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On Non-linear Characterizations of Classical Orthogonal Polynomials

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Abstract. Classical orthogonal polynomials are known to satisfy seven equivalent properties, namely the Pearson equation for the linear functional, the second-order differential/difference/q-differential/ divided-difference equation, the orthogonality of the derivatives, the Rodrigues formula, two types of structure relations, and the Riccati equation for the formal Stieltjes function. In this work, following previous work by Kil et al. (J Differ Equ Appl 4:145–162, 1998a; Kyungpook Math J 38:259–281, 1998b), we state and prove a non-linear characterization result for classical orthogonal polynomials on non-uniform lattices. Next, we give explicit relations for some families of these classes.

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1. Introduction

Univariate orthogonal polynomials (or orthogonal polynomials for short) are systems of polynomials $(p_n)_n$ with $\deg(p_n) = n$, satisfying a certain orthogonality relation. They are very useful in practice in various domains of mathematics, physics, engineering, image processing and so on, because of the many properties and relations they satisfy. As examples of areas where orthogonal polynomials play important roles, we could cite approximation theory (see for example [6,31]) and also numerical analysis (see [14,15]).

It is known that any family of orthogonal polynomials $(p_n)_{n\geq 0}$ satisfies a three-term recurrence relation of the form

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \quad p_{-1}(x) = 0. \tag{1.1}$$

If $h_n = \langle \mathcal{L}, p_n^2 \rangle$, where $\mathcal{L}$ is the corresponding linear functional with respect to the sequence $(p_n)_{n\geq 0}$ and k_n is the leading coefficient of $p_n(x)$ (see [17]), then

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$$A_n = \frac{k_{n+1}}{k_n}, \quad C_n = \frac{A_n}{A_{n+1}} \frac{h_n}{h_{n+1}}, \quad n \geq 1$$

and we set $C_0 = 1$.

The systems of orthogonal polynomials associated with the names of Hermite, Laguerre, Jacobi and Bessel (including the special cases named after Tchebychev, Legendre, and Gegenbauer) are the most extensively and widely applied systems.

An orthogonal polynomial system $(p_n)_{n\geq 0}$ with respect to a weight function $\rho(x)$ is called classical if it satisfies one of the equivalent assertions (see [17]):

- $(p_n)_{n\geq 0}$ satisfies a second-order linear differential equation of the Sturm–Liouville type
$$\phi(x)y''(x) + \psi(x)y'(x) + \lambda_n y(x) = 0, \tag{1.2}$$
where $\phi(x)$ is a polynomial of degree ≤ 2 and $\psi(x)$ is a polynomial of exact degree 1, both independent on n and λ_n is independent on x.
- The derivatives $(p'_{n+1})_{n\geq 0}$ form an orthogonal polynomial system.
- The p_ns have the Rodrigues representation
$$p_n(x) = \frac{D_n}{\rho(x)} \left(\phi^n(x)\rho(x)\right)^{(n)}, \quad n \geq 0. \tag{1.3}$$
- The weight function $\rho(x)$ satisfies a Pearson-type equation
$$(\phi(x)\rho(x))' = \psi(x)\rho(x). \tag{1.4}$$
- The p_ns satisfy a difference-differential equation (or structure relation) of the form
$$\pi(x)p'_n(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x). \tag{1.5}$$

In his paper [2], Al-Salam has obtained an expression for the derivative of the product of two consecutive Bessel polynomials and has shown that this expression does, in fact, characterize the Bessel polynomials. Based on this paper, McCarthy in [25] proved that there is an analogous characterization for very classical orthogonal polynomials (Hermite, Laguerre and Jacobi polynomials). This characterization can be stated as

- $(p_n)_{n\geq 0}$ satisfies a non-linear equation of the form:
$$\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}(p_n(x)p_{n-1}(x)) = (\alpha_n x + \beta_n)p_n(x)p_{n-1}(x) + \gamma_n p_n^2(x) + \delta_n p_{n-1}^2(x), \tag{1.6}$$
where α_n, β_n, γ_n and δ_n are independent on x.

Note that several other characterizations of classical orthogonal polynomials with respect to the derivative operator can be found in [23].

Very close to the very classical orthogonal polynomials (classical orthogonal polynomials of a continuous variable) are the classical orthogonal polynomials of a discrete variable. An orthogonal polynomial system $(p_n)_{n\geq 0}$ of a discrete variable with respect to a weight function $\rho(x)$ is called classical if it satisfies one of the equivalent assertions (see [1,8,13]):

- $(p_n)_{n\geq 0}$ satisfies a second-order linear difference equation of the Sturm–Liouville type

$$\phi(x)\Delta\nabla y(x) + \psi(x)\Delta y(x) + \lambda_n y(x) = 0, \tag{1.7}$$

where $\phi(x)$ is a polynomial of degree ≤ 2 and $\psi(x)$ is a polynomial of exact degree 1, both independent on n and λ_n is independent on x.
- The sequence of difference polynomials $(\Delta p_{n+1})_{n\geq 0}$ form an orthogonal polynomial system of discrete variable.
- The p_ns have the Rodrigues representation

$$p_n(x) = \frac{D_n}{\rho(x)}\Delta^n\left(\phi^n(x)\rho(x)\right), \quad n \geq 0. \tag{1.8}$$

- The weight function $\rho(x)$ satisfies a Pearson-type equation

$$\Delta[\phi(x)\rho(x)] = \psi(x)\rho(x). \tag{1.9}$$

- The p_ns satisfy a difference equation (or structure relation) of the form

$$\pi(x)\nabla p_n(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x), \tag{1.10}$$

or otherwise stated (see [19])

$$\phi(x)\nabla p_n(x) = \tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x). \tag{1.11}$$

- For each $n \geq 1$, p_n and p_{n-1} satisfy a relation of the form (see [21, Theorem 5.2])

$$\begin{aligned}&\pi(x)\left[p_n(x)\nabla p_{n-1}(x) + p_{n-1}(x)\nabla p_n(x)\right]\\&\quad = U_n p_n^2(x) + V_n p_{n-1}^2(x) + (W_n x + Y_n)p_n(x)p_{n-1}(x),\end{aligned}$$

where the coefficients U_n, V_n, W_n and Y_n are independent on x and π is a polynomial of degree less or equal to 2.

It should be noted that the operators Δ and ∇ are respectively defined by

$$\begin{aligned}\Delta f(x) &= f(x+1) - f(x),\\ \nabla f(x) &= f(x) - f(x-1).\end{aligned}$$

Close to the classical discrete orthogonal polynomials are classical orthogonal polynomials of a q-discrete variable. An orthogonal polynomial system $(p_n)_{n\geq 0}$ of a q-discrete variable with respect to a weight function $\rho(x)$ is called classical if it satisfies one of the equivalent assertions (see [8, 18, 19]):

- $(p_n)_{n\geq 0}$ satisfies a second-order linear q-difference equation of the Sturm–Liouville type

$$\phi(x)\mathcal{D}_q\mathcal{D}_{\frac{1}{q}}y(x) + \psi(x)\mathcal{D}_q y(x) + \lambda_n y(x) = 0, \tag{1.12}$$

where $\phi(x)$ is a polynomial of degree less than or equal to 2 and $\psi(x)$ is a polynomial of exact degree 1, both independent on n and λ_n is independent on x.
- The sequence of q-difference polynomials $(\mathcal{D}_q p_{n+1})_{n\geq 0}$ form an orthogonal polynomial system of a q-discrete variable.

- The p_ns have the Rodrigues representation

$$p_n(x) = \frac{D_n}{\rho(x)} \mathcal{D}_q^n \left(\phi^n(x)\rho(x) \right), \quad n \geq 0. \tag{1.13}$$

- The weight function $\rho(x)$ satisfies a Pearson-type equation

$$\mathcal{D}_q[\phi(x)\rho(x)] = \psi(x)\rho(x). \tag{1.14}$$

- The p_ns satisfy a q-difference equation (or structure relation) of the form (see [19])

$$\phi(x)\mathcal{D}_{\frac{1}{q}} p_n(x) = \tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x). \tag{1.15}$$

- For each $n \geq 1$, p_n and p_{n-1} satisfy a relation of the form (see [22, Theorem 3.5])

$$\begin{aligned} &\tilde{\pi}(x) \left[p_n(x)\mathcal{D}_{\frac{1}{q}} p_{n-1}(x) + p_{n-1}(x)\mathcal{D}_{\frac{1}{q}} p_n(x) \right] \\ &\quad = \tilde{U}_n p_n^2(x) + \tilde{V}_n p_{n-1}^2(x) + (\tilde{W}_n x + \tilde{Y}_n) p_n(x) p_{n-1}(x), \end{aligned}$$

  where the coefficients $\tilde{U}_n$, $\tilde{V}_n$, $\tilde{W}_n$ and $\tilde{Y}_n$ are independent on x and $\tilde{\pi}$ is a polynomial of degree less than or equal to 2.

  It should be noted that the q-derivative D_q is defined as

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x} & \text{if } q \neq 1 \text{ and } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}.$$

The difference operator Δ and the q-derivative D_q are both special cases of the Hahn's operator $D_{q,\omega}$ (see [7]) which is defined as

$$D_{q,\omega} f(x) = \frac{f(qx+\omega) - f(x)}{(qx+\omega) - x}.$$

More precisely, $D_q = D_{q,0}$ and $\Delta = D_{1,1}$.

In this paper, we prove equivalent non-linear characterization results similar to (1.6) for classical orthogonal polynomials on non-uniform lattices (including Wilson and Askey–Wilson polynomials). Also, we prove such a non-linear characterization for Meixner–Pollaczek and Continuous Hahn polynomials. Indeed, we give explicitly the coefficients of these relations for some families of classical orthogonal polynomials on non-uniform lattices.

2. Preliminaries

This section contains some preliminary definitions and results that are useful for a better reading of this article. The q-hypergeometric series, a fractional q-derivative and fractional q-integral are defined. The reader will consult the reference [18] for more informations about these concepts.

2.1. The Hypergeometric Series

In what follows, the symbol $(a)_n$ denotes the so-called Pochhammer symbol and is defined by

$$(a)_m = \begin{cases} 1 \quad \text{if} \quad m = 0 \\ a(a+1)\cdots(a+m-1) \quad \text{if} \quad m = 1, 2, \dots \end{cases}$$

and the hypergeometric series is defined as

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

2.2. The q-Hypergeometric Series

The basic hypergeometric or q-hypergeometric series ${}_r\phi_s$ is defined by the series

$${}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}\,\middle|\, q; z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left((-1)^n q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^n}{(q;q)_n},$$

where

$$(a_1, \dots, a_r; q)_n := (a_1; q)_n \cdots (a_r; q)_n,$$

with

$$(a_i; q)_n = \begin{cases} \prod_{j=0}^{n-1} (1 - a_i q^j) & \text{if} \quad n = 1, 2, 3, \dots \\ 1 & \text{if} \quad n = 0 \end{cases}.$$

For $n = \infty$, we set

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \;\; |q| < 1.$$

The notation $(a; q)_n$ is the so-called q-Pochhammer symbol.

2.3. Difference and Divided-Difference Operators

2.3.1. The Operators $\mathcal{D}$ and $\mathcal{S}$. We define the difference operator $\mathcal{D}$ (see [26,28]) and its companion operator $\mathcal{S}$ as follows:

$$\mathcal{D}f(x) = f\left(x + \frac{i}{2}\right) - f\left(x - \frac{i}{2}\right), \quad \mathcal{S}f(x) = \frac{f\left(x + \frac{i}{2}\right) + f\left(x - \frac{i}{2}\right)}{2},$$

with $i^2 = -1$.

The operator $\mathcal{D}$ transforms a polynomial of degree n ($n \geq 1$) in x into a polynomial of degree $n-1$ in x and a polynomial of degree 0 into the zero polynomial. The operator $\mathcal{S}$ transforms a polynomial of degree n in x into a polynomial of degree n in x.

The operators $\mathcal{D}$ and $\mathcal{S}$ fulfill the following properties.

Proposition 2.1. (See [26,30]) *The operators $\mathcal{D}$ and $\mathcal{S}$ satisfy the following product rules*

$$\mathcal{D}(fg) = \mathcal{D}f\mathcal{S}g + \mathcal{S}f\mathcal{D}g, \tag{2.1}$$

$$\mathcal{S}(fg) = \frac{1}{4}\mathcal{D}f\mathcal{D}g + \mathcal{S}f\mathcal{S}g, \tag{2.2}$$

$$\mathcal{D}\mathcal{S} = \mathcal{S}\mathcal{D}, \tag{2.3}$$

$$\mathcal{S}^2 = \frac{1}{4}\mathcal{D}^2 + \mathbf{I}, \tag{2.4}$$

where $\mathbf{I}f = f$.

2.3.2. The Operators D and S. We define the difference operator $\mathbf{D}$ (see [27]) and its companion operator $\mathbf{S}$ as follows:

$$\mathbf{D}f(x^2) = \frac{f\left((x+\frac{i}{2})^2\right) - f\left((x-\frac{i}{2})^2\right)}{2ix}, \quad \mathbf{S}f(x^2) = \frac{f\left((x+\frac{i}{2})^2\right) + f\left((x-\frac{i}{2})^2\right)}{2},$$

with $i^2 = -1$. The operator $\mathbf{D}$ transforms a polynomial of degree n ($n \geq 1$) in x^2 into a polynomial of degree $n-1$ in x^2 and a polynomial of degree 0 into the zero polynomial. The operator $\mathbf{S}$ transforms a polynomial of degree n in x^2 into a polynomial of degree n in x^2.

The operators $\mathbf{D}$ and $\mathbf{S}$ fulfill the following properties.

Proposition 2.2. (See [27]) *The operators* $\mathbf{D}$ *and* $\mathbf{S}$ *satisfy the following product rules*

$$\mathbf{D}(fg) = \mathbf{D}f\mathbf{S}g + \mathbf{S}f\mathbf{D}g, \tag{2.5}$$

$$\mathbf{S}(fg) = -x^2\mathbf{D}f\mathbf{D}g + \mathbf{S}f\mathbf{S}g, \tag{2.6}$$

$$\mathbf{DS} = \mathbf{SD} - \frac{1}{2}\mathbf{D}^2, \tag{2.7}$$

$$\mathbf{S}^2 = -x^2\mathbf{D}^2 - \frac{1}{2}\mathbf{SD} + \mathbf{I}, \tag{2.8}$$

where $\mathbf{I}f = f$.

2.3.3. The Operators $\mathbb{D}_x$ and $\mathbb{S}_x$. We define the operator $\mathbb{D}_x$ (called divided-difference operator) and its companion operator $\mathbb{S}_x$ (called mean operator) as [5,9,11,29]

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})},$$

$$\mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

where $x(s)$ is a non-uniform lattice (see [9]). The operator $\mathbb{D}_x$ transforms a polynomial of degree n ($n \geq 1$) in $x(s)$ into a polynomial of degree $n-1$ in $x(s)$ and a polynomial of degree 0 into the zero polynomial. The operator $\mathbb{S}_x$ transforms a polynomial of degree n in $x(s)$ into a polynomial of degree n in $x(s)$.

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the product rules

$$\mathbb{D}_x\left(f(x(s))g(x(s))\right)=\mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s))+\mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{2.9}$$

$$\mathbb{S}_x\left(f(x(s))g(x(s))\right)=U_2(x(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s))+\mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{2.10}$$

$$\mathbb{D}_x\mathbb{S}_x f=\alpha\,\mathbb{S}_x\mathbb{D}_x f+U_1\,\mathbb{D}_x^2 f, \tag{2.11}$$

$$\mathbb{S}_x^2 f=U_1\mathbb{S}_x\,\mathbb{D}_x f+\alpha U_2\,\mathbb{D}_x^2 f+f, \tag{2.12}$$

where U_2 is a polynomial of degree 2

$$U_2(x(s))=(\alpha^2-1)\,x^2(s)+2\,\beta\,(\alpha+1)\,x(s)+\delta_x, \tag{2.13}$$

and δ_x is a constant depending on α, β and the initial values $x(0)$ and $x(1)$ of $x(s)$:

$$\delta_x=\frac{x^2(0)+x^2(1)}{4\alpha^2}-\frac{(2\alpha^2-1)}{2\alpha^2}x(0)\,x(1)-\frac{\beta\,(\alpha+1)}{\alpha^2}(x(0)+x(1))+\frac{\beta^2\,(\alpha+1)^2}{\alpha^2},$$

and

$$U_1(s):=U_1(x(s))=(\alpha^2-1)\,x(s)+\beta\,(\alpha+1),\quad U_2(s):=U_2(x(s)). \tag{2.14}$$

Note that

$$\mathbb{D}_x F_n(x(s))=\gamma_n F_{n-1}(x(s)),$$
$$\mathbb{S}_x F_n(x(s))=\alpha_n F_n(x(s))+\frac{\gamma_n}{2}\nabla x_{n+1}(\varepsilon)F_{n-1}(x(s)),$$

where $F_n(x(s))$ is a function defined in [24]. More properties of the non-uniform lattices $x(s)$, the properties of the divided-difference operator $\mathbb{D}_x$ and its companion $\mathbb{S}_x$ can be found in [10–12, 16, 24] : $x(s)$ satisfies the conditions

$$x(s+k)-x(s)=\gamma_k\nabla x_{k+1}(s), \tag{2.15}$$

$$\frac{x(s+k)-x(s)}{2}=\alpha_k x_k(s)+\beta_k, \tag{2.16}$$

for $k=0,1,\ldots$, with

$$\alpha_0=1,\alpha_1=\alpha,\beta_0=0,\beta_1=\beta,\gamma_0=0,\gamma_1=1,$$

and the sequences (α_k), (β_k), (γ_k) satisfy the following relations

$$\alpha_{k+1}-2\alpha\alpha_k+\alpha_{k-1}=0,$$
$$\beta_{k+1}-2\beta_k+\beta_{k-1}=2\beta\alpha_k,$$
$$\gamma_{k+1}-\gamma_{k-1}=2\alpha_k,$$

for $k=0,1,\ldots$.

3. Non-linear Characterization for Meixner–Pollaczek and Continuous Hahn Polynomials

The Meixner–Pollaczek polynomials $P_n^{(\lambda)}(x;\varphi)$ and the Continuous Hahn polynomials $p_n(x;a,b,c,d)$, respectively, have the hypergeometric representation (see [18]):

$$P_n^{(\lambda)}(x;\varphi) = \frac{(2\lambda)_n}{n!} e^{in\varphi} {}_2F_1\left(\left. \begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \right| 1 - e^{-2i\varphi} \right), \tag{3.1}$$

$$\frac{p_n(x;a,b,c,d)}{(a+d)_n(a+c)_n} = \frac{i^n}{n!} {}_3F_2\left(\left. \begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} \right| 1 \right). \tag{3.2}$$

They are known to satisfy the second-order difference equation (see [30])

$$\phi(x)\mathcal{D}^2 y(x) + \psi(x)\mathcal{S}\mathcal{D} y(x) + \lambda_n y(x) = 0, \tag{3.3}$$

where ϕ and ψ are polynomials of degree 2 and 1, respectively, and λ is a constant depending on the degree of the polynomial solution and the parameters involved in the polynomials.

Note that for the Meixner–Pollaczek polynomials, we have (see [30])

$$\phi(x) = i(\lambda \sin\varphi - x\cos\varphi), \tag{3.4}$$

$$\psi(x) = 2(\lambda\cos\varphi + x\sin\varphi), \tag{3.5}$$

and

$$\lambda_n = -2in\sin\varphi,$$

and for the Continuous Hahn polynomials we have (see [30])

$$\begin{aligned} \phi(x) &= -x^2 + \frac{i}{2}(a+b-c-d)\,x + \frac{1}{2}(ab+cd), \\ \psi(x) &= -i\,(a+b+c+d)\,x + cd - ab, \end{aligned}$$

and

$$\lambda = \lambda_n = -n(n+a+b+c+d-1).$$

Theorem 3.1. (Non-linear characterization) *Let $(P_n)_{n\geq 0}$ be a sequence of classical orthogonal polynomials on non-uniform lattice. Then, for $n \geq 1$, $P_n(x)$ and $P_{n-1}(x)$ satisfy*

$$\begin{aligned} &\phi(x)\Big[P_n(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + P_{n-1}(x)\mathcal{S}\mathcal{D}P_n(x)\Big] \\ &\quad + \psi(x)\Big[P_n(x)\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2 P_n(x)\Big] \\ &= \Bigg[(\psi_1 + 2i\phi_2)x + \psi_0 + \psi_1\left(\frac{B_n}{2A_n} - \frac{B_{n-1}}{2A_{n-1}}\right) \\ &\quad + i\phi_2\left(n\frac{B_n}{A_n} - (n-2)\frac{B_{n-1}}{A_{n-1}}\right)\Bigg] P_n(x)P_{n-1}(x) \\ &\quad + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)P_n^2(x) - \frac{C_n}{A_n}(\psi_1 + (2n-1)i\phi_2)P_{n-1}^2(x). \end{aligned} \tag{3.6}$$

Furthermore, if $(Q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials such that $Q_0(x) = P_0(x)$ and, for $n \geq 1$, $Q_n(x)$ and $Q_{n-1}(x)$ satisfy (3.6). Then $Q_n(x) = P_n(x)$, for all $n \geq 0$.

Proof. Using the fact that the sequence $(P_n)_{n\geq 0}$ is a classical orthogonal polynomial sequence, for all non-negative integer n, $P_{n+1}(x)$ satisfies (3.3), namely:

$$\phi(x)\mathcal{D}^2 P_{n+1}(x) + \psi(x)\mathcal{S}\mathcal{D}P_{n+1}(x) + \lambda_{n+1}P_{n+1}(x) = 0, \tag{3.7}$$

with

$$\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0;\quad \psi(x) = \psi_1 x + \psi_0;\quad \lambda_n = n(n-1)\phi_2 - in\psi_1.$$

In (1.1), using the relations (2.1), (2.2), (2.3) and (2.4), we obtain:

$$\mathcal{D}^2 P_{n+1}(x) = 2iA_n\mathcal{S}\mathcal{D}P_n(x) + (A_n x + B_n)\mathcal{D}^2 P_n(x) - C_n\mathcal{D}^2 P_{n-1}(x) \tag{3.8}$$

and

$$\mathcal{S}\mathcal{D}P_{n+1}(x) = 2iA_n\mathcal{S}^2 P_n(x) - iA_nP_n(x) + (A_n x + B_n)\mathcal{S}\mathcal{D}P_n(x) - C_n\mathcal{S}\mathcal{D}P_{n-1}(x). \tag{3.9}$$

Using (1.1), (3.8) and (3.9) to replace $\mathcal{D}^2 P_{n+1}(x)$, $\mathcal{S}\mathcal{D}P_{n+1}(x)$ and $P_{n+1}(x)$ in (3.7), we obtain:

$$\begin{aligned}\phi(x)\mathcal{S}\mathcal{D}P_n(x) + \psi(x)\mathcal{S}^2 P_n(x) &= -\frac{C_n}{2iA_n}(\lambda_{n-1} - \lambda_{n+1})P_{n-1}(x)\\ &+ \left[\frac{1}{2}\psi(x) + \left(\frac{1}{2i}x + \frac{B_n}{2iA_n}\right)(\lambda_n - \lambda_{n+1})\right]P_n(x).\end{aligned} \tag{3.10}$$

For $n \geq 2$, we replace n by $n-1$ in (3.10) and obtain:

$$\begin{aligned}&\phi(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + \psi(x)\mathcal{S}^2 P_{n-1}(x)\\ &= -\frac{C_{n-1}}{2iA_{n-1}}(\lambda_{n-2} - \lambda_n)P_{n-2}(x)\\ &\quad + \left[\frac{1}{2}\psi(x) + \left(\frac{1}{2i}x + \frac{B_{n-1}}{2iA_{n-1}}\right)(\lambda_{n-1} - \lambda_n)\right]P_{n-1}(x).\end{aligned} \tag{3.11}$$

We replace again n by $n-1$ in (1.1) and use the resulting relation to replace $P_{n-2}(x)$ in (3.11) to obtain:

$$\begin{aligned}&\phi(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + \psi(x)\mathcal{S}^2 P_{n-1}(x)\\ &= -\frac{1}{2iA_{n-1}}(\lambda_{n-2} - \lambda_n)P_n(x)\\ &\quad + \left[\frac{1}{2}\psi(x) + \left(\frac{1}{2i}x + \frac{B_{n-1}}{2iA_{n-1}}\right)(\lambda_{n-1} - \lambda_{n-2})\right]P_{n-1}(x).\end{aligned} \tag{3.12}$$

If we multiply (3.10) by $P_{n-1}(x)$, (3.12) by $P_n(x)$ and add the resulting expression, we get:

$$\begin{aligned}
&\phi(x)\left[P_n(x)\mathcal{S}\mathcal{D}P_{n-1}(x)+P_{n-1}(x)\mathcal{S}\mathcal{D}P_n(x)\right]\\
&\quad+\psi(x)\left[P_n(x)\mathcal{S}^2P_{n-1}(x)+P_{n-1}(x)\mathcal{S}^2P_n(x)\right]\\
&\quad=\Bigg[(\psi_1+2i\phi_2)x+\psi_0+\psi_1\left(\frac{B_n}{2A_n}-\frac{B_{n-1}}{2A_{n-1}}\right)\\
&\quad+i\phi_2\left(n\frac{B_n}{A_n}-(n-2)\frac{B_{n-1}}{A_{n-1}}\right)\Bigg]P_n(x)P_{n-1}(x)\\
&\quad+\frac{1}{A_{n-1}}(\psi_1+(2n-3)i\phi_2)P_n^2(x)-\frac{C_n}{A_n}(\psi_1+(2n-1)i\phi_2)P_{n-1}^2(x).
\end{aligned}$$

This proves the first part of Theorem 3.1.

Now, we prove the second part.

Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of a quadratic variable such that $Q_0(x)=P_0(x)$ and, for $n\geq 1$, $Q_n(x)$ and $Q_{n-1}(x)$ satisfy

$$\begin{aligned}
&\phi(x)\left[Q_n(x)\mathcal{S}\mathcal{D}Q_{n-1}(x)+Q_{n-1}(x)\mathcal{S}\mathcal{D}Q_n(x)\right]\\
&\quad+\psi(x)\left[Q_n(x)\mathcal{S}^2Q_{n-1}(x)+Q_{n-1}(x)\mathcal{S}^2Q_n(x)\right]\\
&=\left[(\psi_1+2i\phi_2)x+\psi_0+\psi_1\left(\frac{B_n}{2A_n}-\frac{B_{n-1}}{2A_{n-1}}\right)+i\phi_2\left(n\frac{B_n}{A_n}-(2-n)\frac{B_{n-1}}{A_{n-1}}\right)\right]\\
&\quad Q_n(x)Q_{n-1}(x)\\
&\quad+\frac{1}{A_{n-1}}(\psi_1+(2n-3)i\phi_2)Q_n^2(x)-\frac{C_n}{A_n}(\psi_1+(2n-1)i\phi_2)Q_{n-1}^2(x). \qquad (3.13)
\end{aligned}$$

Let b_n be the leading coefficient of $Q_n(x)$. We shall firstly show by induction that $k_n=b_n$ for all $n\geq 0$. We have $b_0=k_0$ and we assume that $n\geq 1$ and $b_{n-1}=k_{n-1}$. If we compare the coefficients of x^{2n} in (3.13), we find that we must consider two cases according as the degree of ϕ is less than two or equal to two.

- If the degree of ϕ is less than two then we have

$$2\psi_1 b_n k_{n-1}=\psi_1 b_n k_{n-1}+\frac{\psi_1}{A_{n-1}}(b_n)^2$$

  and $b_n\neq 0$ implies that for the quadratic or q-quadratic variable, we have $b_n=A_{n-1}k_{n-1}=k_n$.
- If the degree of ϕ is equal to two, then we have

$$\begin{aligned}
&\phi_2\left((n-1)ib_nk_{n-1}+nib_nk_{n-1}\right)+2\psi_1b_nk_{n-1}\\
&\quad=(\psi_1+2i\phi_2)b_nk_{n-1}+\frac{1}{A_{n-1}}\left(\psi_1+(2n-3)i\phi_2\right)b_n^2,
\end{aligned}$$

  and the regularity of the corresponding linear functional with respect to the sequence $(Q_n)_{n\geq 0}$ implies that $\psi_1+(2n-3)i\phi_2\neq 0$ and $b_n\neq 0$ we have $b_n=A_{n-1}k_{n-1}=k_n$.

We have by assumption $Q_0(x)=P_0(x)$. Assume further that $n\geq 1$ and $Q_{n-1}(x)=P_{n-1}(x)$ but $Q_n(x)\neq P_n(x)$.

Then $Q_n(x) = P_n(x) + g(x)$ where $g(x) = c(x^r + \cdots)$, $c \neq 0$. Since $Q_n(x)$ and $P_n(x)$ have the same degree and the same leading coefficient, we must have $r < n$. From (3.13), we get

$$\begin{aligned}
&\phi(x)\left[(P_n(x)+g(x))\mathcal{S}\mathcal{D}P_{n-1}(x)+P_{n-1}(x)\mathcal{S}\mathcal{D}(P_n(x)+g(x))\right]\\
&\quad+\psi(x)\left[(P_n(x)+g(x))\mathcal{S}^2P_{n-1}(x)+P_{n-1}(x)\mathcal{S}^2(P_n(x)+g(x))\right]\\
&=\Bigg[(\psi_1+2i\phi_2)x+\psi_0+\psi_1\left(\frac{B_n}{2A_n}-\frac{B_{n-1}}{2A_{n-1}}\right)\\
&\quad+i\phi_2\left(n\frac{B_n}{A_n}-(2-n)\frac{B_{n-1}}{A_{n-1}}\right)\Bigg](P_n(x)+g(x))P_{n-1}(x)\\
&\quad+\frac{1}{A_{n-1}}(\psi_1+(2n-3)i\phi_2)(P_n(x)+g(x))^2\\
&\quad-\frac{C_n}{A_n}(\psi_1+(2n-1)i\phi_2)P_{n-1}^2(x).
\end{aligned}$$

Using the fact that $P_n(x)$ and $P_{n-1}(x)$ satisfy (3.6) we obtain

$$\begin{aligned}
&\phi(x)\left[g(x)\mathcal{S}\mathcal{D}P_{n-1}(x)+P_{n-1}(x)\mathcal{S}\mathcal{D}g(x)\right]\\
&\quad+\psi(x)\left[g(x)\mathcal{S}^2P_{n-1}(x)+P_{n-1}(x)\mathcal{S}^2g(x)\right]\\
&=\Bigg[(\psi_1+2i\phi_2)x+\psi_0+\psi_1\left(\frac{B_n}{2A_n}-\frac{B_{n-1}}{2A_{n-1}}\right)\\
&\quad+i\phi_2\left(n\frac{B_n}{A_n}-(2-n)\frac{B_{n-1}}{A_{n-1}}\right)\Bigg]g(x)P_{n-1}(x)\\
&\quad+\frac{1}{A_{n-1}}(\psi_1+(2n-3)i\phi_2)(2P_n(x)g(x)+g(x)^2).
\end{aligned}\tag{3.14}$$

We compare the coefficients of x^{n+r} in (3.14). Let us consider two cases:

1. If the degree of ϕ is less than two, then we get

$$2\psi_1cb_{n-1}=\psi_1cb_{n-1}+2\frac{b_n}{A_{n-1}}c\psi_1$$

which is equivalent to

$$2\psi_1cb_{n-1}=\psi_1cb_{n-1}+2c\psi_1b_{n-1}.$$

Then, the fact that $\psi_1b_{n-1}\neq 0$ implies that this is impossible if $c\neq 0$.

2. If the degree of ϕ is equal to two, then we get

$$\begin{aligned}
&\phi_2\left(ci(n-1)b_{n-1}+ricb_{n-1}\right)+2c\psi_1b_{n-1}\\
&\quad=(\psi_1+2i\phi_2)cb_{n-1}+\frac{2}{A_{n-1}}\left(\psi_1+(2n-3)i\phi_2\right)cb_n,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\phi_2\left(ci(n-1)b_{n-1}+ricb_{n-1}\right)+2c\psi_1b_{n-1}\\
&\quad=(\psi_1+2i\phi_2)cb_{n-1}+2\left(\psi_1+(2n-3)i\phi_2\right)cb_{n-1}.
\end{aligned}$$

The regularity of the corresponding linear functional with respect to the sequence (P_n) implies that $\psi_1+(3n-3-r)i\phi_2\neq 0$ and the previous equation is impossible if $c\neq 0$.

The proof is therefore completed. □

The following corollaries give explicit coefficients for the non-linear characterization of the Meixner–Pollaczek and the Continuous Dual Hahn polynomials.

Corollary 3.2. *The Meixner–Pollaczek polynomials are characterized by the following non-linear difference equation*

$$\begin{aligned} &i(\lambda\sin\varphi - x\cos\varphi)\left[P_n(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + P_{n-1}(x)\mathcal{S}\mathcal{D}P_n(x)\right]\\ &\quad + 2(\lambda\cos\varphi + x\sin\varphi)\left[P_n(x)\mathcal{S}^2P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2P_n(x)\right]\\ &= \left[2\sin\varphi x + 3\lambda\cos\varphi\right]P_n(x)P_{n-1}(x) + nP_n^2(x) - (n+2\lambda-1)P_{n-1}^2(x). \end{aligned}$$

Corollary 3.3. *The Continuous Hahn polynomials are characterized by the following non-linear difference equation*

$$\begin{aligned} &\left(-x^2 + \frac{i}{2}(a+b-c-d)x + \frac{1}{2}(ab+cd)\right)\left[P_n(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + P_{n-1}(x)\mathcal{S}\mathcal{D}P_n(x)\right]\\ &\quad + \left(-i(a+b+c+d+2)x + cd - ab\right)\left[P_n(x)\mathcal{S}^2P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2P_n(x)\right]\\ &= -\frac{n(b+c+n-1)(b+d+n-1)}{2n+a+b+c+d-2}P_{n-1}^2(x)\\ &\quad + \left(-i(a+b+c+d+2)x + D_n\right)P_n(x)P_{n-1}(x)\\ &\quad + \frac{(n-2+a+b+c+d)(n-1+a+c)(n-1+a+d)}{2n+a+b+c+d-2}P_n^2(x), \end{aligned}$$

where D_n depends on n, a, b, c and d.

4. Non-linear Characterization for Wilson and Continuous Dual Hahn Polynomials

The Wilson polynomials $W_n(x^2; a, b, c, d)$ and Continuous Dual Hahn polynomials $S_n(x^2; a, b, c)$, respectively, have the hypergeometric representation (see [18]):

$$\frac{W_n(x^2; a, b, c, d)}{(a+b, a+c, a+d)_n} = {}_4F_3\left(\left.\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}\,\right|\, 1\right), \tag{4.1}$$

$$\frac{S_n(x^2; a, b, c)}{(a+b, a+c)_n} = {}_3F_2\left(\left.\begin{matrix} -n, a-ix, a+ix \\ a+b, a+c \end{matrix}\,\right|\, 1\right). \tag{4.2}$$

They are known to satisfy the second-order divided-difference equation (see [27])

$$\phi(x^2)\mathbf{D}^2y(x^2) + \psi(x^2)\mathbf{SD}y(x^2) + \lambda_n y(x^2) = 0, \tag{4.3}$$

and these two families satisfy the three-term recurrence relation

$$P_{n+1}(x^2) = (A_n x^2 + B_n)P_n(x^2) - C_n P_{n-1}(x^2), \quad P_{-1}(x^2) = 0. \tag{4.4}$$

Theorem 4.1. (Non-linear characterization) *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of classical orthogonal polynomials on a non-uniform lattice. Then, for $n\geq 1$, $P_n(x^2)$ and $P_{n-1}(x^2)$ satisfy*

$$\begin{aligned}
&\phi(x^2)\Big[P_n(x^2)\mathbf{DS}P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\Big]\\
&\quad+\psi(x^2)\Big[P_n(x^2)\mathbf{S}^2P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{S}^2P_n(x^2)\Big]\\
&=\left[(\psi_1+2\phi_2)x^2+\psi_0+\psi_1\left(\frac{B_n}{A_n}-\frac{B_{n-1}}{A_{n-1}}\right)+\phi_2\left(n\frac{B_n}{A_n}+(2-n)\frac{B_{n-1}}{A_{n-1}}\right)\right]\\
&\quad\times P_n(x^2)P_{n-1}(x^2)+\frac{1}{A_{n-1}}(\psi_1+(2n-3)\phi_2)P_n^2(x^2)\\
&\quad-\frac{C_n}{A_n}(\psi_1+(2n-1)\phi_2)P_{n-1}^2(x^2).
\end{aligned}\tag{4.5}$$

Furthermore, if $(Q_n(x^2))_{n\in\mathbb{N}}$ is a sequence of polynomials such that $Q_0(x^2)=P_0(x^2)$ and, for $n\geq 1$, $Q_n(x)$ and $Q_{n-1}(x)$ satisfy (4.5). Then $Q_n(x^2)=P_n(x^2)$, for all $n\geq 0$.

Proof. For all integers n, $P_{n+1}(x^2)$ satisfies (4.3), namely:

$$\phi(x^2)\mathbf{D}^2P_{n+1}(x^2)+\psi(x^2)\mathbf{SD}P_{n+1}(x^2)+\lambda_{n+1}P_{n+1}(x^2)=0,\tag{4.6}$$

with

$$\phi(x^2)=\phi_2x^4+\phi_1x^2+\phi_0;\quad \psi(x^2)=\psi_1x^2+\psi_0;\quad \lambda_n=-n(n-1)\phi_2-n\psi_1.$$

From (4.4), using the relations (2.5), (2.6), (2.7) and (2.8), we obtain:

$$\mathbf{D}^2P_{n+1}(x^2)=2A_n\mathbf{DS}P_n(x^2)+(A_nx^2+B_n)\mathbf{D}^2P_n(x^2)-C_n\mathbf{D}^2P_{n-1}(x^2)\tag{4.7}$$

and

$$\mathbf{SD}P_{n+1}(x^2)=2A_n\mathbf{S}^2P_n(x^2)-A_nP_n(x^2)+(A_nx^2+B_n)\mathbf{SD}P_n(x^2)-C_n\mathbf{SD}P_{n-1}(x^2).\tag{4.8}$$

We use (4.4), (4.7) and (4.8) to replace $\mathbf{D}^2P_{n+1}(x^2)$, $\mathbf{SD}P_{n+1}(x^2)$ and $P_{n+1}(x^2)$ in (4.6); we obtain:

$$\begin{aligned}
&\phi(x^2)\mathbf{DS}P_n(x^2)+\psi(x^2)\mathbf{S}^2P_n(x^2)\\
&\quad=-\frac{C_n}{2A_n}(\lambda_{n-1}-\lambda_{n+1})P_{n-1}(x^2)\\
&\qquad+\left[\frac{1}{2}\psi(x^2)+\left(\frac{1}{2}x^2+\frac{B_n}{2A_n}\right)(\lambda_n-\lambda_{n+1})\right]P_n(x^2).
\end{aligned}\tag{4.9}$$

For $n\geq 2$, we replace n by $n-1$ in (4.9) and we obtain:

$$\begin{aligned}
&\phi(x^2)\mathbf{DS}P_{n-1}(x^2)+\psi(x^2)\mathbf{S}^2P_{n-1}(x^2)\\
&\quad=-\frac{C_{n-1}}{2A_{n-1}}(\lambda_{n-2}-\lambda_n)P_{n-2}(x^2)\\
&\qquad+\left[\frac{1}{2}\psi(x^2)+\left(\frac{1}{2}x^2+\frac{B_{n-1}}{2A_{n-1}}\right)(\lambda_{n-1}-\lambda_n)\right]P_{n-1}(x^2).
\end{aligned}\tag{4.10}$$

We replace again n by $n-1$ in (4.4) and we use the resulting relation to replace $P_{n-2}(x^2)$ in (4.10) to obtain:

$$\begin{aligned}
&\phi(x^2)\mathbf{DS}P_{n-1}(x^2) + \psi(x^2)\mathbf{S}^2P_{n-1}(x^2) \\
&\quad = \frac{1}{2A_{n-1}}(\lambda_{n-2}-\lambda_n)P_n(x^2) \\
&\qquad + \left[\frac{1}{2}\psi(x^2) + \left(\frac{1}{2}x^2 + \frac{B_{n-1}}{2A_{n-1}}\right)(\lambda_{n-1}-\lambda_{n-2})\right]P_{n-1}(x^2). \quad (4.11)
\end{aligned}$$

If we multiply (4.9) by $P_{n-1}(x^2)$, (4.11) by $P_n(x^2)$ and add the resulting expressions, we obtain:

$$\begin{aligned}
&\phi(x^2)\left[P_n(x^2)\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\right] \\
&\quad + \psi(x^2)\left[P_n(x^2)\mathbf{S}^2P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2P_n(x^2)\right] \\
&\quad = \left[(\psi_1+2\phi_2)x^2 + \psi_0 + \psi_1\left(\frac{B_n}{A_n} - \frac{B_{n-1}}{A_{n-1}}\right) + \phi_2\left(n\frac{B_n}{A_n} + (2-n)\frac{B_{n-1}}{A_{n-1}}\right)\right] \\
&\quad \times P_n(x^2)P_{n-1}(x^2) + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)\phi_2)P_n^2(x^2) \\
&\quad - \frac{C_n}{A_n}(\psi_1 + (2n-1)\phi_2)P_{n-1}^2(x^2).
\end{aligned}$$

This proves the first part of Theorem 4.1.

Now, we prove the second part.

Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of a quadratic variable such that $Q_0(x^2) = P_0(x^2)$ and, for $n \geq 1$, $Q_n(x^2)$ and $Q_{n-1}(x^2)$ satisfy (4.5). Let b_n be the leading coefficient of $Q_n(x^2)$. We shall first show by induction that $k_n = b_n$ for all $n \geq 0$. We have $b_0 = k_0$ and we assume that $n \geq 1$ and $b_{n-1} = k_{n-1}$. If we compare the coefficients of x^{4n} in (4.5), we find that we must consider two cases whether the degree of ϕ is less than two or equal to two.

- If the degree of ϕ is less than two then, we have

$$2\psi_1 b_n k_{n-1} = \psi_1 b_n k_{n-1} + \frac{\psi_1}{A_{n-1}}(b_n)^2$$

  and $b_n \neq 0$ implies that for the quadratic or q-quadratic variable, we have $b_n = A_{n-1}k_{n-1} = k_n$.
- If the degree of ϕ is equal to two then, we have

$$\begin{aligned}
&\phi_2\left((n-1)b_nk_{n-1} + nb_nk_{n-1}\right) + 2\psi_1 b_n k_{n-1} \\
&\quad = (\psi_1+2\phi_2)b_nk_{n-1} + \frac{1}{A_{n-1}}\left(\psi_1 + (2n-3)\phi_2\right)b_n^2,
\end{aligned}$$

  and the regularity of the corresponding linear functional with respect to the sequence $(Q_n)_{n\geq 0}$ implies that $\psi_1 + (2n-3)\phi_2 \neq 0$ and $b_n \neq 0$ we have $b_n = A_{n-1}k_{n-1} = k_n$.

We have by assumption $Q_0(x^2) = P_0(x^2)$. Assume further that $n \geq 1$ and $Q_{n-1}(x^2) = P_{n-1}(x^2)$ but $Q_n(x^2) \neq P_n(x^2)$.

Then $Q_n(x^2) = P_n(x^2) + g(x^2)$ where $g(x^2) = c(x^{2r} + \cdots)$, $c \neq 0$. Since $Q_n(x^2)$ and $P_n(x^2)$ have the same degree and the same leading coefficient, we must have $r < n$. From (4.5), we get

$$\begin{aligned}
&\phi(x^2)\left[(P_n(x^2)+g(x^2))\mathbf{DS}P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{DS}(P_n(x^2)+g(x^2))\right]\\
&\quad+\psi(x^2)\left[(P_n(x^2)+g(x^2))\mathbf{S}^2P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{S}^2(P_n(x^2)+g(x^2))\right]\\
&\ =\Bigg[(\psi_1+2\phi_2)x^2+\psi_0+\psi_1\left(\frac{B_n}{A_n}-\frac{B_{n-1}}{A_{n-1}}\right)\\
&\quad+\phi_2\left(n\frac{B_n}{A_n}+(2-n)\frac{B_{n-1}}{A_{n-1}}\right)\Bigg](P_n(x^2)+g(x^2))P_{n-1}(x^2)\\
&\quad+\frac{1}{A_{n-1}}(\psi_1+(2n-3)\phi_2)(P_n(x^2)+g(x^2))^2\\
&\quad-\frac{C_n}{A_n}(\psi_1+(2n-1)\phi_2)P_{n-1}^2(x^2).
\end{aligned}$$

Using the fact that $P_n(x)$ and $P_{n-1}(x)$ satisfy (4.5), we obtain

$$\begin{aligned}
&\phi(x^2)\left[g(x^2)\mathbf{DS}P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{DS}g(x^2)\right]\\
&\quad+\psi(x^2)\left[g(x^2)\mathbf{S}^2P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{S}^2g(x^2)\right]\\
&\ =\left[(\psi_1+2\phi_2)x^2+\psi_0+\psi_1\left(\frac{B_n}{A_n}-\frac{B_{n-1}}{A_{n-1}}\right)+\phi_2\left(n\frac{B_n}{A_n}+(2-n)\frac{B_{n-1}}{A_{n-1}}\right)\right]\\
&\quad\times g(x^2)P_{n-1}(x^2)+\frac{1}{A_{n-1}}(\psi_1+(2n-3)\phi_2)(2P_n(x)g(x)+g(x)^2).
\end{aligned}\tag{4.12}$$

We compare the coefficients of x^{2n+2r} in (4.12) and consider two cases:

1. If the degree of ϕ is less than two, then
$$2\psi_1 cb_{n-1} = \psi_1 cb_{n-1} + 2\frac{b_n}{A_{n-1}}c\psi_1$$
which is equivalent to
$$2\psi_1 cb_{n-1} = \psi_1 cb_{n-1} + 2c\psi_1 b_{n-1}.$$
Then, the fact that $\psi_1 b_{n-1} \neq 0$ implies that this is impossible if $c \neq 0$.
2. If the degree of ϕ is equal to two, then
$$\begin{aligned}&\phi_2\left(c(n-1)b_{n-1}+rcb_{n-1}\right)+2c\psi_1 b_{n-1}\\ &\quad=(\psi_1+2\phi_2)cb_{n-1}+\frac{2}{A_{n-1}}\left(\psi_1+(2n-3)\phi_2\right)cb_n\end{aligned}$$
which is equivalent to
$$\begin{aligned}&\phi_2\left(c(n-1)b_{n-1}+rcb_{n-1}\right)+2c\psi_1 b_{n-1}\\ &\quad=(\psi_1+2\phi_2)cb_{n-1}+2\left(\psi_1+(2n-3)\phi_2\right)cb_{n-1}.\end{aligned}$$
The regularity of the corresponding linear functional with respect to the sequence $(P_n)_{n\geq 0}$ implies that $\psi_1 + (3n-3-r)\phi_2 \neq 0$ and the previous equality is impossible if $c \neq 0$.

$\square$

Corollary 4.2. *The Wilson polynomials are characterized by the following non-linear difference equation*

$$\begin{aligned}
&\phi(x^2)\left[P_n(x^2)\mathbf{DS}P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\right]\\
&\quad+\psi(x^2)\left[P_n(x^2)\mathbf{S}^2P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{S}^2P_n(x^2)\right]\\
&\quad=-\left[\frac{n(b+c+n-1)(b+d+n-1)(c+d+n-1)}{a+b+c+d+2n-2}\right]P_{n-1}^2(x^2)\\
&\quad+\left[\frac{(a+b+c+d+n-2)(a+b+n-1)(a+c+n-1)(a+d+n-1)}{a+b+c+d+2n-2}\right]P_n^2(x^2)\\
&\quad+\left[(a+b+c+d+2)x^2+D_n\right]P_n(x^2)P_{n-1}(x^2),
\end{aligned}$$

where D_n depends on n, a, b, c and d.

Corollary 4.3. *The Continuous Dual Hahn polynomials are characterized by the following non-linear difference equation*

$$\begin{aligned}
&(-(a+b+c)x^2+abc)\left[P_n(x^2)\mathbf{DS}P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\right]\\
&\quad+(x^2-ab-ac-bc)\left[P_n(x^2)\mathbf{S}^2P_{n-1}(x^2)+P_{n-1}(x^2)\mathbf{S}^2P_n(x^2)\right]\\
&\quad=\left(x^2+D_n\right)P_n(x^2)P_{n-1}(x^2)-(a+b+n-1)(a+c+n-1)P_n^2(x^2)\\
&\quad+n(b+c+n-1)P_{n-1}^2(x^2),
\end{aligned}$$

where D_n depends on n, a, b, c and d.

5. Non-linear Characterization for Orthogonal Polynomials on q-Quadratic Lattices

A family $p_n(x)$ of polynomials of degree n is a family of classical q-quadratic orthogonal polynomials (also known as orthogonal polynomials on non-uniform lattices) if it is the solution of a divided-difference equation of the type (see [8,9])

$$\phi(x(s))\mathbb{D}_x^2y(x(s))+\psi(x(s))\mathbb{S}_x\mathbb{D}_xy(x(s))+\lambda_ny(x(s))=0, \tag{5.1}$$

where ϕ is a polynomial of maximal degree two and ψ is a polynomial of exact degree one, λ_n is a constant depending on the integer n and the leading coefficients ϕ_2 and ψ_1 of ϕ and ψ:

$$\lambda_n=-\gamma_n(\gamma_{n-1}\phi_2+\alpha_{n-1}\psi_1)$$

and $x(s)$ is a non-uniform lattice defined by

$$x(s)=c_1q^s+c_2q^{-s}+c_3,\quad c_1c_2\neq 0, \tag{5.2}$$

and the sequences (α_n) and (γ_n) are given explicitly by :

$$\alpha_n=\frac{1}{2}\left(q^{\frac{n}{2}}+q^{-\frac{n}{2}}\right),\gamma_n=\frac{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}.$$

5.1. General Theorem

In this section, we state and prove a non-linear characterization result for classical orthogonal polynomials on non-uniform lattices. The result is stated in the following theorem.

Theorem 5.1. *Let $(P_n)_{n\geq 0}$ be a sequence of classical orthogonal polynomials on a non-uniform lattice. Then, for $n \geq 1$, $P_n(x(s))$ and $P_{n-1}(x(s))$ satisfy*

$$\begin{aligned}
&\psi(x(s))\Big[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\Big] \\
&\quad + \phi(x(s))\Big[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\Big] \\
&= \Big[(D_{n-1} + D_n - G_{n-1}A_{n-1})x(s) + E_{n-1} \\
&\quad - G_{n-1}B_{n-1} + E_n\Big] P_n(x(s))P_{n-1}(x(s)) \\
&\quad + G_{n-1}\left(P_n(x(s))\right)^2 - C_nG_n\left(P_{n-1}(x(s))\right)^2, \qquad (5.3)
\end{aligned}$$

where

$$\begin{aligned}
D_n &= \frac{1}{2}(\lambda_n - \lambda_{n+1} + \psi_1), \quad E_n = \frac{1}{2}\left((\lambda_n - \lambda_{n+1})\frac{B_n}{A_n} + \psi_0\right), \\
G_n &= \frac{1}{2A_n}\left(\lambda_{n-1} - \lambda_{n+1}\right). \qquad (5.4)
\end{aligned}$$

Furthermore, if $(Q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials a on non-uniform lattice such that $Q_0(x) = P_0(x)$ and, for $n \geq 1$, $Q_n(x(s))$ and $Q_{n-1}(x(s))$ satisfy (5.3). Then $Q_n(x(s)) = P_n(x(s))$, for all $n \geq 0$.

Proof. Using the fact that $(P_n(x(s))_{n\geq 0}$ is a classical q-orthogonal polynomial sequence on non-uniform lattice, substituting n by $n+1$ in (5.1) we obtain

$$\phi(x(s))\mathbb{D}_x^2 P_{n+1}(x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x P_{n+1}(x(s)) + \lambda_{n+1}P_{n+1}(x(s)) = 0. \qquad (5.5)$$

In (1.1), using the product rules given in [12, page 407], in [11, pages 741-742] or in [10, page 4], we obtain:

$$\begin{aligned}
\mathbb{D}_x^2 P_{n+1}(x(s)) &= \left[A_n\alpha^2 x(s) + A_n\beta(\alpha+1) + B_n\right]\mathbb{D}_x^2 P_n(x(s)) \\
&\quad + 2A_n\mathbb{D}_x\mathbb{S}_x P_n(x(s)) - A_nU_1(x(s))\mathbb{D}_x^2 P_n(x(s)) \\
&\quad - C_n\mathbb{D}_x^2 P_{n-1}(x(s)) \qquad (5.6)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{S}_x\mathbb{D}_x P_{n+1}(x(s)) &= \left[A_n\alpha^2 x(s) + A_n\beta(\alpha+1) + B_n\right]\mathbb{S}_x\mathbb{D}_x P_n(x(s)) \\
&\quad + 2A_n\mathbb{S}_x^2 P_n(x(s)) - A_nU_1(x(s))\mathbb{S}_x\mathbb{D}_x P_n(x(s)) \\
&\quad - A_nP_n(x(s)) - C_n\mathbb{S}_x\mathbb{D}_x P_{n-1}(x(s)). \qquad (5.7)
\end{aligned}$$

Using (1.1), (5.6) and (5.7) to replace $\mathbb{D}_x^2 P_{n+1}(x(s))$, $\mathbb{S}_x\mathbb{D}_x P_{n+1}(x(s))$ and $P_{n+1}(x(s))$ in (5.5), we obtain:

$$\begin{aligned}&\psi(x(s))\mathbb{S}_x^2 P_n(x(s)) + \phi(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\\ &\quad= -\frac{C_n}{2A_n}\left[\lambda_{n-1} - \lambda_{n+1}\right] P_{n-1}(x(s))\\ &\qquad+ \frac{1}{2}\left[(\lambda_n - \lambda_{n+1} + \psi_1)x(s) + (\lambda_n - \lambda_{n+1})\frac{B_n}{A_n} + \psi_0\right] P_n(x(s)),\ \forall n \geq 1,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\psi(x(s))\mathbb{S}_x^2 P_n(x(s)) + \phi(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\\ &\quad= \left[D_n x(s) + E_n\right] P_n(x(s)) - C_n G_n P_{n-1}(x(s)),\ \forall n \geq 1,\end{aligned} \tag{5.8}$$

where D_n, E_n and G_n are defined in (5.4). For $n \geq 2$, we replace n by $n-1$ in (5.8) and we obtain:

$$\begin{aligned}&\psi(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + \phi(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s))\\ &\quad= \left[D_{n-1}x(s) + E_{n-1}\right] P_{n-1}(x(s)) - C_{n-1}G_{n-1}P_{n-2}(x(s)),\ \forall n \geq 2.\end{aligned} \tag{5.9}$$

We also replace n by $n-1$ in (1.1) and use the resulting relation to replace $P_{n-2}(x(s))$ in (5.9) to obtain:

$$\begin{aligned}&\psi(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + \phi(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) = G_{n-1}P_n(x(s))\\ &\quad+ \left[(D_{n-1} - G_{n-1}A_{n-1})x(s) + E_{n-1} - G_{n-1}B_{n-1}\right] P_{n-1}(x(s)),\ \forall n \geq 1.\end{aligned} \tag{5.10}$$

If we multiply (5.8) by $P_{n-1}(x(s))$ and (5.10) by $P_n(x(s))$ and add the resulting expressions, we obtain:

$$\begin{aligned}&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]\\ &\qquad+ \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]\\ &\quad= \left[(D_{n-1} - G_{n-1}A_{n-1} + D_n)x(s) + E_{n-1}\right.\\ &\qquad\left.-G_{n-1}B_{n-1} + E_n\right] P_n(x(s))P_{n-1}(x(s))\\ &\qquad+ G_{n-1}\left(P_n(x(s))\right)^2 - C_n G_n \left(P_{n-1}(x(s))\right)^2.\end{aligned}$$

This proves the first part of Theorem 5.1.

Now, we prove the second part.

Let $(Q_n(x(s)))_{n\in\mathbb{N}}$ be a sequence of polynomials of a q-quadratic variable such that $Q_0(x(s)) = P_0(x(s))$ and, for $n \geq 1$, $Q_n(x(s))$ and $Q_{n-1}(x(s))$ satisfy

$$\begin{aligned}&\psi(x(s))\left[Q_n(x(s))\mathbb{S}_x^2 Q_{n-1}(x(s)) + Q_{n-1}(x(s))\mathbb{S}_x^2 Q_n(x(s))\right]\\ &\qquad+\phi(x(s))\left[Q_n(x(s))\mathbb{D}_x\mathbb{S}_x Q_{n-1}(x(s)) + Q_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x Q_n(x(s))\right]\\ &\quad= \left[(D_{n-1} - G_{n-1}A_{n-1} + D_n)x(s) + E_{n-1} - G_{n-1}B_{n-1} + E_n\right]\\ &\qquad\times Q_n(x(s))Q_{n-1}(x(s)) + G_{n-1}\left(Q_n(x(s))\right)^2 - C_n G_n \left(Q_{n-1}(x(s))\right)^2.\end{aligned} \tag{5.11}$$

Let a_n be the leading coefficient of $Q_n(x(s))$. We shall firstly show by induction that $k_n = a_n$ for all $n \geq 0$. We have $a_0 = k_0$ and we assume that $n \geq 1$ and $a_{n-1} = k_{n-1}$. If we compare the coefficients of $F_{2n}(x(s))$ in (5.11), we find that we must consider two cases whether the degree of ϕ is less than two or equal to two.

- If the degree of $\phi(x(s))$ is less than two then, we have

$$\psi_1((\alpha_{n-1})^2 + (\alpha_n)^2)a_n k_{n-1} = (D_n + D_{n-1} - A_{n-1}G_{n-1})a_n k_{n-1} + G_{n-1}(a_n)^2$$

  and $a_n \neq 0$ implies that for the q-quadratic variable, we have $a_n = A_{n-1}k_{n-1} = k_n$.

- If the degree of $\phi(x(s))$ is equal to two then we get

$$\begin{aligned}&\psi_1((\alpha_{n-1})^2 + (\alpha_n)^2)a_n k_{n-1} + \phi_2(\alpha_{n-1}\gamma_{n-1} + \alpha_n\gamma_n)a_n k_{n-1}\\ &\quad = (D_n + D_{n-1} - A_{n-1}G_{n-1})a_n k_{n-1} + G_{n-1}(a_n)^2\end{aligned}$$

  and $a_n \neq 0$ implies that for the quadratic case or the q-quadratic case, we have $a_n = A_{n-1}k_{n-1} = k_n$.

We have by assumption $Q_0(x(s)) = P_0(x(s))$. Assume further that $n \geq 1$ and we $Q_{n-1}(x(s)) = P_{n-1}(x(s))$ but $Q_n(x(s)) \neq P_n(x(s))$.

Then $Q_n(x(s)) = P_n(x(s)) + g(x(s))$, where $g(x(s)) = c(F_r(x(s)) + \cdots)$, $c \neq 0$. Since $Q_n(x(s))$ and $P_n(x(s))$ have the same degree and the same leading coefficient, we must have $r < n$. From (5.11), we get

$$\begin{aligned}&\psi(x(s))\left[(P_n(x(s)) + g(x(s)))\mathbb{S}_x^2 P_{n-1}(x(s))\right.\\ &\qquad \left.+P_{n-1}(x(s))(\mathbb{S}_x^2 P_n(x(s)) + \mathbb{S}_x^2 g(x(s)))\right]\\ &\qquad + \phi(x(s))\left[(P_n(x(s)) + g(x(s)))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s))\right.\\ &\qquad \left.+P_{n-1}(x(s))(\mathbb{D}_x\mathbb{S}_x P_n(x(s)) + \mathbb{D}_x\mathbb{S}_x g(x(s)))\right]\\ &\quad = \left[(D_{n-1} - G_{n-1}A_{n-1} + D_n)x(s) + E_{n-1} - G_{n-1}B_{n-1} + E_n\right]\\ &\qquad \times (P_n(x(s))P_{n-1}(x(s)) + g(x(s))P_{n-1}(x(s))) + G_{n-1}\Big((P_n(x(s)))^2\\ &\qquad +2g(x(s))P_n(x(s)) + (g(x(s)))^2\Big) - C_n G_n\left(P_{n-1}(x(s))\right)^2.\end{aligned}$$

Using the fact that $P_n(x(s))$ and $P_{n-1}(x(s))$ satisfy (5.3), we obtain

$$\begin{aligned}&\psi(x(s))\left[g(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 g(x(s))\right]\\ &\qquad +\phi(x(s))\left[g(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x g(x(s))\right]\\ &\quad = \left[(D_{n-1} - G_{n-1}A_{n-1} + D_n)x(s) + E_{n-1}\right.\\ &\qquad \left.-G_{n-1}B_{n-1} + E_n\right] g(x(s))P_{n-1}(x(s)))\\ &\qquad +G_{n-1}\left(2g(x(s))P_n(x(s)) + (g(x(s)))^2\right).\end{aligned} \tag{5.12}$$

We compare the coefficients of $F_{n+r}(x(s))$ in (5.12). Two cases arise:

1. If the degree of $\phi(x(s))$ is less than two, then we get

$$\psi_1\left((\alpha_{n-1})^2 + (\alpha_r)^2\right) ck_{n-1} = (D_n + D_{n-1} - G_{n-1}A_{n-1})ck_{n-1} + 2ck_n G_{n-1},$$

   which is equivalent to

$$\psi_1\left((\alpha_{n-1})^2 + (\alpha_r)^2\right) ck_{n-1} = (D_n + D_{n-1} - G_{n-1}A_{n-1})ck_{n-1} + 2ck_{n-1}G_{n-1}A_{n-1}.$$

   Then, for the q-quadratic variable, this is impossible if $c \neq 0$.

2. If the degree of $\phi(x(s))$ is equal to two, then we get
$$\begin{aligned}&\psi_1\left((\alpha_{n-1})^2+(\alpha_r)^2\right)ck_{n-1}+\phi_2\left(\alpha_{n-1}\gamma_{n-1}+\alpha_r\gamma_r\right)ck_{n-1}\\&\quad=(D_n+D_{n-1}-G_{n-1}A_{n-1})ck_{n-1}+2ck_nG_{n-1},\end{aligned}$$
which is equivalent to
$$\begin{aligned}&\psi_1\left((\alpha_{n-1})^2+(\alpha_r)^2\right)ck_{n-1}+\phi_2\left(\alpha_{n-1}\gamma_{n-1}+\alpha_r\gamma_r\right)ck_{n-1}\\&\quad=(D_n+D_{n-1}-G_{n-1}A_{n-1})ck_{n-1}+2ck_{n-1}G_{n-1}A_{n-1}.\end{aligned}$$
Again this is impossible if $c\neq 0$.

□

5.2. Special Cases

We can specialize the above result to the various classical orthogonal polynomials on non-uniform lattice, namely Askey–Wilson, q-Racah, Continuous dual q-Hahn, Continuous q-Hahn, Dual q-Hahn, Al-Salam Chihara, q-Meixner–Pollaczek, Continuous q-Jacobi, Dual q-Krawtchouk, Continuous big q-Hermite, Continuous q-Laguerre and Continuous q-Hermite polynomials. Note that the results for the Askey–Wilson and the q-Racah polynomials would be enough since the other families can be obtained by some limit transitions. But here, we would like to provide a complete database for all these polynomials orthogonal on a q-quadratic lattices.

5.2.1. Askey–Wilson Polynomials. The Askey–Wilson polynomials have the q-hypergeometric representation [18, P. 415]
$$\frac{a^np_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n}={}_4\phi_3\left(\left.\begin{matrix}q^{-n},abcdq^{n-1},ae^{i\theta},ae^{-i\theta}\\ab,ac,ad\end{matrix}\right|q;q\right),\quad x=\cos\theta.$$
They satisfy the divided-difference equation (5.1) with
$$\begin{aligned}\phi\left(x(s)\right)&=2\left(dcba+1\right)x^2\left(s\right)-\left(a+b+c+d+abc+abd+acd+bcd\right)x\left(s\right)\\&\quad+ab+ac+ad+bc+bd+cd-abcd-1,\\\psi\left(x(s)\right)&=\frac{4\left(abcd-1\right)q^{\frac12}\,x\left(s\right)}{q-1}+\frac{2\left(a+b+c+d-abc-abd-acd-bcd\right)q^{\frac12}}{q-1}.\end{aligned}$$

The monic Askey–Wilson polynomials are characterized by the following non-linear recurrence relation
$$\begin{aligned}&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2P_n(x(s))\right]\\&\qquad+\phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_xP_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_xP_n(x(s))\right]\\&\quad=\left[K_nx(s)+M_n\right]P_n(x(s))P_{n-1}(x(s))\\&\qquad-\frac{2(q+1)(abcdq^{n-\frac32}-q^{\frac32-n})}{q-1}\left(P_n(x(s))\right)^2\\&\qquad-\frac{2(q+1)(abcdq^{n-\frac12}-q^{\frac12-n})}{q-1}C_n\left(P_{n-1}(x(s))\right)^2,\end{aligned}$$
where

$$
\begin{aligned}
K_n &= \frac{2\sqrt{q}[abcd\left(q^{2n-2}(q+1)^2+2q^n\right)-\left(2q^n+(q+1)^2\right)]}{(q-1)q^n},\\
C_n &= \frac{1}{4}\frac{(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-abcdq^{n-1})}{(1-abcdq^{2n-3})(1-abcdq^{2n-1})}\\
&\quad\times\frac{(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})^2},\\
M_n &= \frac{2\sqrt{q}(q^{2n}-abcd)}{q^n(q-1)}\left(a+a^{-1}-(\widetilde{A_n}+\widetilde{C_n})\right)\\
&\quad+\left\{\frac{2\sqrt{q}[q^{2n-1}-abcdq+(q+1)(q^{2n-2}-abcdq)]}{q^n(q-1)}\right\}\\
&\quad\times\left(a+a^{-1}-(\widetilde{A}_{n-1}+\widetilde{C}_{n-1})\right)\\
&\quad+\frac{2\left(a+b+c+d-abc-abd-acd-bcd\right)q^{\frac{1}{2}}}{q-1},
\end{aligned}
$$

with

$$
\begin{aligned}
\widetilde{A_n} &= \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},\\
\widetilde{C_n} &= \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n-2})}.
\end{aligned}
$$

5.2.2. *q*-Racah Polynomials. The q-Racah polynomials have the q-hypergeometric representation [18, P. 422]

$$
R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\phi_3\left(\begin{matrix} q^{-n},\alpha\beta q^{n+1},q^{-x},\delta\gamma q^{x+1}\\ \alpha q,\beta\delta q,\gamma q\end{matrix}\,\middle|\, q;q\right), n=0,1,2,\ldots,N
$$

where

$$
\mu(x) := q^{-x}+\delta\gamma q^{x+1}
$$

and

$$
\alpha q = q^{-N} \quad \text{or} \quad \beta\delta q = q^{-N} \quad \text{or} \quad \gamma q = q^{-N},
$$

with N a non-negative integer. They satisfy (5.1) with

$$
\begin{aligned}
\phi(x(s)) &= \left(\beta\,\alpha\,q^2+1\right)x(s)^2 - q\left(\gamma\,q\alpha+\gamma\,q\beta\,\delta+q\alpha\,\beta\,\delta+q\alpha\,\beta+\beta\,\delta+\gamma\,\delta+\gamma+\alpha\right)x(s)\\
&\quad+2\,q\left(-q^2\alpha\,\beta\,\delta\,\gamma+\gamma\,q\alpha+\gamma^2q\delta+\gamma\,q\delta^2\beta+q\alpha\,\beta\,\delta+q\gamma\,\delta\,\alpha+\gamma\,q\beta\,\delta-\gamma\,\delta\right),\\
\psi(x(s)) &= 2\sqrt{q}\left(\frac{(\beta\,\alpha\,q^2-1)}{q-1}\right)x(s)\\
&\quad-2\,\frac{q^{3/2}\left(\gamma\,q\alpha+\gamma\,q\beta\,\delta-\gamma\,\delta-\gamma+q\alpha\,\beta\,\delta+q\alpha\,\beta-\alpha-\beta\,\delta\right)}{q-1}.
\end{aligned}
$$

The monic q-Racah polynomials are characterized by the following relation

$$\begin{aligned}
&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2P_n(x(s))\right]\\
&\quad+\phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_xP_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_xP_n(x(s))\right]\\
&=\left[K_nx(s)+M_n\right]P_n(x(s))P_{n-1}(x(s))-\frac{(q+1)\sqrt{q}(\alpha\beta q^{2n}-q)}{(q-1)q^n}\left(P_n(x(s))\right)^2\\
&\quad-\frac{(q+1)\sqrt{q}(\alpha\beta q^{2n+1}-1)}{(q-1)q^n}C_n\left(P_{n-1}(x(s))\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
K_n&=\frac{\alpha\beta\sqrt{q}\,(q^n(q+1)^2+2q^2)}{q-1}-\frac{\sqrt{q}\,((q+1)^2+2q^n)}{q^n(q-1)},\\
C_n&=\frac{q(1-\alpha q^n)(1-\alpha\beta q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n})^2},\\
M_n&=\frac{\sqrt{q}(\alpha\beta q^{2n}-1)}{q^n(q-1)}\left(\widetilde{A_n}+\widetilde{C_n}-q\gamma\delta-1\right)\\
&\quad+\left(\frac{q^{\frac{3}{2}}(\alpha\beta q^{2n}-1)+\sqrt{q}(q+1)(\alpha\beta q^{2n}-q)}{q^n(q-1)}\right)\left(\widetilde{A}_{n-1}+\widetilde{C}_{n-1}-q\gamma\delta-1\right)\\
&\quad-2\frac{(q\gamma\alpha+q\gamma\beta\delta-\gamma\delta-\gamma+q\alpha\beta\delta+q\alpha\beta-\alpha-\beta\delta)\,q^{\frac{3}{2}}}{q-1}
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{A_n}&=\frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})},\\
\widetilde{C_n}&=\frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}.
\end{aligned}$$

5.2.3. Continuous Dual q-Hahn Polynomials. The Continuous Dual q-Hahn polynomials have the q-hypergeometric representation [18, P. 429]

$$\frac{a^np_n(x;a,b,c|q)}{(ab,ac;q)_n}={}_3\phi_2\left(\begin{matrix}q^{-n},ae^{i\theta},ae^{-i\theta}\\ ab,ac\end{matrix}\;\middle|\;q,q\right),\quad x=\cos\theta.$$

They satisfy (5.1) with

$$\begin{aligned}
\phi(x(s))&=2(x(s))^2-(a+b+c+abc)x(s)-1+bc+ab+ac,\\
\psi(s(s))&=-\frac{4\sqrt{q}}{q-1}x(s)+\frac{2(a+b+c-abc)\sqrt{q}}{q-1}.
\end{aligned}$$

The monic continuous dual q-Hahn polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\left(-\frac{4\sqrt{q}}{q-1}x(s)+\frac{2(a+b+c-abc)\sqrt{q}}{q-1}\right)\left[P_n(x(s))\mathbb{S}_x^2P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2P_n(x(s))\right]\\
&\quad+\left(2(x(s))^2-(a+b+c+abc)x(s)-1+bc+ab+ac\right)\\
&\quad\times\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_xP_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_xP_n(x(s))\right]\\
&=\left[-\left(\frac{2\left((q+1)^2+2q^n\right)\sqrt{q}}{q^n(q-1)}\right)x(s)-\frac{2q^{\frac{1}{2}}(B_n+(q^2+2q)B_{n-1})}{q^n(q-1)}+\frac{2(a+b+c-abc)\sqrt{q}}{q-1}\right]\\
&\quad\times P_n(s)P_{n-1}(s)+\frac{2(q+1)q^{\frac{3}{2}}}{q^n(q-1)}\left(P_n(x(s))\right)^2+\frac{2(q+1)q^{\frac{1}{2}}}{q^n(q-1)}C_n\left(P_{n-1}(x(s))\right)^2,
\end{aligned}$$

where

$$B_n = -\frac{1}{2}\left(a + a^{-1} - a^{-1}(1-abq^n)(1-acq^n) - a(1-q^n)(1-bcq^{n-1})\right)$$
$$C_n = \frac{1}{4}(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-bcq^{n-1}).$$

5.2.4. Continuous q-Hahn Polynomials. The Continuous q-Hahn polynomials have the q-hypergeometric representation [18, P. 415] or [8, P. 75]

$$\frac{(ae^{i\varphi})^n P_n(x;a,b,c,d;q)}{(ab,ac,ad;q)_n} = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i(\theta+2\varphi)}, ae^{-i\theta} \\ abe^{2i\varphi}, ac, ad \end{matrix}\middle| q;q\right),$$

here $x = \cos(\theta+\varphi)$. They satisfy the divided-difference equation (5.1) with

$$\begin{aligned}\phi(x(s)) &= 2(dcba+1)x^2(s) - \frac{(d+dcb+at^2+bt^2ad+abct^2+c+acd+bt^2)x(s)}{t} \\ &\quad + \frac{cat^2+bt^2d-t^2cbad+cbt^2+cd+t^2+bt^4a+t^2ad}{t^2}, \\ \psi(x(s)) &= \frac{4(abcd-1)q^{\frac{1}{2}}x(s)}{q-1} - 2\sqrt{q}\frac{(-c-d+cda-bt^2-at^2+dcb+cbat^2+bt^2q)}{(q-1)t},\end{aligned}$$

where $t = e^{i\varphi}$.

The monic Continuous q-Hahn polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right] \\ &\quad + \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right] \\ &= \left[K_n x(s) + M_n\right] P_n(x(s))P_{n-1}(x(s)) \\ &\quad + \frac{2(q+1)(-abcdq^{2n-1}+q^2)}{\sqrt{q}(q-1)q^n}\left(P_n(x(s))\right)^2 \\ &\quad + \frac{2(q+1)(abcdq^{2n}-q)}{\sqrt{q}(q-1)q^n} C_n\left(P_{n-1}(x(s))\right)^2,\end{aligned}$$

where

$$\begin{aligned}K_n &= \frac{2abcd[\left(q^{2n}(q+1)^2+2q^{n+2}\right)]}{q^{\frac{3}{2}}(q-1)q^n} - \frac{2\sqrt{q}\left((q+1)^2+2q^n\right)}{(q-1)q^n}, \\ C_n &= \frac{1}{4}\frac{(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1}e^{-2i\varphi})(1-abcdq^{n-2})}{(1-abcdq^{2n-1})(1-abcdq^{2n-3})} \\ &\quad\times\frac{(1-abq^{n-1}e^{2i\varphi})(1-acq^{n-1})(1-adq^{n-1})}{(1-abcdq^{2n-2})^2}, \\ M_n &= \frac{\sqrt{q}(1-abcdq^{2n})}{q^n(q-1)}\left(ae^{i\varphi}+a^{-1}e^{i\varphi}-(\widetilde{A_n}+\widetilde{C_n})\right) \\ &\quad - \left\{(q+2)\frac{(q^2-q^{2n-1}abcd)}{\sqrt{q}(q-1)q^n}\right\}\left(ae^{i\varphi}+a^{-1}e^{i\varphi}-(\widetilde{A}_{n-1}+\widetilde{C}_{n-1})\right) \\ &\quad -2\sqrt{q}\frac{(-c-d+cda-bt^2-at^2+dcb+cbat^2+bt^2q)}{(q-1)t}\end{aligned}$$

with $t = e^{i\varphi}$,

$$\widetilde{A_n} = \frac{(1-abq^n e^{2i\varphi})(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{ae^{i\varphi}(1-abcdq^{2n-1})(1-abcdq^{2n})},$$
$$\widetilde{C_n} = \frac{ae^{i\varphi}(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1}e^{-2i\varphi})}{(1-abcdq^{2n-1})(1-abcdq^{2n-2})}.$$

5.2.5. Dual *q*-Hahn Polynomials. The Dual q-Hahn polynomials have the q-hypergeometric representation ([18, P. 450] or [8, P. 76]

$$R_n(x(s);\gamma,\delta,N|q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-s}, \gamma\delta q^{s+1} \\ \gamma q, q^{-N}\end{matrix}\,\middle|\, q;q\right), \quad n=0,1,\ldots,N,$$

where $x(s) = q^{-s} + \gamma\delta q^{s+1}$ and N a non-negative integer. They satisfy (5.1) with

$$\begin{aligned}\phi(x(s)) &= (x(s))^2 - \frac{(\gamma q + q^{N+1}\gamma\delta + \gamma q^{N+1} + 1)x(s)}{2q^N} \\ &\quad + \frac{\gamma(q^{N+1}\gamma\delta - \delta q^N + \delta + 1)q}{2q^N}, \\ \psi(s(s)) &= -\frac{2\sqrt{q}}{q-1}x(s) + \frac{(q^{N+1}\gamma\delta + \gamma q^{N+1} - \gamma q + 1)\sqrt{q}}{(q-1)q^N}.\end{aligned}$$

The monic Dual q-Hahn polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right] \\ &\quad + \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right] \\ &= \left[K_n x(s) + M_n\right] P_n(x(s))P_{n-1}(x(s)) + \frac{(q+1)q^{\frac{3}{2}}}{(1-q)q^n}\left(P_n(x(s))\right)^2 \\ &\quad + \frac{(1+q)\sqrt{q}}{(1-q)q^n} C_n \left(P_{n-1}(x(s))\right)^2,\end{aligned}$$

where

$$\begin{aligned}K_n &= \frac{\sqrt{q}\left(2q^n - q^2 + 1\right)}{q^n(1-q)}, \\ M_n &= \frac{\sqrt{q}(-q^2\gamma\delta + 2\gamma\delta q - q^2\gamma + 2\gamma q - 2q^{-N+1}\gamma + q^{-N}\gamma - q^{N+1} + 2q^{-N})}{q-1}, \\ C_n &= \gamma q(1-q^{n-N-1})(1-\gamma q^n)(1-q^n)(\delta - q^{n-N-1}).\end{aligned}$$

5.2.6. Al-Salam–Chihara Polynomials. The Al-Salam–Chihara polynomials have the q-hypergeometric representation [18, P. 455] or [8, P. 77]

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0\end{matrix}\,\middle|\, q;q\right), x = \cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\begin{aligned}\phi(x(s)) &= 2(x(s))^2 - (a+b)x(s) + ab - 1, \\ \psi(s(s)) &= -\frac{4\sqrt{q}x(s)}{q-1} + \frac{2(a+b)\sqrt{q}}{q-1}.\end{aligned}$$

The monic Al-Salam–Chihara polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]\\
&\quad+\phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]\\
&=\left[K_n x(s)+M_n\right]P_n(x(s))P_{n-1}(x(s))+\frac{2(q+1)q^{\frac{3}{2}}}{(1-q)q^n}\left(P_n(x(s))\right)^2\\
&\quad+\frac{(1+q)(1-q^n)(1-abq^{n-1})}{2(1-q)q^{n-\frac{1}{2}}}\left(P_{n-1}(x(s))\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
K_n&=\frac{2\sqrt{q}\left(2q^n-q^2+1\right)}{q^n(1-q)},\\
M_n&=\frac{(a+b)\sqrt{q}(q-3)}{1-q}.
\end{aligned}$$

5.2.7. *q*-Meixner–Pollaczek Polynomials. The q-Meixner–Pollaczek polynomials [18, P. 460] or [8, P. 78]

$$P_n(x;a|q)=a^{-n}e^{-in\varphi}\frac{(a^2;q)_n}{(q;q)_n}\,{}_3\phi_2\left(\left.\begin{matrix}q^{-n},ae^{i(\theta+2\varphi)},ae^{-i\theta}\\ a^2,0\end{matrix}\,\right|q,q\right),x=\cos(\theta+\varphi).$$

They satisfy (5.1) with

$$\begin{aligned}
\phi(x(s))&=2(x(s))^2-2a\cos\varphi x(s)+a^2-1,\\
\psi(s(s))&=-\frac{4\sqrt{q}}{q-1}x(s)+\frac{4a\sqrt{q}\cos\varphi}{q-1}.
\end{aligned}$$

The monic q-Meixner–Pollaczek polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\left(-\frac{4\sqrt{q}}{q-1}x(s)+\frac{4a\sqrt{q}\cos\varphi}{q-1}\right)\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]\\
&\quad+\left(2(x(s))^2-2a\cos\varphi x(s)+a^2-1\right)\\
&\quad\times\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]\\
&=\left[-\left(\frac{2\left((q+1)^2+2q^n\right)\sqrt{q}}{q^n(q-1)}\right)x(s)+\frac{2a\sqrt{q}\cos\varphi(q+5)}{q-1}\right]P_n(x(s))P_{n-1}(x(s))\\
&\quad+\frac{2(q+1)q^{\frac{3}{2}}}{q^n(q-1)}\left(P_n(x(s))\right)^2+\left(\frac{\sqrt{q}(q+1)(1-q^n)(1-a^2q^{n-1})}{2q^n(q-1)}\right)\left(P_{n-1}(x(s))\right)^2.
\end{aligned}$$

5.2.8. Continuous *q*-Jacobi Polynomials. The Continuous q-Jacobi polynomials have the q-hypergeometric representation [18, P. 463] or [8, P. 78]

$$\begin{aligned}
&P_n^{(\alpha,\beta)}(x|q)\\
&=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_4\phi_3\left(\left.\begin{matrix}q^{-n},q^{n+\alpha+\beta+1},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}\\ q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)}\end{matrix}\,\right|q;q\right),x=\cos\theta.
\end{aligned}$$

They satisfy the divided-difference equation (5.1) with

$$\phi\left(x(s)\right) = \left(p^{2\alpha+2\beta+4}+1\right)x^{2}\left(s\right)+\frac{1}{2}(p+1)p^{\frac{1}{2}}\left(p^{2\alpha+2\beta+2}-p^{\alpha}-p^{\alpha+2\beta+2}+p^{\beta}\right)x\left(s\right)$$
$$-\frac{1}{2}\left(p^{2\alpha+2\beta+4}+p^{\alpha+\beta+3}-p^{2\alpha+2}+p^{\alpha+\beta+2}-p^{2\beta+2}+p^{\alpha+\beta+1}+1\right),$$
$$\psi\left(x(s)\right) = \frac{4p\left(p^{2\alpha+2\beta+4}-1\right)x\left(s\right)}{(p-1)(p+1)}-\frac{\left(-p^{2\alpha+\beta+2}-p^{\alpha}+p^{\alpha+2\beta+2}+p^{\beta}\right)p^{\frac{3}{2}}}{p-1},$$

with $p=q^2$.

The monic Continuous q-Jacobi polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2P_n(x(s))\right]\\
&\quad+\phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_xP_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_xP_n(x(s))\right]\\
&=\left[K_nx(s)+M_n\right]P_n(x(s))P_{n-1}(x(s))\\
&\quad-\frac{q^{\frac{3}{2}}(q+1)(q^{\alpha+\beta+2n-1}-1)}{(q-1)q^n}\left(P_n(x(s))\right)^2\\
&\quad-\frac{\sqrt{q}(q+1)(q^{\alpha+\beta+2n+1}-1)}{(q-1)q^n}C_n\left(P_{n-1}(x(s))\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
K_n&=\frac{\sqrt{q}[\left((q+1)^2\left(-1+q^{\alpha+\beta+2}\right)\right)+\left(2q^n\left(-1+q^{\alpha+\beta+2}\right)\right)]}{(q-1)q^n},\\
C_n&=\frac{1}{4}\frac{(1-q^n)(1-q^{n+\alpha})(1-q^{n+\beta})(1-q^{n+\alpha+\beta})\left(1-q^{n+\frac{1}{2}(\alpha+\beta-1)}\right)}{(1-q^{2n-1+\alpha+\beta})(1-q^{2n+1+\alpha+\beta})}\\
&\quad\times\frac{\left(1+q^{n+\frac{1}{2}(\alpha+\beta+1)}\right)\left(1-q^{n+\frac{1}{2}(\alpha+\beta)}\right)^2}{(1-q^{2n+\alpha+\beta})^2},\\
M_n&=-\frac{\sqrt{q}(q+1)(q^{\alpha+\beta+2n+2}-1)}{2(q-1)q^n}\left(q^{\frac{1}{2}\alpha+\frac{1}{4}}+q^{-\frac{1}{2}\alpha-\frac{1}{4}}-(\widetilde{A_n}+\widetilde{C_n})\right)\\
&\quad-\left\{\frac{q^{\frac{3}{2}}[q^{\alpha+\beta+2n}-1+(q+1)(q^{\alpha+\beta+2n+-1}-1)]}{2q^n(q-1)}\right\}\\
&\quad\times\left(q^{\frac{1}{2}\alpha+\frac{1}{4}}+q^{-\frac{1}{2}\alpha-\frac{1}{4}}-(\widetilde{A}_{n-1}+\widetilde{C}_{n-1})\right)\\
&\quad+\frac{\left(-p^{2\alpha+\beta+2}-p^{\alpha}+p^{\alpha+2\beta+2}+p^{\beta}\right)p^{\frac{3}{2}}}{p-1},
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{A_n}&=\frac{(1-q^{n+\alpha+1})(1-q^{\alpha+\beta+n+1})(1-q^{n+\frac{1}{2}(\alpha+\beta+1)})(1-q^{n+\frac{1}{2}(\alpha+\beta+2)})}{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^{\alpha+\beta+2n+1})(1-q^{\alpha+\beta+2n+2})},\\
\widetilde{C_n}&=\frac{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^n)(1-q^{n+\beta})\left(1+q^{n+\frac{1}{2}(\alpha+\beta)}\right)\left(1+q^{n+\frac{1}{2}(\alpha+\beta+1)}\right)}{(1-q^{\alpha+\beta+2n})(1-q^{\alpha+\beta+2n+1})}.
\end{aligned}$$

5.2.9. Dual q-Krawtchouk Polynomials. The Dual q-Krawtchouk polynomials [18, P. 505] or [8, P. 80]

$$K_n(x(s);c,N|q)={}_3\phi_2\left(\left.\begin{matrix}q^{-n},q^{-s},cq^{s-N}\\q^{-N},0\end{matrix}\,\right|q,q\right),\quad n=0,1,\ldots,N,$$

where $x(s)=q^{-s}+cq^{s-N}$. They satisfy (5.1) with

$$\phi(x(s)) = (x(s))^2-(c+1)q^{-N}x(s)-2c(q^{-N}-q^{-2N}),$$
$$\psi(s(s)) = -\frac{2\sqrt{q}}{q-1}x(s)+\frac{2(c+1)\sqrt{q}}{(q-1)q^N}.$$

The monic Dual q-Krawtchouk polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\left(-\frac{2\sqrt{q}}{q-1}x(s)+\frac{2(c+1)\sqrt{q}}{(q-1)q^N}\right)\left[P_n(x(s))\mathbb{S}_x^2P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2P_n(x(s))\right]\\
&\quad+\left((x(s))^2-(c+1)q^{-N}x(s)-2c(q^{-N}-q^{-2N})\right)\\
&\quad\times\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_xP_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_xP_n(x(s))\right]\\
&=\left[-\left(\frac{\left(2q^n+(1+q)^2\right)\sqrt{q}}{q^n(q-1)}\right)x(s)+\frac{(c+1)\sqrt{q}(q+5)}{(q-1)q^N}\right]P_n(x(s))P_{n-1}(x(s))\\
&\quad+\frac{(q+1)q^{\frac{3}{2}}}{q^n(q-1)}\left(P_n(x(s))\right)^2+\left(\frac{c\sqrt{q}(q+1)(1-q^n)(1-q^{n-N-1})}{q^{n+N}(q-1)}\right)\left(P_{n-1}(x(s))\right)^2.
\end{aligned}$$

5.2.10. Continuous Big *q*-Hermite Polynomials. The continuous big q-Hermite polynomials [18, P. 509]

$$H_n(x;a,|q)=a^{-n}{}_3\phi_2\left(\begin{matrix}q^{-n},ae^{i\theta},ae^{-i\theta}\\0,0\end{matrix}\middle|\,q,q\right),\quad x=\cos\theta.$$

They satisfy (5.1) with

$$\phi(x(s)) = 2(x(s))^2-ax(s)-1,$$
$$\psi(s(s)) = -\frac{4\sqrt{q}}{q-1}x(s)+\frac{2a\sqrt{q}}{q-1}.$$

The Continuous big q-Hermite polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\left(-\frac{4\sqrt{q}}{q-1}x(s)+\frac{2a\sqrt{q}}{q-1}\right)\left[P_n(x(s))\mathbb{S}_x^2P_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{S}_x^2P_n(x(s))\right]\\
&\quad+\left(2(x(s))^2-ax(s)-1\right)\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_xP_{n-1}(x(s))+P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_xP_n(x(s))\right]\\
&=\left[-\left(\frac{2\left((q+1)^2+2q^n\right)\sqrt{q}}{q^n(q-1)}\right)x(s)+\frac{a\sqrt{q}(q+5)}{q-1}\right]P_n(x(s))\\
&\quad\times P_{n-1}(x(s))+\frac{2(q+1)q^{\frac{3}{2}}}{q^n(q-1)}\left(P_n(x(s))\right)^2-\frac{(q+1)(q^n-1)\sqrt{q}}{2q^n(q-1)}\left(P_{n-1}(x(s))\right)^2.
\end{aligned}$$

5.2.11. Continuous *q*-Laguerre Polynomials. The Continuous q-Laguerre polynomials have the q-hypergeometric representation [18, P. 514] or [8, P. 81]

$$P_n^{(\alpha)}(x|q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{matrix}q^{-n},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}\\q^{\alpha+1},0\end{matrix}\middle|\,q;q\right),x=\cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - \frac{p^{\alpha+\frac{1}{2}}(p+1)x(s)}{2q^N} + p^{2\alpha+2} - 1,$$
$$\psi(s(s)) = -\frac{4p}{p^2-1}x(s) + \frac{p^{\alpha+\frac{3}{2}}}{(p-1)},$$

with $p = q^2$.

The monic Continuous q-Laguerre polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]\\
&\quad + \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]\\
&= \left[K_n x(s) + M_n\right]P_n(x(s))P_{n-1}(x(s))\\
&\quad - G_{n-1}\left(P_n(x(s))\right)^2 - G_n C_n\left(P_{n-1}(x(s))\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
\xi_n K_n &= q^{n^2+2} - q^{11} + q^{10} + q^{n^2+5} + q^9 - pq^{\frac{15}{2}} + 2pq^{\frac{13}{2}} - q^{n^2+4} - q^8 - 2pq^{\frac{9}{2}}\\
&\quad - pq^{n^2+\frac{3}{2}} - q^{n^2+3} - q^{n^2+6} + pq^{\frac{7}{2}} + q^7 - q^5 - q^6 + q^4 - q^{n^2+9}\\
&\quad + q^{n^2+8} + pq^{n^2+\frac{11}{2}} + 4pq^{n+\frac{11}{2}} - 2pq^{n^2+\frac{9}{2}} - 8pq^{n+\frac{9}{2}}\\
&\quad + q^{n^2+7} + 4pq^{n+\frac{7}{2}} + 2pq^{n^2+\frac{5}{2}},
\end{aligned}$$

with $\xi_n = -(q-1)^2(p^2-1)q^{n+\frac{7}{2}}$,

$$\begin{aligned}
\nu_n M_n &= q^{2n+\frac{1}{2}\alpha+\frac{1}{2}} + q^{2n+\frac{1}{2}\alpha+1} + q^{2n+\frac{1}{2}\alpha+\frac{3}{2}} + q^{2n+\frac{1}{2}\alpha+2} + q^{\frac{1}{2}\alpha+\frac{7}{2}} + q^{\frac{1}{2}\alpha+4}\\
&\quad + q^{2n+\frac{1}{2}\alpha+3} + q^{2n+\frac{1}{2}\alpha+\frac{5}{2}}4q^{\alpha+\frac{3}{2}}p^{\frac{11}{4}} + 4q^{\alpha+\frac{5}{2}}p^{\frac{11}{4}} - q^{2n+\frac{1}{2}\alpha+3}p^2 - q^{2n+\frac{1}{2}\alpha+\frac{5}{2}}p^2\\
&\quad - q^{2n+\frac{1}{2}\alpha+\frac{1}{2}}p^2 - q^{2n+\frac{1}{2}\alpha+\frac{1}{2}}p + q^{2n+\frac{1}{2}\alpha+\frac{7}{2}}p + q^{2n+\frac{1}{2}\alpha+3}p - q^{2n+\frac{1}{2}\alpha+2}p^2\\
&\quad - q^{2n+\frac{1}{2}\alpha+\frac{3}{2}}p^2 - q^{2n+\frac{1}{2}\alpha+1}p^2 - q^{2n+\frac{1}{2}\alpha}p - q^{\frac{1}{2}\alpha+\frac{9}{2}}p - q^{\frac{1}{2}\alpha+4}p^2\\
&\quad - q^{\frac{1}{2}\alpha+4}p - q^{\frac{1}{2}\alpha+\frac{7}{2}}p^2 + q^{\frac{1}{2}\alpha+\frac{7}{2}}p + q^{\frac{1}{2}\alpha+3}p,
\end{aligned}$$

with $\nu_n = 2(p^2-1)q^{\frac{11}{4}}$,

$$\begin{aligned}
\tau_n G_n &= q^{n^2+\frac{7}{2}} + q^{\frac{9}{2}}p - pq^{n^2+\frac{5}{2}} - pq^{\frac{7}{2}} - q^{n^2+7} - pq^{n^2+\frac{3}{2}} - pq^{\frac{5}{2}} + q^8 + pq^{n^2+5}\\
&\quad + q^{n^2+3} + pq^{n^2+\frac{1}{2}} + pq^{\frac{3}{2}} - q^6 - q^4 + q^2 - q^{n^2+1},
\end{aligned}$$

where $\tau_n = -(q-1)^2(p^2-1)q^{n+\frac{3}{2}}$,

$$C_n = \frac{1}{4}(1-q^n)(1-q^{n+\alpha}).$$

5.2.12. Continuous q-Hermite Polynomials. The Continuous q-Hermite polynomials have the q-hypergeometric representation [18, P. 540] or [8, P. 82]

$$H_n(x|q) = e^{in\theta}{}_2\phi_0\left(\left.\begin{array}{c} q^{-n}, 0 \\ - \end{array}\right| q; q^n e^{-2in\theta}\right), x = \cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - 1,$$
$$\psi(s(s)) = -\frac{4\sqrt{q}}{q-1}.$$

The monic Continuous q-Hermite polynomials are characterized by the following non-linear recurrence relation

$$\begin{aligned}
&\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]\\
&\quad + \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]\\
&= \left[\frac{\sqrt{q}\left(2q^n - q^2 + 1\right)x(s)}{q^n(1-q)}\right] P_n(x(s))P_{n-1}(x(s)) + \frac{(1+q)q^{\frac{3}{2}}}{(1-q)q^n}\left(P_n(x(s))\right)^2\\
&\quad - \frac{(1+q)\sqrt{q}}{2(1-q)q^n}\left(P_{n-1}(x(s))\right)^2.
\end{aligned}$$

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Declarations

Conflict of interest None of the authors has competing interests of a financial or personal nature.

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