Duplication Coefficients via Generating Functions

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Abstract

In this paper, we solve the duplication problem

$$P_n(ax) = \sum_{m=0}^{n} C_m(n,a) P_m(x),$$

where $\{P_n\}_{n\geq 0}$ belongs to a wide class of polynomials, including the classical orthogonal polynomials (Hermite, Laguerre, Jacobi) as well as the classical discrete orthogonal polynomials (Charlier, Meixner, Krawtchouk) for the specific case $a = -1$. We give closed-form expressions as well as recurrence relations satisfied by the duplication coefficients.

Key words. Duplication coefficients, generating functions, Brenke polynomials, Boas-Buck polynomials, Brafman polynomials, Chaunday polynomials, Gould-Hopper polynomials, Hermite polynomials, Laguerre polynomials, Jacobi polynomials, Charlier polynomials, Meixner polynomials, Krawtchouk polynomials, classical discrete orthogonal polynomials

1 Introduction

Let $\mathscr{P}$ be the linear space of polynomials with complex coefficients. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathscr{P}$ is called a *polynomial set* if and only if $\deg P_n = n$ for all nonnegative integers n.

Given a polynomial set $\{P_n\}_{n\geq 0}$, the so-called *duplication or multiplication problem* associated to this family asks to find the coefficients $C_m(n,a)$ in the expansion

$$P_n(ax) = \sum_{m=0}^{n} C_m(n,a) P_m(x), \tag{1.1}$$

where a designates a nonzero complex number.

Such identities have applications in many problems in pure and applied mathematics, especially in combinatorial analysis. This problem may be viewed as a special case of the so-called *connection*

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problem between two polynomial sets where the first member of (1.1) is replaced by a polynomial set $Q_n(x)$.

The solution of this problem is known for some particular polynomial sets. For instance, the so-called Fields-Wimp expansion [10] gives a solution for some hypergeometric polynomials [17].

A general method, based on lowering operators and generating functions, was developed in [2, 3] to solve connection and linearization problems [4, 7]. The purpose of this work is to use this approach to express explicitly the coefficients $C_m(n,a)$. The method depends on simple manipulations of formal power series. The approach we shall propose in this paper does not need the orthogonality of the polynomials involved in the problem, and in this way the formulae obtained are still valid outside the range of orthogonality of the parameters.

2 The method

Definition 2.1. *Let $\{P_n\}_{n\geq 0}$ be a polynomial set. $\{P_n\}_{n\geq 0}$ is said to have a generating function of Boas-Buck type (or is called a Boas-Buck polynomial set) if there exists a sequence of nonzero numbers $(\lambda_n)_{n\geq 0}$ such that*

$$\sum_{n=0}^{\infty} \lambda_n P_n(x) t^n = A(t)B(xC(t)), \tag{2.1}$$

where A, B, C are three formal power series such that

$$A(0)C'(0) \neq 0,\ C(0) = 0 \quad \textit{and} \quad B^{(k)}(0) \neq 0,\ k \in \mathbb{N}. \tag{2.2}$$

The choice of $C(t) = t$ gives the class of Brenke polynomials.

It is obvious to see that if the normalization is changed, say: $P_n = c_n \widetilde{P}_n$, then the new duplication coefficients $\widetilde{C}_m(n,a)$ are given by

$$\widetilde{C}_m(n,a) = \frac{c_m}{c_n} C_m(n,a).$$

That means that there is not loss of generality if we limit ourselves to the case $\lambda_n = \dfrac{1}{n!}$ in (2.1).

Theorem 2.2. *Let $\{P_n\}_{n\geq 0}$ be a Boas-Buck polynomial set generated by (2.1). Then the associated duplication coefficients defined by (1.1) are given by*

$$\mathcal{F}(t) = \frac{A(t)}{A(\Phi(t))} \Phi^m(t) = \sum_{n=m}^{\infty} \frac{m!}{n!} C_m(n,a) t^n \tag{2.3}$$

where $\Phi(t) = C^{-1}(aC(t))$ and C^{-1} is the inverse of C, i.e. $C^{-1}(C(t)) = C(C^{-1}(t)) = t$.

Proof. The proof of this result is based on the following lemma.

Lemma 2.3 (see [3], Corollary 3.9). *Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two polynomial sets of Boas-Buck type that are generated, respectively, by*

$$A_1(t)B(xC_1(t)) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n \quad \textit{and} \quad A_2(t)B(xC_2(t)) = \sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} t^n.$$

Then the connection coefficients in

$$Q_n(x) = \sum_{m=0}^{n} C_m(n) P_m(x), \tag{2.4}$$

are given by

$$\frac{A_2(t)}{A_1(\Phi(t))}\Phi^m(t) = \sum_{k=m}^{\infty} \frac{m!}{k!} C_m(k) t^k, \tag{2.5}$$

where $\Phi(t) = C_1^{-1}(C_2(t))$.

In order to derive (2.3) from this lemma, we set $A_1 = A_2 = A,\ C_1 = C$ and $C_2 = aC$ in (2.5). $\square$

This result shows that the duplication coefficients of a Boas-Buck polynomial set generated by (2.1) depend only on C and A.

3 Applications

3.1 Brenke polynomials

Corollary 3.1. *The Brenke polynomials* $\{P_n\}_{n\geq 0}$*, generated by*

$$A(t)B(xt) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n,$$

possess a duplication formula of the form

$$P_n(ax) = \sum_{m=0}^{n} \binom{n}{m} a^m \beta_{n-m}(a) P_m(x) \tag{3.1}$$

where

$$\frac{A(t)}{A(at)} = \sum_{k=0}^{\infty} \frac{\beta_k(a)}{k!} t^k. \tag{3.2}$$

Proof. Applying Theorem 2.2 with $C(t) = t$, we obtain $\Phi(t) = at$ and

$$\frac{A(t)}{A(at)} a^m t^m = \sum_{n=m}^{\infty} \frac{m!}{n!} C_m(n,a) t^n.$$

Put $\frac{A(t)}{A(at)} = \sum_{k=0}^{\infty} \frac{\beta_k(a)}{k!} t^k$. It follows, by identification, that $C_m(n,a) = \binom{n}{m} a^m \beta_{n-m}$. $\square$

For $A(t) = e^t$, Corollary 3.1 is reduced to Carlitz Formula [6]

$$P_n(ax) = \sum_{m=0}^{n} \binom{n}{m} a^m (1-a)^{n-m} P_m(x). \tag{3.3}$$

Let us mention that Corollary 3.1 was already given in [3] and applied essentially to a some q-polynomial sets.

Next, we consider some examples.

3.1.1 Brafman polynomials

The Brafman polynomials defined by

$$\mathscr{B}_n^1((a_p),(b_q);x) = {}_{p+1}F_q\left(\begin{matrix} -n,(a_p) \\ (b_q) \end{matrix}\middle| x\right) \tag{3.4}$$

are generated by ([5, 8])

$$\sum_{n=0}^{\infty} \mathscr{B}_n^1((a_p),(b_q);x)\frac{t^n}{n!} = e^t {}_pF_q\left(\begin{matrix} (a_p) \\ (b_q) \end{matrix}\middle| -xt\right). \tag{3.5}$$

For the definition of the generalized hypergeometric function ${}_pF_q$ see [18] or [15].

In the given case, we have $A(t) = e^t$. According to (3.3), we obtain

$$\mathscr{B}_n^1((a_p),(b_q);ax) = \sum_{m=0}^{n} \binom{n}{m} a^m(1-a)^{n-m}\mathscr{B}_m^1((a_p),(b_q);x). \tag{3.6}$$

A particular case of the Brafman polynomials are the Laguerre polynomials generated by ([18], p. 201)

$$e^t {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}\middle| -xt\right) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n}t^n. \tag{3.7}$$

According to (3.3) or (3.6), we find the well-known formula ([18], p. 209)

$$L_n^{(\alpha)}(ax) = \sum_{m=0}^{n} \frac{(\alpha+1)_n}{(n-m)!(\alpha+1)_m}a^m(1-a)^{n-m}L_m^{(\alpha)}(x).$$

3.1.2 Chaunday polynomials

The Chaunday hypergeometric polynomials

$$P_n^{\lambda}(x) = {}_{p+1}F_{q+1}\left(\begin{matrix} -n,(a_p) \\ 1-\lambda-n,(b_q) \end{matrix}\middle| x\right)$$

are generated by [8]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}P_n^{\lambda}(x)t^n = (1-t)^{-\lambda}\ {}_pF_q\left(\begin{matrix} (a_p) \\ (b_q) \end{matrix}\middle| xt\right).$$

For this case we have $A(t) = (1-t)^{-\lambda}$ and

$$\frac{A(t)}{A(at)} = \left(\frac{1-t}{1-at}\right)^{-\lambda}$$

It follows, using a Cauchy product,

$$\begin{aligned} (1-t)^{-\lambda}(1-at)^{\lambda} &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}t^n \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!}a^n t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(\lambda)_{n-k}}{(n-k)!}\frac{(-\lambda)_k}{k!}a^k\right) t^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\sum_{k=0}^{n} \frac{(-n)_k(-\lambda)_k}{(1-\lambda-n)_k}\frac{a^k}{k!}\right) t^n = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n,-\lambda \\ 1-\lambda-n \end{matrix}\middle| a\right)t^n\ . \end{aligned} \tag{3.8}$$

Note that this type of computation can be done completely automatically by the `Sumtohyper` command of the Maple `hsum` package [15]. We obtain from (3.1)

$$(\lambda)_n\, P_n^\lambda(ax) = \sum_{m=0}^{n} \binom{n}{m} a^m (\lambda)_{n-m}\, {}_2F_1\left(\left.\begin{matrix} m-n, -\lambda \\ 1-\lambda-n+m \end{matrix}\right| a\right) (\lambda)_m P_m^\lambda(x). \tag{3.9}$$

It is easy to obtain the recurrence equation

$$a\, m\, (m+1)\, \beta_m(a) - (\lambda + a\, m + a + m + 1 - \lambda\, a)\, \beta_{m+1}(a) + \beta_{m+2}(a) = 0$$

for the coefficients $\beta_k(a)$ defined by (3.2) in the given situation. This can be accomplished at least in two ways: Either we use the given generating function $F(t) := A(t)/A(at)$ of $\beta_k(a)/k!$, compute the holonomic differential equation (i. e. linear, homogeneous with polynomial coefficients)

$$(-1+t)\,(-1+a\,t)\, F'(t) + \lambda\,(a-1)\, F(t) = 0$$

for $F(t)$ and convert this differential equation into the above holonomic recurrence equation for the corresponding series coefficients by the Maple `FPS` package (see [12] and [14]).

Or we use the hypergeometric representation of $\beta_k(a)$ given by (3.8) and (3.2) together with Zeilberger's algorithm (see e. g. [15]) via the Maple `sumrecursion` command which yields the same recurrence.

3.1.3 Gould-Hopper polynomials

Recall that the Gould-Hopper polynomials are generated by [11]

$$e^{ht^d} \exp(xt) = \sum_{n=0}^{\infty} g_n^d(x,h) \frac{t^n}{n!}, \quad d \in \mathbb{N}. \tag{3.10}$$

For this case we have $A(t) = e^{ht^d}$ and

$$\frac{A(t)}{A(at)} = e^{h(1-a^d)t^d} = \sum_{k=0}^{\infty} \frac{h^k(1-a^d)^k}{k!} t^{kd}\ , \tag{3.11}$$

which, by virtue of (3.1), gives

$$g_n^d(ax,h) = \sum_{m=0}^{[\frac{n}{d}]} \frac{a^n n!}{m!(n-dm)!} h^m (a^{-d}-1)^m g_{n-dm}^d(x,h). \tag{3.12}$$

This family contains as special case the Hermite polynomials $H_n(x) = g_n^2(2x,-1)$, so, (3.12) is reduced to

$$H_n(ax) = \sum_{m=0}^{[\frac{n}{2}]} \frac{a^n n!}{(n-2m)!m!} (1-a^{-2})^m H_{n-2m}(x).$$

3.2 Shifted Jacobi polynomials

The shifted Jacobi polynomials defined by [18]

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(1-x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left(\begin{matrix} -n, \alpha+\beta+n+1 \\ \alpha+1 \end{matrix} \middle| \frac{x}{2}\right),$$

are generated by

$$(1-t)^{-\lambda}{}_2F_1\left(\begin{matrix} \frac{\lambda}{2}, \frac{\lambda+1}{2} \\ \alpha+1 \end{matrix} \middle| \frac{-2xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} \frac{(\lambda)_n R_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n} t^n,$$

where $\lambda = \alpha+\beta+1$.

To solve the duplication problem for the shifted Jacobi case, we need the following lemma

Lemma 3.2 (Lagrange's inversion formula [20]). *Let ξ be a function of t implicitly defined by*

$$\xi = t(1+\xi)^{s+1}, \quad \xi(0) = 0. \tag{3.13}$$

Then we have

$$(1+\xi(t))^r = \sum_{n=0}^{\infty} \frac{r}{r+(s+1)n} \binom{r+(s+1)n}{n} t^n, \tag{3.14}$$

where r and s are complex numbers not independent of n.

For this case we have

$$A(t) = (1-t)^{-\lambda} \text{ and } \quad C(t) = \frac{-t}{(1-t)^2}\ .$$

C^{-1} is implicitly defined by

$$(1-C^{-1}(t))^2 t = -C^{-1}(t)\ .$$

Using (3.14), with $\xi = -C^{-1},\ s=1$ and $r = \lambda+2m$, we obtain

$$\begin{aligned} \frac{[C^{-1}]^m(t)}{A(C^{-1}(t))} &= (-1)^m (1-C^{-1}(t))^{2m+\lambda} t^m \\ &= (-1)^m \sum_{n=0}^{\infty} \frac{\lambda+2m}{\lambda+2n+2m} \binom{2n+2m+\lambda}{n} t^{n+m}\ . \end{aligned}$$

Replacing t by $aC(t)$ and multiplying by $A(t)$, we get moreover

$$\begin{aligned}
\mathcal{F}(t) &= (1-t)^{-\lambda}(-1)^m \sum_{n=0}^{\infty} \frac{\lambda+2m}{\lambda+2n+2m}\binom{2n+2m+\lambda}{n}(aC(t))^{n+m} \\
&= \sum_{n=0}^{\infty} \frac{\lambda+2m}{\lambda+2n+2m}\binom{2n+2m+\lambda}{n} a^{n+m}(-1)^n \frac{t^{n+m}}{(1-t)^{2n+2m+\lambda}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2n+2m+\lambda)_k}{k!}\frac{\lambda+2m}{\lambda+2n+2m}\binom{2n+2m+\lambda}{n} a^{n+m}(-1)^n t^{n+m+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(2k+2m+\lambda)_{n-k}}{(n-k)!}\frac{\lambda+2m}{\lambda+2m+2k}\binom{2m+2k+\lambda}{k} a^{m+k}(-1)^k t^{n+m} \\
&= \sum_{n=m}^{\infty}\left(\sum_{k=0}^{n-m} \frac{(2m+2k+\lambda)_{n-m-k}}{(n-m-k)!}\frac{\lambda+2m}{\lambda+2m+2k}\binom{2m+2k+\lambda}{k} a^{m+k}(-1)^k\right) t^n \,.
\end{aligned}$$

The duplication formula associated to shifted Jacobi polynomials can therefore be written in terms of hypergeometric functions as follows

$$R_n^{(\alpha,\beta)}(ax) = \sum_{m=0}^{n} \frac{a^m}{(n-m)!}\frac{(1+\alpha)_n}{(1+\alpha)_m}\frac{(\lambda+n)_m}{(\lambda+m)_m}\, {}_2F_1\left(\left.\begin{matrix} m-n, m+n+\lambda \\ 2m+\lambda+1\end{matrix}\right| a\right) R_m^{(\alpha,\beta)}(x). \quad (3.15)$$

For $C_m(n,a)$ we get (again by Zeilberger's algorithm) the recurrence

$$\begin{aligned}
&a\,(m-n)\,(\alpha+\beta+5+2\,m)\,(\alpha+\beta+2+m)\,(\alpha+\beta+1+m) \\
&(2\,m+4+\alpha+\beta)\,(m+n+\alpha+\beta+1)\,C_m(n)-(1+\alpha+m) \\
&(\alpha+\beta+5+2\,m)\,(\alpha+\beta+1+2\,m)\,(\alpha+\beta+2+m)(2\,a\,m^2-4\,m^2 \\
&-4\,m\,\beta+2\,a\,m\,\alpha-12\,m+2\,a\,m\,\beta+6\,a\,m-4\,m\,\alpha-8+2\,a\,\alpha\,n \\
&+2\,a\,n+2\,a\,n^2+2\,a\,\beta\,n+4\,a\,\beta+4\,a\,\alpha+2\,a\,\beta\,\alpha+4\,a-6\,\alpha \\
&-6\,\beta+a\,\alpha^2+a\,\beta^2-\alpha^2-2\,\beta\,\alpha-\beta^2)C_{m+1}(n)+a\,(2+m-n) \\
&(2+\alpha+m)\,(1+\alpha+m)\,(2\,m+\alpha+\beta+2)\,(\alpha+\beta+1+2\,m) \\
&(m+\alpha+3+\beta+n)\,C_{m+2}(n)=0
\end{aligned}$$

w. r. t. the variable m. The initial values for this recurrence are given by $C_n(n) = a^n$ and $C_{n+1}(n) = 0$. In a similar way, the recurrence

$$\begin{aligned}
&(2+\alpha+n)\,(1+\alpha+n)\,(4+2\,n+\beta+\alpha)\,(m-n) \\
&(m+n+\alpha+\beta+1)\,C_m(n)-(2+\alpha+n)\,(\alpha+3+2\,n+\beta) \\
&(\alpha+\beta+1+n)(a\,\alpha^2-\alpha^2+2\,a\,\beta\,\alpha-2\,\beta\,\alpha-2\,\alpha\,n-4\,\alpha-2\,m\,\alpha \\
&+6\,a\,\alpha+4\,a\,\alpha\,n-4-2\,m^2-2\,m\,\beta-2\,n^2-2\,\beta\,n+12\,a\,n \\
&+4\,a\,n^2+4\,a\,\beta\,n+6\,a\,\beta+8\,a-2\,m-4\,\beta-6\,n+a\,\beta^2-\beta^2) \\
&C_m(n+1)+(2\,n+\alpha+\beta+2)\,(\alpha+\beta+2+n)\,(\alpha+\beta+1+n) \\
&(m-n-2)\,(m+\alpha+3+\beta+n)\,C_m(n+2)=0
\end{aligned}$$

w. r. t. n is obtained, where we have replaced λ by $\alpha+\beta+1$, again.

3.3 Classical discrete orthogonal polynomials

In this section, we limit ourselves to the following particular duplication problem

$$P_n(-x) = \sum_{k=0}^{n} C_k(n) P_k(x). \tag{3.16}$$

3.3.1 Charlier polynomials

The monic Charlier polynomial set [9, Chapter VI, (1.2)]

$$\widetilde{C}_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n}{m} (-\alpha)^{n-m} m! \binom{x}{m} \quad (n \geq 0)$$

is generated by [9, Chapter VI, (1.2)]

$$G(x,t) = e^{-\alpha t} e^{x \log(1+t)} = \sum_{n=0}^{\infty} \frac{\widetilde{C}_n^{(\alpha)}(x)}{n!} t^n \ . \tag{3.17}$$

For this case we have:

$$A(t) = e^{-\alpha t},\ C(t) = \log(1+t),\ C^{-1}(t) = e^t - 1 \text{ and } \Phi(t) = -\frac{t}{t+1}.$$

It follows that

$$\mathcal{F}(t) = \frac{A(t)}{A\left(\Phi(t)\right)} \Phi^m(t) = (1+t)^{-m} \exp(-\alpha \frac{t}{t+1}) e^{-\alpha t} (-t)^m \ . \tag{3.18}$$

By sum manipulations, we get

$$\begin{aligned} (1+t)^{-m} \exp(-\alpha \frac{t}{t+1}) &= \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \frac{t^n}{(1+t)^{n+m}} \\ &= \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} t^n \left(\sum_{k=0}^{\infty} (-1)^k \frac{(n+m)_k}{k!} t^k \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^n \frac{(m)_n}{(m)_k} \frac{\alpha^k}{k!(n-k)!} \right) t^n. \end{aligned}$$

It follows

$$\begin{aligned} \mathcal{F}(t) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^n \frac{(m)_n}{(m)_k} \frac{\alpha^k}{k!(n-k)!} \right) t^n \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} t^n (-t)^m \\ &= \sum_{n=m}^{\infty} (-1)^n \left(\sum_{p=0}^{n-m} \sum_{k=0}^{p} \frac{(m)_p}{(m)_k} \frac{\alpha^{n-m+k-p}}{k!(p-k)!(n-m-p)!} \right) t^n. \end{aligned} \tag{3.19}$$

Then the duplication coefficient in (3.16) is given by

$$C_m(n) = (-1)^n \alpha^{n-m} \binom{n}{m} \sum_{k=0}^{n-m} \frac{(-n+m)_k (m)_k}{k!} \, {}_1F_1\left(\left. \begin{matrix} -k \\ m \end{matrix} \right| -\alpha \right) (-\alpha)^{-k}. \tag{3.20}$$

Recently, a fourth order recurrence relation to calculate the connection coefficients in this last duplication formula was given in [1] where the authors used a different approach based on the so-called `Navima` algorithm. Unfortunately their recurrence [1, p. 386] contains a misprint and is wrong.

Using the above double sum and a Fasenmyer type algorithm [15] to deduce recurrence equations for multiple hypergeometric series ([19], see also [22]) we get – using Sprenger's `multsum` package – the following much simpler third order recurrence for $C_m(n)$:

$$\begin{aligned}&-\alpha\,(m+2)\,(m+1)\,(m+3)\,C_{m+3}(n)\\&-(m+2)\,(m+1)\,(1+m+2\,\alpha)\,C_{m+2}(n)\\&+(m+1)\,(-1-2\,m-2\,\alpha+n)\,C_{m+1}(n)+(n-m)\,C_m(n)=0\end{aligned}$$

w. r. t. m with initial values $C_n(n)=(-1)^n$ and $C_{n+1}(n)=C_{n+2}(n)=0$, as well as[1]

$$\begin{aligned}&(n+3-m)\,C_m(n+3)+(n+2)\,(n+3)\,(n+1+2\,\alpha)\,C_m(n+1)\\&+(n+2)\,\alpha\,(n+1)\,(n+3)\,C_m(n)+(n+3)\,(2\,n+4+2\,\alpha-m)\,C_m(n+2)\\&=0\end{aligned}$$

w. r. t. n.

3.3.2 Meixner polynomials

The monic Meixner polynomial set [16, Theorem 6]

$$\widetilde{M}_n(x;\alpha,c)=(\alpha)_n\left(\frac{c}{c-1}\right)^n{}_2F_1\left(\left.\begin{matrix}-n,-x\\\alpha\end{matrix}\right|1-\frac{1}{c}\right)\quad(n\geq 0)$$

is generated by [13, (1.9.11)]:

$$G(x,t)=\sum_{n=0}^{\infty}\left(\frac{c-1}{c}\right)^n\frac{\widetilde{M}_n(x;\alpha,c)}{n!}\,t^n=\frac{1}{(1-t)^{\alpha}}\exp(x\ln\frac{1-\frac{t}{c}}{1-t}).$$

For this case, we have

$$A(t)=\frac{1}{(1-t)^{\alpha}},\ C(t)=\ln\frac{1-\frac{t}{c}}{1-t},\ C^{-1}(t)=\frac{c\,(e^t-1)}{e^t\,c-1}\ \text{and}\ \Phi(t)=\frac{c\,t}{-c+c\,t+t}\ .$$

It follows that

$$\mathcal{F}(t)=\frac{A(t)}{A\,(\Phi(t))}\Phi^m(t)=\left(\frac{1-\frac{t}{c}}{1-t}\right)^{\alpha}\frac{(-t)^m}{\left(1-(1+\frac{1}{c})t\right)^{m+\alpha}}.$$

By sum manipulations (as in (3.8)), we get

$$\mathcal{F}(t)=(-t)^m\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}{}_2F_1\left(\left.\begin{matrix}-n,-\alpha\\1-\alpha-n\end{matrix}\right|\frac{1}{c}\right)t^n\ \sum_{n=0}^{\infty}\frac{(m+\alpha)_n}{n!}(1+\frac{1}{c})^n t^n\ ,$$

which gives

$$\mathcal{F}(t)=(-t)^m\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(\alpha)_k}{k!}\,{}_2F_1\left(\left.\begin{matrix}-k,-\alpha\\1-\alpha-k\end{matrix}\right|\frac{1}{c}\right)\frac{(m+\alpha)_{n-k}}{(n-k)!}(1+\frac{1}{c})^{n-k}\right)t^n.$$

[1] Note that Maple sorts expressions by their memory allocation, therefore the output is not always in the usual order.

Then the duplication coefficient in (3.16) is given by

$$C_m(n) = (-1)^m \frac{n!}{m!} \left(\frac{c+1}{c-1}\right)^{n-m} \sum_{k=0}^{n-m} \frac{(\alpha)_k (m+\alpha)_{n-m-k}}{k!(n-m-k)!} (1+\frac{1}{c})^{-k} {}_2F_1\left(\begin{matrix} -k, -\alpha \\ 1-\alpha-k \end{matrix} \middle| \frac{1}{c}\right). \tag{3.21}$$

Note that in [1, p. 385] a similar representation was obtained.

Using the `multsum` package, we get the recurrence

$$\begin{aligned}
&-(c-1)^3\,(n-m)\,C_m(n)\\
&-(m+2)(m+1)(c-1)(cn - c^2m - c^2 - 2\alpha c^2 - 3cm - 4c - 2c\alpha - 1 - m)C_{m+2}(n)\\
&-c(m+3)(m+2)(m+1)(m+2+\alpha)(c+1)C_{m+3}(n)\\
&+(c-1)^2(m+1)(cn + n - 2cm - c - 2c\alpha - 2m - 1)C_{m+1}(n) = 0
\end{aligned}$$

w. r. t. m with initial values $C_n(n) = (-1)^n$ and $C_{n+1}(n) = C_{n+2}(n) = 0$, as well as

$$\begin{aligned}
&c(n+2)(n+1)(n+3)(\alpha+n)(c+1)C_m(n)\\
&-(n+3)(n+2)(c-1)(nc^2 + c^2 + 3cn + 3c + n + 1 + 2c^2\alpha + 2c\alpha - cm)C_m(n+1)\\
&+(c-1)^2(n+3)(2cn + 4c + 2n + 4 - cm + 2c\alpha - m)C_m(n+2)\\
&-(c-1)^3(n+3-m)C(n+3) = 0
\end{aligned}$$

w. r. t. n.

3.3.3 Krawtchouk polynomials

The monic Krawtchouk polynomial set [16, Theorem 6]

$$\widetilde{K}_n^p(x,N) = p^n\,(-N)_n\,{}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| \frac{1}{p}\right) \quad (n \geq 0)$$

is generated by [21]

$$G(x,t) = (1-pt)^N \left(\frac{1+qt}{1-pt}\right)^x = \sum_{n=0}^{\infty} \frac{\widetilde{K}_n^p(x,N)}{n!}\, t^n\ ,$$

where q satisfies $p + q = 1$. For this case, we have

$$A(t) = (1-pt)^N,\ C(t) = \ln\frac{1+qt}{1-pt},\ C^{-1}(t) = \frac{e^t - 1}{e^t\,p - p + 1} \text{ and } \Phi(t) = \frac{t}{2\,p\,t - 1 - t}\ .$$

It follows that

$$\mathcal{F}(t) = \frac{A(t)}{A\left(\Phi(t)\right)}\Phi^m(t) = \left(\frac{1-pt}{1-(p-1)t}\right)^N \frac{(-t)^m}{(1-(2p-1)t)^{m-N}}.$$

Again, by sum manipulation, we get

$$\mathcal{F}(t) = (-1)^m \sum_{n=m}^{\infty} \left(\sum_{k=0}^{n} \frac{(N)_k}{k!}\ {}_2F_1\left(\begin{matrix} -k, -N \\ 1-N-k \end{matrix} \middle| \frac{p}{p-1}\right) \frac{(m-N)_{n-k}}{(n-k)!}(p-1)^k(2p-1)^{n-k}\right) t^n.$$

Then the duplication coefficient in (3.16) is given by

$$C_m(n) = (-1)^m \frac{n!}{m!}(2p-1)^{n-m} \sum_{k=0}^{n-m} \frac{(N)_k (m-N)_{n-m-k}}{k!(n-m-k)!} (\frac{p-1}{2p-1})^k {}_2F_1\left(\begin{matrix} -k, -N \\ 1-N-k \end{matrix} \middle| \frac{p}{p-1} \right). \tag{3.22}$$

One can find such a representation also by the formula

$$\widetilde{K}_n^p(x,N) = \widetilde{M}_n(x; -N, \frac{p}{p-1})$$

which follows from the hypergeometric representations. Therefore from representation (3.21) it follows for the multiplication coefficients of the Krawtchouk polynomials

$$C_m(n) = (-1)^n \frac{n!}{m!} (2p-1)^{n-m} \sum_{k=0}^{n-m} \frac{(-N)_k (m-N)_{n-m-k}}{k! (n-m-k)!} \left(\frac{p}{2p-1} \right)^k {}_2F_1\left(\begin{matrix} -k, N \\ 1+N-k \end{matrix} \middle| \frac{p-1}{p} \right).$$

Note that this representation differs from (3.22) modulo some hypergeometric identity.

Using the `multsum` package, we get the recurrence

$$\begin{aligned}
&p\,(m+3)\,(2\,p-1)\,(p-1)\,(m+2)\,(m+1)\,(m+2-N)\,C_{m+3}(n)-\\
&(m+2)\,(m+1)\\
&(m+1+5\,p^2\,m+6\,p^2-5\,p\,m-6\,p-4\,p^2\,N+2\,N\,p-n\,p^2+p\,n)\\
&C_{m+2}(n)\\
&+(m+1)\,(4\,p\,m+2\,p-2\,m-1-2\,N\,p-2\,p\,n+n)\,C_{m+1}(n)\\
&+(n-m)\,C_m(n)=0
\end{aligned}$$

w. r. t. m with initial values $C_n(n) = (-1)^n$ and $C_{n+1}(n) = C_{n+2}(n) = 0$, as well as

$$\begin{aligned}
&p\,(n+2)\,(n+1)\,(n+3)\,(2\,p-1)\,(p-1)\,(n-N)\,C_m(n)+(n+3)\,(n+2)\\
&(p^2\,m-p\,m+5\,p+5\,p\,n-5\,p^2-5\,n\,p^2-1-n+4\,p^2\,N-2\,N\,p)\\
&C_m(n+1)+(m-n-3)\,C_m(n+3)\\
&-(n+3)\,(2\,p\,m-m-8\,p-4\,p\,n+4+2\,n+2\,N\,p)\,C_m(n+2)=0
\end{aligned}$$

w. r. t. n.

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