Received: 1 July 2023 | Accepted: 22 August 2023

DOI: 10.1002/pamm.202300235

PAMM Proceedings in Applied Mathematics and Mechanics

RESEARCH ARTICLE

Numerical detection of synchronization phenomena in quasi-periodic solutions

Alexander Seifert | **Simon Baeuerle** | **Hartmut Hetzler**

Engineering Dynamics Group, University of Kassel, Kassel, Germany

Correspondence
Alexander Seifert, Engineering Dynamics Group, University of Kassel, D 34125 Kassel, Germany.
Email: alexander.seifert@uni-kassel.de

Abstract
In science and technology, dynamical systems can show so-called quasi-periodic solutions. These solutions are composed of two or more base frequencies. The solution in the time domain can be represented by an invariant manifold. To parametrize the invariant manifold, we choose the hyper-time parametrization. If quasi-periodic solutions branches are continued by means of a path continuation, the phenomenon of synchronization may occur. This is important, because the hyper-time parametrization is only valid, as long as the number of base frequencies remains unchanged. Therefore, it is essential to detect a parametrization to a synchronization point. Synchronization can happen in different types. We address the mechanism of suppression, where one base frequency becomes suppressed until its amplitude vanishes. This corresponds to the quasi-periodic solution ending in a Neimark–Sacker bifurcation. We present a method to derive a scalar measure from the quasi-periodic solution in the hyper-time parametrization, to detect an approach to a Neimark–Sacker bifurcation while continuing the solution branch.

1 | INTRODUCTION

In science and technology, the dynamic behavior of systems plays a key role. To ensure a safe operation it is essential to understand the mechanisms which determine the behavior of solutions of dynamical systems. Therefore, the mechanisms which can excite stationary solutions require a thorough investigation.

In literature, there are many sources that describe the behavior of systems that are excited by one mechanism for example, external excitation, parametric excitation, or self-excitation [1, p. 16]. The occurrence of more than one excitation mechanism may lead to periodic, quasi-periodic, or chaotic solutions. Recently, quasi-periodic solutions have become the center of interest in research. The occurrence of quasi-periodic solutions depends on the frequencies which excite the system.

For stable-state and periodic solutions, there exists a bifurcation theory based on the stability of the solution [2, p. 346]. However, for quasi-periodic solutions, there is no known systematic approach to detect bifurcations. Therefore, this contribution will present an approach to detect Neimark–Sacker bifurcations in quasi-periodic solutions, which are parametrized by hyper-time parametrization.

TABLE 1 Parameter sets.

	Parameter set 1	Parameter set 2
s (Excitation strength)	0.9	0.9
ε (Nonlinearity)	0.5	0.5
Ω (Excitation frequency)	1.05	1.5

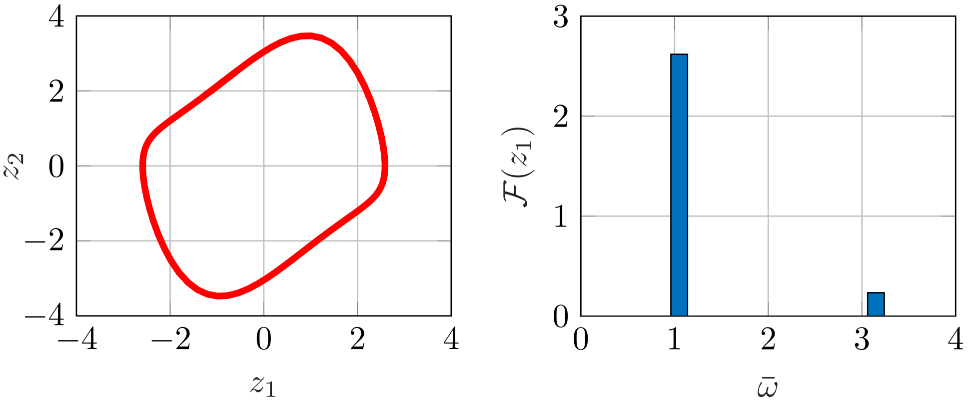

FIGURE 1 Periodic solution of the forced van-der-Pol equation in state-space, with the states z_1 and z_2 and in the frequency domain, with the value of the Fourier transform $\mathcal{F}(z_1)$ over the occuring frequencies $\bar{\omega}$, for parameter set 1.

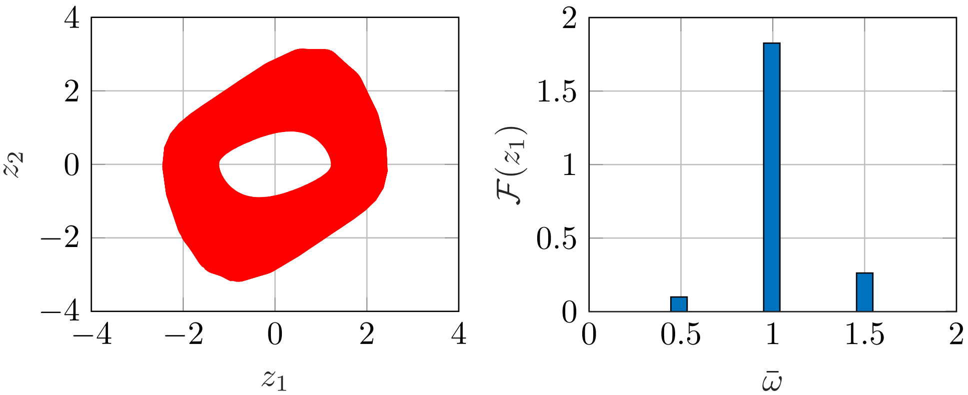

FIGURE 2 Quasi-periodic solution of the forced van-der-Pol equation in state-spacewith the states z_1 and z_2 and in the frequency domain, with the value of the Fourier transform $\mathcal{F}(z_1)$ over the occuring frequencies $\bar{\omega}$, for parameter set 2.

2 | QUASI-PERIODIC SOLUTIONS

Quasi-periodic solutions can occur in a dynamical system that experiences more than one mechanism of excitation. We, therefore, demonstrated the occurrence of a quasi-periodic solution on the example of the forced van-der-Pol equation

$$y'' + \varepsilon\left(y^2 - 1\right)y' + y = s\cos(\Omega t). \tag{2.1}$$

For the parameter sets in Table 1 we can derive the state-space of the solution by time integration and the Fourier transform, which are depicted in Figures 1 and 2.

In Figures 1 and 2 we see that for both parameter-sets the spectrum remains discrete, however, the trajectory in state-space appears completely different. While parameter set 1 leads to a periodic solution (i.e., a closed orbit in the state-space), parameter set 2 leads to a solution that is no longer periodic, but, contrary to a chaotic solution, retains a discrete Fourier spectrum.

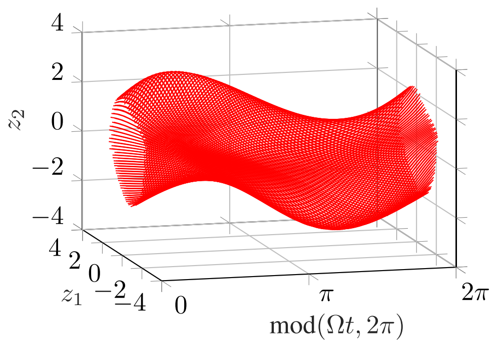

FIGURE 3 Quasi-periodic solution of the forced van-der-Pol equation in autonomous state-space for parameter set 2, showing the states z_1 and z_2 over the time Ωt modulo 2π.

In the spectrum of parameter set 2, we can identify two base frequencies $\boldsymbol{\omega} = (\omega_1, \omega_2)^\top$, from which all other occurring frequencies Ω_i can be composed by the linear combination $\langle \boldsymbol{a}_i, \boldsymbol{\omega} \rangle = \Omega_i$ with a constant $\boldsymbol{a}_i \in \mathbb{Z}^2$. For the identified frequencies $\boldsymbol{\omega}$, the following statement holds. The elements of a set of frequencies $\boldsymbol{\omega} \in \mathbb{R}^p$ are called *base frequencies*, if the equation

$$\langle \boldsymbol{c}, \boldsymbol{\omega} \rangle = \boldsymbol{0}, \qquad \boldsymbol{c} \in \mathbb{Z}^p \tag{2.2}$$

is only satisfied for the constatnt $\boldsymbol{c} = \boldsymbol{0}$ [3, p. 15]. The number $p \in \mathbb{N}$ being the number of base frequencies. We restrict further investigations to $p = 2$. In the case of the forced van-der-Pol equation, we can identify the two base frequencies as $\boldsymbol{\omega} = (\Omega, \omega)^\top$, where Ω represents the external excitation frequency and ω the self-excitation frequency. From the time solution in Figure 2 we cannot extract key information about the solution because we cannot know, for example, if the solution has yet passed the maximum amplitude or velocity.

We therefore transform the solution to parameter set 2 into autonomous state-space and introduce a periodicity corresponding to the external (known) excitation frequency Ω in the time axis by $z_3 = \mathrm{mod}(\Omega t, 2\pi)$. If we transform the solution into an autonomous state-space we derive the graphic shown in Figure 3. We can identify an invariant object, which is represented by the surface of an opened torus. The invariant object will be smooth because the number of base frequencies is finite [4, p. 56], therefore the object satisfies the requirements for a manifold. The quasi-periodic solution will fill this manifold densely if t approaches infinity [3, p. 17]. Given the ergodic theorem [5, p. 146–147] we can identify the quasi-periodic time solution with the geometric invariant manifold as the carrier of the solution. To calculate the invariant manifold, we need a parametrization, which represents the invariant manifold.

3 | HYPER-TIME PARAMETRIZATION

If we want to describe the invariant manifold, we need a parametrization. Such a parametrization is given by the hyper-time parametrization. Given a quasi-periodic solution with the frequency base $\boldsymbol{\omega} = (\Omega_1, \Omega_2, \dots, \Omega_p)^\top$, the hyper-time parametrization introduces a multiple time axis $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_p)^\top$, the so-called hyper-time coordinates. The connection between the physical time and the hyper-time is given by $\theta_i = \mathrm{mod}(\Omega_i t, 2\pi)$. Here, the discrete nature of the Fourier spectrum is used to introduce periodicities in the hyper-time axis [2, p. 364]. The quasi-periodic solution in the hyper-time parametrization represents the stationary solution in the state space [3, p. 53].

The quasi-periodic solution in the hyper-time parametrization for parameter set 2 is depicted in Figure 4. For each state variable, there exists one ≫surface≪. To determine the solution in the hyper-time parametrization, we need an equation. Due to the fact that the solution domain is no longer scalar but p-dimensional, the equation will be a partial differential equation (PDE). The solution in physical time is described by the system of ordinary differential equations

$$\boldsymbol{z}' = \boldsymbol{f}(t, \boldsymbol{z}), \tag{3.1}$$

with $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. We changed the domain of the mapping; therefore, we have to replace the derivative operator with a partial differential operator [3, p. 54]. Keeping the restriction to $p = 2$ we introduce the torus function $\boldsymbol{Z} : \mathbb{T}^2 \to \mathbb{R}^n$ and

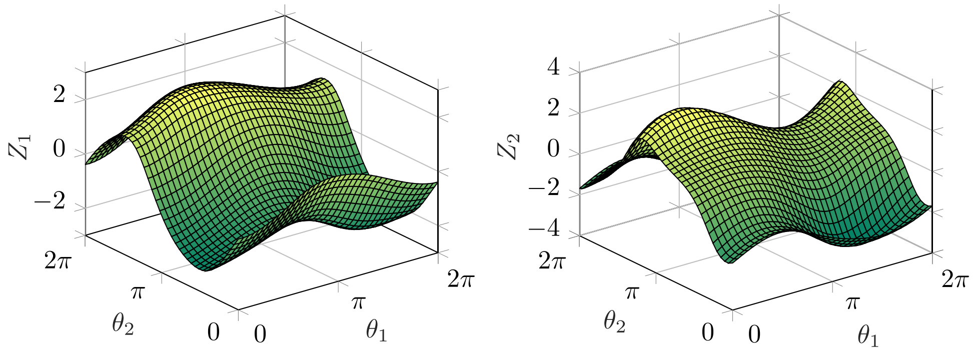

FIGURE 4 Quasi-periodic solution of the forced van-der-Pol equation in hyper-time parametrization for parameter set 2, showing the values of the torus function Z_i over the hyper-time coodinates θ_k.

derive the *invariance equation*

$$\Omega \frac{\partial \boldsymbol{Z}}{\partial \theta_1} + \omega \frac{\partial \boldsymbol{Z}}{\partial \theta_2} = \boldsymbol{f}(\theta, \boldsymbol{Z}), \tag{3.2}$$

which describes the quasi-periodic solution in the hyper-time parametrization. By the introduction of the hyper-time coordinates, we achieve a 2π-periodicity $\boldsymbol{Z}(\theta_1, \theta_2) = \boldsymbol{Z}(\theta_1 + 2\pi, \theta_2)$ and $\boldsymbol{Z}(\theta_1, \theta_2) = \boldsymbol{Z}(\theta_1, \theta_2 + 2\pi)$. To solve Equation (3.2) any numerical method for solving PDE's is suitable, but given the periodicity the quasi-periodic shooting algorithm and the quasi-periodic Fourier–Galerkin method are particularly suitable [3, p. 65]. The major drawback of the hyper-time parametrization, however, is that this parametrization is only valid as long as the number of base frequencies remains unchanged (cf. [3, p. 53–54]). This is particularly critical if a quasi-periodic solution branch is continued and a synchronization point is approached. We need a mechanism to detect an approach to such bifurcation points based on the properties of the quasi-periodic solution.

4 | SYNCHRONIZATION

If a dynamical system produces different types of solutions, such as periodic or quasi-periodic solutions, the phenomenon of synchronization may occur. Here, a dynamical system may change its solution type from quasi-periodic to periodic as a parameter changes. This behavior is called synchronization. Restricting to the case of the synchronization of quasi-periodic solutions with $p = 2$ and combined forced and self-excited systems (i.e., the forced van-der-Pol equation), we can find two major mechanisms of synchronization [6, p. 56–62].

The first mechanism is called *entrainment* or *frequency locking*. Here, the dynamical system adjusts the base frequencies of its solution so that they are rationally dependent and therefore the number of base frequencies reduces to $p = 1$. The quasi-periodic solution branch ends in a saddle-node bifurcation [6, p. 33] [7, p. 194]. The second mechanism is called *suppression of the natural dynamic* or short *suppression*. Here, the dynamical system suppresses one base frequency of its solution until its amplitude in the Fourier spectrum vanishes and only one base frequency remains. The quasi-periodic solution branch ends in a Neimark–Sacker bifurcation [8, p. 330]. In the following, we will focus on the mechanism of suppression.

5 | NEIMARK–SACKER BIFURCATION

Given the forced van-der-Pol equation (2.1), we can derive the bifurcation diagram shown in Figure 5 for the parameters $s = 0.9, \varepsilon = 0.5$.

The periodic solution loses its stability due to a pair of complex conjugated Floquet multipliers leaving the unit circle [8, p. 329–330]. The unstable periodic solution continues to exist [8, p. 330]. From the bifurcation point branches off a quasi-

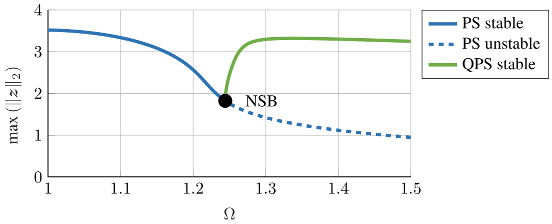

FIGURE 5 Bifurcation diagram of the forced van-der-Pol equation with Neimark–Sacker bifurcation (NSB), showing the maximal value of the solutions $max\,(|\boldsymbol{z}|_2)$ over the excitation frequency Ω.

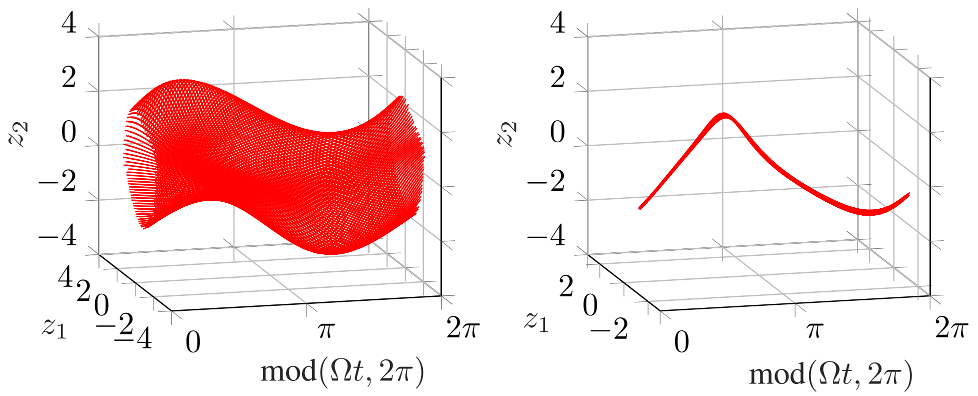

FIGURE 6 Comparison between a quasi-periodic solution far from the synchronization $\Omega = 1.5$ (left), and close to the synchronization point $\Omega = 1.244$ (right), showing the states z_1 and z_2 over the time Ωt modulo 2π.

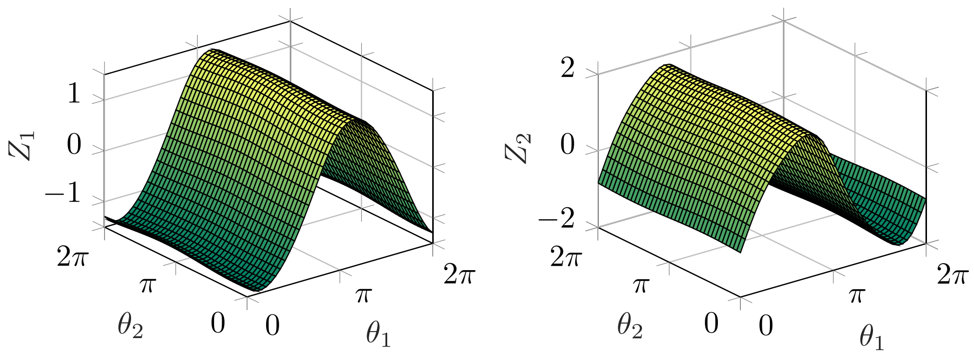

FIGURE 7 Quasi-periodic solution of the forced van-der-Pol equation in hyper-time parametrization for $\Omega \approx \Omega_{\text{sync}}$ close to synchronization, showing the values of the torus function Z_i over the hyper-time coodinates θ_k.

periodic solution branch. To detect Neimark–Sacker bifurcations while continuing the periodic branch is easily doable, because the solution continues to exist after the bifurcation. Thus, it is easy to iterate the bifurcation point.

The major difference is that the quasi-periodic solution starts/ends in the Neimark–Sacker bifurcation and does not continue beyond the bifurcation point. Furthermore, there exists no test function based merely on the quasi-periodic solution. Therefore, it is not possible to simply detect a sign change. For a quasi-periodic solution far from the synchronization point, we find that the solution in Figure 6 fills an invariant manifold. If we now vary the parameter Ω and approach the bifurcation, we find that the torus shrinks until only a line remains.

This corresponds to the suppression of the self-excitation frequency. If we visualize the approach to the Neimark–Sacker bifurcation in the hyper-time parametrization, we can see that the hyper-time manifold becomes constant in θ_2 which likewise corresponds to the suppression of the self-excitation frequency (Figure 7).

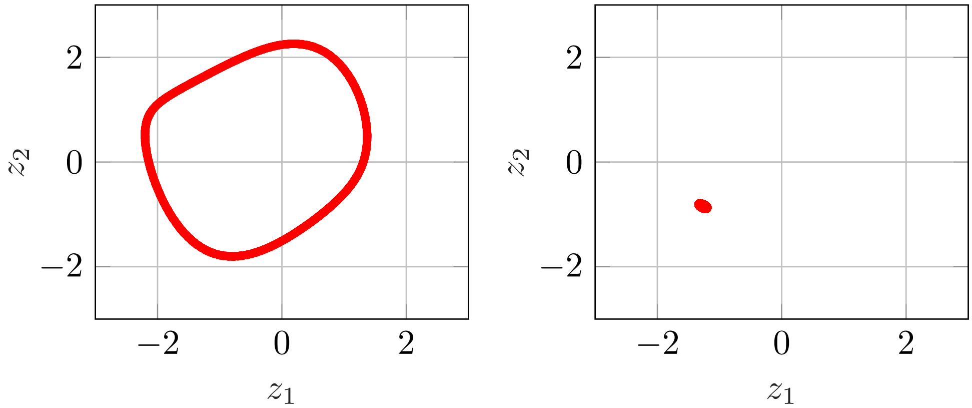

FIGURE 8 Comparison of the Poincaré sections between a quasi-periodic solution far from the synchronization $\Omega = 1.5$ (left), and close to the synchronization point $\Omega = 1.244$ (right), showing the cross-section of the state-space with the states z_1 and z_2.

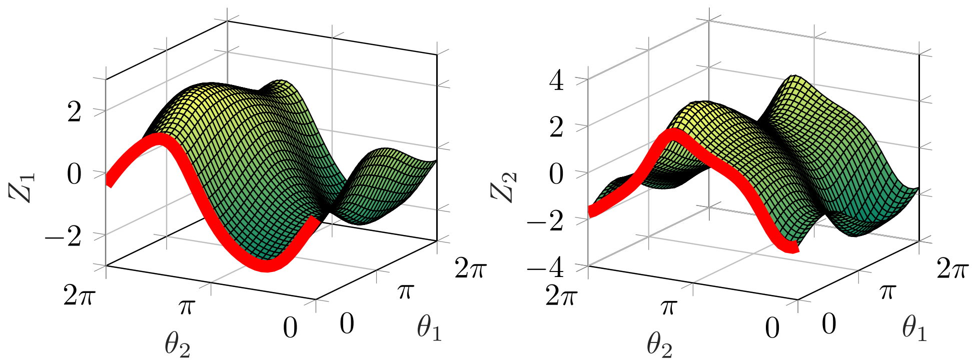

FIGURE 9 Derivation of the Poincaré sections from the quasi-periodic solution of the forced van-der-Pol equation in the hyper-time parametrization for $\Omega = 1.5$, showing the values of the torus functions Z_i over the hyper-time coordinates θ_k.

6 | SYNTHESIZING POINCARÉ SECTION FROM A SOLUTION IN THE HYPER-TIME PARAMETRIZATION

Poincaré sections Σ represent a sub-manifold of the state-space. In the analysis of dynamical systems, these sections are usually executed as stroboscopic sections, [2, p. 321] which means that for every integer multiple of a predetermined time T, the current state is represented in the section. Therefore, the stroboscopic Poincaré section of a periodic orbit with period T, is represented by a single point. The stroboscopic Poincaré section of a quasi-periodic solution, with the period of one base-frequency being T, is represented by a densely filled curve.

The closed curve shrinks to eventually become a point once the Neimark–Sacker bifurcation is reached (c.f. Figure 8). These Poincaré sections are derived from a time solution. If we continue the quasi-periodic solution in the hyper-time parametrization we have to derive the Poincaré sections from this representation of the solution.

For the example of the forced van-der-Pol equation, we know that the amplitude along θ_2 corresponds to the self-excitation frequency and that the mechanism of synchronization is the suppression of the self-excitation frequency. We therefore derive sections for constant θ_1 (c.f. Figure 10). If we combine the ≫lines≪ derived in Figure 9 we get the object depicted in Figure 10. We define the Poincaré section for $\theta_\ell = \frac{2\pi}{k}$ as $\Sigma_k^\ell(\boldsymbol{Z})$.

7 | DETECTING NEIMARK-SACKER BIFURCATIONS

As shown before, the Poincaré sections $\Sigma_k^1(\boldsymbol{Z})$ shrink to a point when approaching a Neimark–Sacker bifurcation. We can utilize the area which is enclosed by the Poincaré as a scalar measure to determine an approach to a Neimark–Sacker bifurcation. To get a measure for the whole torus, we take the area of the Poincaré section $A_k = A[\Sigma_k^1(\boldsymbol{Z})]$ and average this

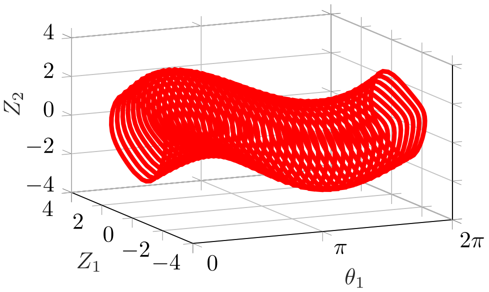

FIGURE 10 Poincaré sections derived from the quasi-periodic solution in the hyper-time parametrization, showing the values of the torus function Z_i over one torus dimension θ_1.

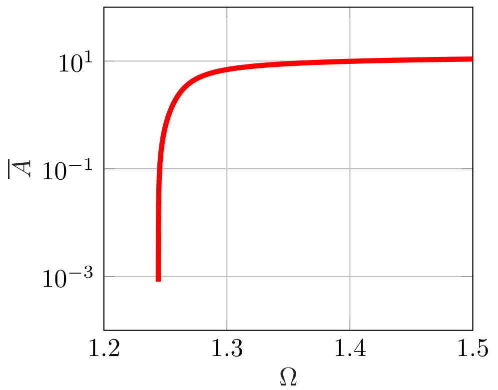

FIGURE 11 Averaged area of Poincaré sections $\bar{A}$, derived from the quasi-periodic solution in the hyper-time parametrization, over the excitation frequency Ω.

area for all sections (m being the number of Poincaré sections) along θ_1 by

$$\bar{A} = \frac{1}{m} \sum_{k=1}^{m} A_k. \tag{7.1}$$

If we plot the averaged area over the external excitation frequency, we get the diagram shown in Figure 11.

The averaged area approaches zero. Terminating the continuation if $\overline{A} < \text{tol} = 10^{-3}$ we can determine an approach to the bifurcation point. Comparing the value for $\Omega_{\text{end}} = 1.2442138$ as the approximate Ω for which synchronization occurs with an iterated bifurcation point based on a periodic continuation $\Omega_{\text{NSB}} = 1.2442044$, we get a relative error

$$\varepsilon = \left(\frac{\Omega_{\text{end}} - \Omega_{\text{NSB}}}{\Omega_{\text{NSB}}} \right) \times 100\% = 7.56 \times 10^{-4}\%. \tag{7.2}$$

8 | CONCLUSION

In this contribution, we present a method to detect Neimark–Sacker bifurcations (synchronization points), while continuing quasi-periodic solution branches with two base frequencies. The method uses the averaged area of Poincaré sections, which are derived from the quasi-periodic solution in the hyper-time parametrization. Here we can show for the forced van-der-Pol equation that the averaged area of the Poincaré sections tends to zero as the Neimark–Sacker bifurcation is approached. We can show that the detected value of the bifurcation parameter for which the bifurcation is detected differs from the actual value by less than $10^{-3}\%$.

ACKNOWLEDGMENTS
Open access funding enabled and organized by Projekt DEAL.

ORCID

Alexander Seifert https://orcid.org/0000-0001-8806-0451

REFERENCES

1. Magnus, K., Popp, K., & Sextro, W. (2008). *Schwingungen: Eine Einführung in die physikalischen Grundlagen und die theoretische Behandlung von Schwingungsproblemen, 8., überarb. aufl. edition, Studium*. Vieweg + Teubner, Wiesbaden.
2. Marx, B., & Vogt, W. (2011). *Dynamische Systeme*. Spektrum Akademischer Verlag.
3. Fiedler, R. (2021). *Numerical analysis of invariant manifolds characterized by quasi-periodic oscillations of nonlinear systems* [Doctoral Dissertation, Berichte des Instituts für Mechanik, Vol. Bericht 2021, 1. Kassel University Press, Kassel].
4. Bäuerle, S. A. (2021). *An approach to non-linear dynamics of rotors with flexible seals* [Doctoral Dissertation, Berichte des Instituts für Mechanik, Vol. Bericht 2021, 2. Kassel University Press, Kassel].
5. Argyris, J. H., Faust, G., Haase, M., & Friedrich, R. (2017). *Die Erforschung des Chaos: Dynamische Systeme* (3rd ed.). Springer Vieweg, Berlin, Heidelberg.
6. Balanov, A., Janson, N., Postnov, D., & Sosnovtseva, O. (2009). *Synchronization: From Simple to Complex*, Springer Complexity. Springer, Berlin, Heidelberg.
7. Pikovsky, A., Kurths, J., & Rosenblum, M. (2001). *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge Nonlinear Science Series (Vol. 12). Cambridge University Press, Cambridge.
8. Seydel, R. (2010). *Practical Bifurcation and Stability Analysis* (3rd ed.). Interdisciplinary Applied Mathematics (Vol. 5). Springer Science+Business Media LLC, New York, NY.

How to cite this article: Seifert, A., Baeuerle, S., & Hetzler, H. (2023). Numerical detection of synchronization phenomena in quasi-periodic solutions. *Proceedings in Applied Mathematics and Mechanics*, *23,* e202300235.
https://doi.org/10.1002/pamm.202300235