ldots, n\}$, this paper considers the efficient solution of MIP problems of the type:

$$\min_{x_1,\cdots,x_n} \sum_{i\in N} f_i(x_i),\ \textit{s.t.:} \sum_{i\in N} A_i x_i \leq b;\ x_i \in X_i \subset \mathbb{R}^{r_i} \times \mathbb{Z}^{z_i},\ \forall i \in N, b \in \mathbb{R}^m. \tag{1}$$

Such problems arise in the context of optimal control of networked hybrid systems with coupling constraints if the following steps are caried out, compare to [4]: time-discretization is introduced, the local vector $x_i$ is defined to comprise all variables (inputs and states) of the subsystem $i$ over the considered time-domain, all local constraints are formulated to linearly depend on $x_i$, and the global costs are defined as the summation of the local cost functions $f_i(x_i)$. Note that $x_i$ contains $r_i$ real-valued entries as well as $z_i$ integer components, and it is bound to a local mixed-integer polyhedral set $X_i = \{x_i \in \mathbb{R}^{r_i} \times \mathbb{Z}^{z_i} | D_i x_i \leq d_i\}$. Note further that a subset of the $m$ inequality constraints may establish coupling constraints $\sum_{i\in N} A_i x_i \leq b$ to be satisfied by several or even all subsystems.

Techniques based on branch-and-bound principles are standard approaches to solve MIP problems such as (1). These tree-search techniques rely on the iterative determination of suitable cost bounds based on which sub-trees which may not include the optimal solution are eliminated from further exploration. The efficiency depends critically on the tightness of these bounds, usually obtained by relaxing the integrality constraints (lower bounds) or from appropriate heuristics (upper bounds). With respect to (1), larger numbers of $N$ often lead to large computation times for centralized branch-and-bound techniques, in particular since the provision of suitable heuristics to determine upper bounds is difficult, and since the distributed structure of the problem is not explicitely considered. Solution techniques which employ problem decomposition require appropriate handling of the coupling constraints. If however, and this is part of the alternative solution technique proposed in this paper, the integrality constraints for the integer components of $x_i$, $\forall i \in N$ are temporarily relaxed (in an iterative scheme), if the local cost functions $f_i(x_i)$ are convex, and if the Slater constraint qualification is satisfied, the decomposed solution by use of the Lagrangian dual is possible. This path is promising as the distributed solution can adopt the well-known scheme of primal-dual iteration [1] to solve the dual problem, and global optimality is ensured due to a zero duality gap. Of course, without relaxing the integrality constraint, a duality gap may exist in the general case, causing the distributed solution to be non-optimal or even infeasible. Nonetheless, recent studies revealed for $n$ being large in (1), aside of an increase of the problem size, that the duality gap vanishes to zero under certain conditions [2]. Accordingly, by making use of the Lagrangian dual of (1), one can indeed determine an optimal or closed-to-optimal solution in a distributed way.

## 2 Efficient Distributed Solution Based on Dual Decomposition

First consider the case that all local cost functions $f_i(x_i)$ are linear, thus leading to (1) in form of a mixed-integer linear program (MILP). Then by replacing $X_i, \forall i \in N$ with its convex hull $Conv(X_i)$, and by relaxing the integrality constraint for $x_i$, a linear programming (LP) problem with zero duality gap is obtained:

$$\min_{x_1,\cdots,x_n} \sum_{i\in N} f_i(x_i),\ \textit{s.t.:} \sum_{i\in N} A_i x_i \leq b;\ x_i \in Conv(X_i) \subset \mathbb{R}^{r_i+z_i},\ \forall i \in N. \tag{2}$$

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Different properties of problem (2) offer its use within a distributed solution procedure for (1): 1.) by dualizing the coupling constraints $\sum_{i\in N} A_i x_i \leq b$ of (2), the primal-dual iteration allows to determine the optimal solution $x_i^{*,LP}$, $i \in N$ in a distributed way; 2.) although the optimal cost $\sum_{i\in N} f_i(x_i^{*,LP})$ of (2) is always lower than or equal to the optimal costs $\sum_{i\in N} f_i(x_i^*)$ of (1), the optimality gap tends to be zero for increasing $n$; 3.) the optimal solution $x_i^{*,LP}$, $i \in N$ of (2) always satisfies the coupling constraints in (1), and, under mild assumptions, it only fails to satisfy the local constraint $x_i \in X_i$ for at most $m+1$ subsystems [2]. Given these properties, one only has to recover the local feasibility of $X_i$ for a small fraction of the subsystems (especially for $n \gg m$) after distributed solution of (2). To this end, a sub-optimal solution of (1) with bounded performance loss is obtained, while most of the computation is carried out in a distributed form. In addition, as the global problem is decomposed into $n$ local problems with local variables only, the overall complexity increases only moderately with the number of subsystems. Drawbacks are, however, that the convergence towards $x_i^{*,LP}$, $i \in N$ is often found to be slow such that a large number of iterations is required. In addition, the success of recovering local feasibility for the up to $m+1$ subsystems, for which $x_i^{*,LP}$ may not be feasible with respect to $X_i$, requires conservative assumptions on the feasibility of (1). With respect to these limitations, a new method has recently been proposed in [3], which relaxes the assumptions and also allows an earlier termination of the primal-dual iteration. Key idea of the new approach is to start from a feasible candidate of (1) and successively generate new candidates, such that the global costs decrease as in gradient methods. In each iteration, a new MILP problem in the form of (1) is formulated for which feasibility is guaranteed by adapted constraints, and in which a definitely improving step $\Delta x$ composed of $\Delta x_i$, $i \in N$ is determined in distributed form.

For quadratic local cost functions $f_i(x_i) = x_i^{\mathrm{T}} Q_i x_i + c_i^{\mathrm{T}} x_i$ with positive-definite $Q_i$, $i \in N$ in (1), the resulting mixed-integer quadratic programming (MIQP) problems constitute a more common case in optimal control. In this case, a solution procedure very similar to the one sketched before can be used, starting from a first feasible candidate $x^{[0]}$ composed of $x_i^{[0]}$, $i \in N$, which can typically be determined with very low effort. In any iteration $k$, the linearization of the global cost function in $x^{[k]}$ leads to a linear approximation $\sum_{i\in N} g_i^{[k]} \Delta x_i$. A permissible ellipsoidal set $\varepsilon^{[k]}$ is chosen for a step $\Delta x^{[k]}$ such that feasibility with respect to the coupling constraints of (1) is obtained. By inner-approximating $\varepsilon^{[k]}$ by a polytopic set $P^{[k]}$, the following MILP problem is formulated to obtain the step $\Delta x^{[k]}$:

$$\min_{\Delta x_1,\cdots,\Delta x_n} \sum_{i\in N} g_i^{[k]} \Delta x_i, \ \textit{s.t.:} \ \sum_{i\in N} A_i(\Delta x_i + x_i^{[k]}) \leq b; \ \Delta x \in P^{[k]}; \ \Delta x_i + x_i^{[k]} \in X_i, \ \forall i \in N. \tag{3}$$

The structure is the same as in (1), if $\Delta x \in P^{[k]}$ is categorized into coupling constraints or in local ones for $i$. Thus, the distributed solution as described above for (2) can be employed, and for any $k$, after obtaining $\Delta x^{[k]}$ and thus the new feasible candidate $x^{[k+1]} = x^{[k]} + \Delta x^{[k]}$, the linear approximation (3) is updated and the iteration continues until no further improvement is found, or is below a specified threshold.

## 3 Numerical Experiment and Conclusion

The proposed distributed solution was tested for a large number of problem instances of type (1). Exemplarily, for an MILP with $n = 40$ subsystems with each $z_i = r_i = 15$ integer and real variables, the solution $x^*$ was found with optimal cost of $-2.17 \cdot 10^5$ in $336sec$ by centralized solution using a standard numerical solver. By employing the proposed distributed solution, $x^{[1]}$ was obtained with costs of $-2.14 \cdot 10^5$ after $0.35sec$, and $x^{[2]}$ with costs of $-2.16 \cdot 10^5$ after $0.48sec$, thus much faster and already very close to the global solution. Tests results for problems with up to 500 subsystems are reported on in [3].

This work has proposed a scheme for the distributed solution of different MIP problems. The dual decomposition makes use of the separable structure of the problem in order to enhance computational efficiency. While the present method has shown its potential for MILP and MIQP problems, succeeding work aims at extending the principles to more general cost functions and constraints.

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