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Measuring the Time Discretization Error in Linear Elasticity

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The present paper considers the spread of longitudinal waves in an elastic rod. On the one hand an analytical solution is presented, on the other hand a spatially and temporally discretized continuum formulation of this problem is treated. With these approaches at hand the appearing time discretization error is analyzed using various quantities.

1 Motivation

Different industrial manufacturing processes are characterized by deformations where dynamic effects can't be neglected any longer, cf. [4]. Thus, the analysis of time discretization schemes and the corresponding errors is of great importance. To assess the later appropriate error measurements have to be derived. Therefore, problems which have an analytical solution, as it is the case for the spread of longitudinal waves in an elastic rod, can be considered.

2 Longitudinal Waves in a Thin Elastic Rod

In order to describe the spread of longitudinal waves in a thin elastic rod with length L mathematically, the second order partial differential equation $(1)_1$ - depending on the time t and the space Z - is considered together with appropriate boundary conditions $(1)_2$-$(1)_3$ and initial conditions $(1)_4$-$(1)_5$

$$\rho\frac{\partial^2 u_Z}{\partial t^2} - E\frac{\partial^2 u_Z}{\partial Z^2} = 0, \quad u_Z(Z=0,t)=0, \quad \frac{\partial u_Z}{\partial Z}(Z=L,t)=0, \quad u_Z(Z,t=0)=0, \quad \frac{\partial u_Z}{\partial t}(Z,t=0)=\frac{Z}{L}\omega. \quad (1)$$

Therein, the density ρ and the YOUNG modulus E of the rod material as well as the displacement field in axial direction u_Z are included. The parameter ω embodies a prescribed velocity within the rod at the beginning. With the abbreviation $c^2 = E/\rho$ a classical wave equation as in [2] is obtained.

2.1 Analytical Solution

For the solution of equation $(1)_1$ a separation approach, assuming $u_Z = W(Z)T(t)$, is used following the scheme in [1]. Hence, series of spatial solutions $(2)_1$ and of temporal solutions $(2)_2$ are derived with $\sqrt{\lambda} = [[2k-1]\pi]/[2L]$ for $k = 1, ..., \infty$:

$$W_k = \sin\left(\sqrt{\lambda}Z\right), \qquad T_k = \frac{16\omega L}{[2k-1]^3\pi^3 c}\sin(\sqrt{\lambda}L)\sin(c\sqrt{\lambda}t), \qquad u_Z(Z,t) = \sum_{k=1}^{\infty} W_k T_k. \quad (2)$$

This leads to the analytical solution of the rod's displacement field $(2)_3$. With these solutions at hand a semianalytical solution can be established. Therefore, the time-dependent part $T(t)$ is solved using certain time discretization schemes.

2.2 Semianalytical Solution with RUNGE-KUTTA Schemes

In the following a semianalytical solution using stiffly accurate implicit RUNGE-KUTTA methods is derived. The transformation of equation $(1)_1$ into a purely time dependent part, exploiting the separation approach of [1], leads to

$$\frac{\partial^2 T_{k,ni}}{\partial t^2} + \left[\frac{[2k-1]\pi}{2L}\right]^2 c^2 T_{k,ni} = 0, \qquad T_k(0) = 0, \qquad \frac{\partial T_k(0)}{\partial t} = \frac{8\omega}{[2k-1]^2\pi^2}\sin\left(\frac{[2k-1]\pi}{2L}L\right). \quad (3)$$

Therein, for each k the considered time interval $[0, ..., T^*]$ is split into time steps $\Delta t = t_{ns} - t_n$, where equation $(3)_1$ is solved at various points in time t_{ni} with the initial conditions $(3)_2$-$(3)_3$, applying the approximations

$$T_{k,ni} = T_{k,n} + \Delta t \sum_{j=1}^{s} a_{ij}\,\frac{\partial T_{k,nj}}{\partial t}, \qquad \frac{\partial T_{k,ni}}{\partial t} = \frac{\partial T_{k,n}}{\partial t} + \Delta t \sum_{j=1}^{s} a_{ij}\,\frac{\partial^2 T_{k,nj}}{\partial t^2}. \quad (4)$$

The number of point in time per time interval is denoted as s, while the parameters a_{ij} represent the RUNGE-KUTTA coefficients, [5]. A multiplication and summation with the spatial solution $(2)_1$ yields the semianalytical solution $u_{Z,\text{SEMI}}(L,t)$.

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2.3 Numerical Solution

But if the elastic rod is modeled as an axisymmetric continuum with appropriate parameters as the constitutive tensor $\boldsymbol{C}$, the external loads $\boldsymbol{t}^*$, and the strain tensor ε, the weak form

$$\int_{\Omega} \rho_0 \delta\dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}}\, dV + \int_{\Omega} \delta\dot{\boldsymbol{\varepsilon}} : \boldsymbol{C} : \boldsymbol{\varepsilon}\, dV - \int_{\Gamma_{t^*}} \delta\dot{\boldsymbol{u}} \cdot \boldsymbol{t}^*\, dA = 0 \tag{5}$$

builds the basis to describe the spread of waves. Following the procedure in [3] and carrying out a spatial discretization of (5), performing a temporal discretization using the methods of chapter 2.2, and solving the obtained system of linear equations, the numerical solution $u_{Z,\text{NUM}}(L,t)$ can be obtained.

3 Time Discretization Error Analysis

With these three types of solution at hand distinct time discretization errors can be formulated. The global error of a quantity X can be determined by subtracting the numerically determined values X_{ns} from the analytical ones X_{ns}^{ana} at the end of the distinct time intervals, see equation $(6)_1$. The local h-error estimator for each time interval is obtained, if a numerical solution is subtracted from a numerical solution calculated with a two-times smaller time interval, see $(6)_2$.

$$e_{\text{h},ns}^{\text{glob}}(X) = X_{ns}^{\text{ana}} - X_{ns}, \qquad e_{\text{h},ns}(X) = X_{ns}^{\Delta t/2} - X_{ns}, \qquad q = \text{mean}(\text{linear fit}(\log(\Delta t), \log(e_{\text{h},ns}))). \tag{6}$$

If these errors are determined for distinct time step sizes and collocated in vectors, expression $(6)_3$ can be used to determine the order of convergence of the time discretization scheme. A more detailed explanation is given in [3, 5] and references therein. For three different RUNGE-KUTTA schemes the quantities in equation (6) are evaluated in **Table 1** in the context of the semianalytical and the numerical solution. It can be seen that if T_k is determined by only considering slow oscillations, theoretical orders of convergence, determined by the local and global error, are only obtained for low order schemes. For the RADAU IIA 3 method its theoretical order is not predicted, since the error height is in general too small. If higher oscillations for T_k are taken into account, only the RADAU IIA method seems to reach its theoretical order. The order of convergence calculated for $u_{Z,\text{SEMI}}(L,t)$ with the global error is for all schemes almost identical to the theoretical order. Within the numerical solution $u_{Z,\text{NUM}}(L,t)$ the global error is not able to determine the order of convergence of the time discretization scheme. Here the spatial error is higher than the temporal and masks the latter. The order of convergence determined by the local error is identical to the theoretical one except for the DIRK 3 method.

Table 1: Orders of convergence obtained for distinct RUNGE-KUTTA schemes

	$e_{\text{h},ns}^{\text{glob}}(T_k)$	$e_{\text{h},ns}(T_k)$	$e_{\text{h},ns}^{\text{glob}}(T_k)$	$e_{\text{h},ns}(T_k)$	$e_{\text{h},ns}^{\text{glob}}(u_{Z,\text{SEMI}}(L,t))$	$e_{\text{h},ns}^{\text{glob}}(u_{Z,\text{NUM}}(L,t))$	$e_{\text{h},ns}(u_{Z,\text{NUM}}(L,t))$
	$k=1$	$k=1$	$k=10$	$k=10$	$k=1,...,10$	$k=1,...,10$	$k=1,...,10$
DIRK 1	$q=1.0$	$q=0.9$	$q=0.0$	$q=-14$	$q=0.9$	$q=0.9$	$q=0.8$
DIRK 3	$q=3.0$	$q=3.0$	$q=2.5$	$q=0.3$	$q=2.5$	$q=-0.4$	$q=1.6$
RADAU IIA 3	$q=0.8$	$q=2.9$	$q=5.0$	$q=5.0$	$q=5.0$	$q=0.0$	$q=4.7$

4 Conclusion and Outlook

The present paper shows different approaches to model and determine the spread of longitudinal waves in elastic rods. Furthermore, applied time discretization methods are evaluated using different quantities. It can be observed, that there are distinct ways to calculate the order of convergence of time integrators. But even within this small example locally and globally determined orders of convergence are only rarely identical and hence no perfect error measurement could be derived. Thus, further investigations have to be carried out. These should also focus on problems where no analytical solutions prevail as it is the case within plasticity.

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