Numerical Methods for Non-Stationary Stokes Flow

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Abstract We consider a first order implicit time stepping procedure (Euler scheme) for the non-stationary Stokes equations in smoothly bounded domains of $\mathbb{R}^3$. Using energy estimates we can prove optimal convergence properties in the Sobolev spaces $H^m(G)$ $(m = 0, 1, 2)$ uniformly in time, provided that the solution of the Stokes equations has a certain degree of regularity. For the solution of the resulting Stokes resolvent boundary value problems we use a representation in form of hydrodynamical volume and boundary layer potentials, where the unknown source densities of the latter can be determined from uniquely solvable boundary integral equations' systems. For the numerical computation of the potentials and the solution of the boundary integral equations a boundary element method of collocation type is used. Some simulations of a model problem are carried out and illustrate the efficiency of the method.

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1 Introduction and Notation

Let $T > 0$ be given and $G \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth compact boundary S. In $(0,T)$ we consider the non-stationary Stokes equations

$$D_t v - \nu \Delta v + \nabla p = F, \quad div\ v = 0, \quad v_{|S} = 0, \quad v_{|t=0} = v_0\,. \tag{1.1}$$

These equations describe the linearized motion of a viscous incompressible fluid: The vector $v = (v_1(t,x),\ v_2(t,x),\ v_3(t,x))$ represents the velocity field and the scalar $p = p(t,x)$ the kinematic pressure function of the fluid at time $t \in (0,T)$ and at position $x \in G$. The constant $\nu > 0$ is the kinematic viscosity, and the external force density F together with the initial velocity v_0 are the given data. The condition div $v = 0$ means the incompressibility of the fluid, and $v = 0$ on the boundary S expresses the no-slip condition, i. e. the fluid adheres to the boundary.

It is the aim of the present paper to develop a method for the numerical solution of (1.1). This method consists of three steps. In the first step, the implicit Euler method in time is used in to transform (1.1) into a finite number of certain boundary value problems. In the second step, these boundary value problems are studied with methods of hydrodynamical potential theory. This leads to a representation of their solutions consisting of volume and surface potentials, where the unknown densities have to be determined from systems of boundary integral equations. In the third step, for the discretization of the boundary integral equations and the numerical computation of the potentials a boundary element method of collocation type and suitable quadrature methods are used.

Let us consider the following semi-discrete first order Euler approximation scheme for the Stokes equations (1.1): Setting

$$h = T/N > 0, \qquad t_k = k\,h \quad (k = 0, 1, \dots, N),$$

we approximate the solution v, p of (1.1) at time t_k by the solution $v^k, p^k (k = 1, 2, \dots, N)$ of the following equations in G:

$$\begin{aligned} &(v^k - v^{k-1})h^{-1} - \nu\Delta v^k + \nabla p^k \;=\; h^{-1} \int\limits_{(k-1)h}^{kh} F(t)dt\,, \\ &\operatorname{div} v^k = 0, \qquad\qquad v^0 = v_0, \qquad\qquad v^k_{|s} = 0\,. \end{aligned}$$

Here F and v_0 are the given data. Thus for every $k = 1, 2, 3, \dots, N$ we have to determine in G the solution v^k, q^k of the Stokes resolvent boundary value problem

$$(\lambda - \Delta)v^k + \nabla q^k = F^{\lambda,k-1}\,, \qquad \operatorname{div} v^k = 0\,, \qquad v^k_{|s} = 0\,,$$

with $\lambda = (\nu h)^{-1} > 0$, $q^k = p^k/\nu$, and

$$F^{\lambda,k-1}(x) \;=\; \lambda \left(v^{k-1}(x) + \int\limits_{(k-1)h}^{kh} F(t,x)dt \right). \tag{1.2}$$

Using methods of hydrodynamical potential theory we find a representation of the solution v^k, q^k in the form

$$(v^k(x), q^k(x)) \;=\; (V_\lambda F^{\lambda,k-1})(x) + (D_\lambda \Psi)(x), \quad x \in G. \tag{1.3}$$

Here $V_\lambda F^{\lambda,k-1}$ is a hydrodynamical volume potential with density $F^{\lambda,k-1}$, and $D_\lambda \Psi$ is a double layer potential with an unknown source density Ψ, which can be determined from the boundary integral equations

$$-(V^*_\lambda F^{\lambda,k-1})(x) \;=\; \frac{1}{2}\Psi(x) + (D^*_\lambda \Psi)(x) - (P_N \Psi)(x)\,, \quad x \in S. \tag{1.4}$$

Here the superscript $*$ indicates the velocity part of the above potentials. $(D_\lambda^* \Psi)$ is the direct value of the hydrodynamical double layer potential for the velocity, and P_N is a one-dimensional perturbation operator, which ensures that the solution Ψ is unique in the space of continuous vector fields on S. For the spatial discretization of (1.4) we use a boundary element method of collocation type as described in [1], [5].

At this point, let us introduce our notations. Throughout the paper, $G \subset \mathbb{R}^3$ is a bounded domain having a compact boundary S of class C^2. In the following, all functions are real valued. As usual, $C_0^\infty(G)$ denotes the space of smooth functions defined in G with compact support, and $L^2(G)$ is equipped with scalar product and norm

$$(f,g) \;=\; \int\limits_G f(x)g(x)dx\,, \qquad ||f|| \;=\; (f,f)^{\frac{1}{2}}\,,$$

respectively. For functions $f,g \in L^2(G)$ we need the following well-known relations:

$$\begin{aligned}(f-g,f+g) &= ||f||^2 - ||g||^2\,,\\ (f-g,2f) &= ||f||^2 - ||g||^2 + ||f-g||^2\,,\\ 2(f,g) \le 2\,||f||\,||g|| &\le\; ||f||^2 + ||g||^2\,.\end{aligned} \tag{1.5}$$

The Sobolev space $H^m(G)$ $(m = 0,1,2,\ldots)$ is the space of functions f such that $D^\alpha f \in L^2(G)$ for all $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 \le m$. Its norm is denoted by

$$||f||_m \;=\; ||f||_{H^m(G)} \;=\; \left(\sum_{|\alpha|\le m} ||D^\alpha f||^2\right)^{\frac{1}{2}}\,,$$

where $D^\alpha \;=\; D_1^{\alpha_1}D_2^{\alpha_2}D_3^{\alpha_3}$ with $D_k = \frac{d}{dx_k}$ $(k = 1,2,3)$ is the distributional derivative. The completion of $C_0^\infty(G)$ with respect to $||\cdot||_m$ is denoted by $H_0^m(G)(H_0^0(G) \;=\; H^0(G) \;=\; L^2(G))$. If $f \in H_0^1(G)$, in particular, we have Poincaré's inequality

$$||f||^2 \;\le\; C_G||\nabla f||^2\,, \tag{1.6}$$

where here the constant $\lambda_1 = C_G^{-1}$ is the smallest eigenvalue of the Laplace operator $-\Delta$ in G with zero boundary condition.

The spaces $C_0^\infty(G)^3, L^2(G)^3, H^m(G)^3, \ldots$ are the corresponding spaces of vector fields $u = (u_1,u_2,u_3)$. Here norm and scalar product are denoted as in the scalar case, i. e. for example,

$$(u,v) = \sum_{k=1}^{3}(u_k,v_k)\,, \quad ||u|| \;=\; (u,u)^{\frac{1}{2}} \;=\; \int\limits_G |u(x)|^2 dx^{\,\frac{1}{2}}\,,$$

where $|u(x)| = (u_1(x)^2 + u_2(x)^2 + u_3(x)^2)^{\frac{1}{2}}$ is the Euclidian norm of $u(x) \in \mathbb{R}^3$. The completion of

$$C_{0,\sigma}^{\infty}(G)^3 \;=\; \{u \in C_0^{\infty}(G)^3 | \text{div } u \;=\; 0\}$$

with respect to the norm $||\cdot||$ and $||\cdot||_1$ are important spaces for the treatment of the Stokes equations. They are denoted by

$$H(G)^3 , \quad V(G)^3 ,$$

respectively. In $H_0^1(G)^3$ and $V(G)^3$ we also use

$$(\nabla u, \nabla v) \;=\; \sum_{k,j=1}^{3} (D_k u_j, D_k v_j) , \qquad ||\nabla u|| \;=\; (\nabla u, \nabla u)^{\frac{1}{2}}$$

as scalar product and norm. Moreover, we need the B-valued spaces $C^m(J,B)$ and $H^m(a,b,B)$, $m \in \mathbb{N}_0$, where $J \subset \mathbb{R}$ with $a,b \in \mathbb{R}$ $(a < b)$, and where B is any of the spaces above. In case of $C^0(,)$ we simply write $C(,)$, and we use $H, V, H^m, \ldots$ instead of $H(G), V(G), H^m(G), \ldots$, if the domain of definition is clear from the context. Finally, let

$$P : L^2(G)^3 \;\longrightarrow\; H(G)^3 \tag{1.7}$$

denote the orthogonal projection. Then we have

$$L^2(G)^3 \;=\; H(G)^3 \oplus \{v \in L^2(G)^3 | v \;=\; \nabla p \text{ for some } p \in H^1(G)\} ,$$

with means

$$(u, \nabla p) \;=\; 0 \quad \text{for all} \quad u \in V(G)^3 \quad \text{and} \quad p \in H^1(G) . \tag{1.8}$$

2 An Implicit Euler Scheme

Because the projection P from (1.7) commutes with the strong time derivative D_t, from the Stokes equations (1.1) we obtain the following evolution equations for the function $t \to v(t) \in H(G)^3$:

$$D_t v(t) - \nu P \Delta v(t) \;=\; PF(t) \; (t \in (0,T)) , \qquad v(0) \;=\; v_0 . \tag{2.1}$$

In this case, the condition div $v = 0$ and the boundary condition $v = 0$ on S are satisfied in the sense that we require $v(t) \in V(G)^3$ for all $t \in (0,T)$. Concerning the solvability of the evolution equations (2.1) it is known that for

$$v_0 \in H^2(G)^3 \cap V(G)^3 , \qquad F \in H^1(0,T,H(G)^3) \tag{2.2}$$

there is a unique solution v of (2.1) in G such that

$$v \in C([0,T], H^2(G)^3 \cap V(G)^3) , \; D_t v \in C([0,T], H(G)^3) \cap L^2(0,T,H^1(G)^3) , \tag{2.3}$$

and that there is some constant K depending only on G, ν, F, v_0 and not on $t \in [0,T]$ such that for all $t \in [0,T]$

$$\int_0^t ||\nabla D_\sigma v(\sigma)||^2 d\sigma \leq K\,, \quad ||v(t)||_2 \leq K\,, \quad ||D_t v(t)|| \leq K\,. \tag{2.4}$$

Let us now consider the discrete equations under the weaker assumptions

$$v_0 \in H(G)^3\,, \quad F \in L^2(0,T,H(G)^3)\,. \tag{2.5}$$

Using P as above and noting that $F = PF$ we obtain in $G(h = T/N > 0)$

$$(v^k - v^{k-1}) - h\nu P\Delta v^k \;=\; \int_{(k-1)h}^{kh} F(t)dt\,, \quad v^0 \;=\; v_0\,. \tag{2.6}$$

It is known that under the above assumptions (2.5) there is a unique solution

$$v^k \in H^2(G)^3 \cap V(G)^3 \qquad (k = 1,2,\ldots,N) \tag{2.7}$$

of (2.6): If we define the Stokes operator A to be the extension of $-P\Delta$ in $H(G)^3$, then its domain of definition $D(A)$ is $H^2(G)^3 \cap V(G)^3$. Because $\lambda = (\nu h)^{-1} > 0$ belongs to the resolvent set of $-A$, the equations

$$v^k \;=\; (\lambda + A)^{-1} F^{\lambda,k-1}\,, \qquad F^{\lambda,k-1} \in H(G)^3$$

(see (1.2)) are uniquely solvable with $v^k \in H^2(G)^3 \cap V(G)^3$, as asserted.

To prove the convergence of the discrete equations (2.6) to the evolution equations (2.1) and to estimate the discretization error, we use the approach "'stability + consistency $\rightarrow$ convergence"'. Let us define

$$\left(\Pi v\right)(t_k) = v(t_k) - v(t_{k-1}) - \nu h P\Delta v(t_k), \quad \left(\Pi\{v^j\}\right)(t_k) = v^k - v^{k-1} - \nu h P\Delta v^k\,.$$

Then the discretization error

$$e^k \;=\; v^k - v(t_k) \tag{2.8}$$

satisfies the identity

$$e^k - e^{k-1} - \nu h P\Delta e^k \;=\; \left(\Pi\{v^j\}\right)(t_k) - \left(\Pi v\right)(t_k) \;=\; R^k\,, \tag{2.9}$$

which is used to obtain estimates of e^k in terms of the right hand side R^k ($\approx$ stability). Then the behavior of

$$
\begin{aligned}
R^k &= \int\limits_{(k-1)h}^{kh} (D_t v(t) - \nu P\Delta v(t))dt - \{(v(t_k) - v(t_{k-1})) - h\nu P\Delta v(t_k)\} \\
&= \int\limits_{(k-1)h}^{kh} -\nu P\Delta(v(t) - v(t_k))dt \qquad (2.10) \\
&= -\nu P\Delta E^k
\end{aligned}
$$

as h tends to zero (≈ consistency) follows from the regularity properties of the exact solution of the Stokes equations (2.1).

Theorem 1. *Let $T > 0$, $N \in \mathbb{N}$, and $G \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary S of class C^2. Assuming (2.2), let v and $v^k (k = 1,2,\ldots,N)$ denote the solution of (2.1) and (2.6), respectively. Then the discretization error e^k (see (2.8)) satisfies the following estimates:*

$$
||e^k||^2 + \sum_{j=1}^{k} (h\nu||\nabla e^j||^2 + ||e^j - e^{j-1}||^2) \ \le \ Kh^2 \, ,
$$

$$
||\nabla e^k||^2 + \sum_{j=1}^{k} (2(h\nu)^{-1}||e^j - e^{j-1}||^2 + \frac{1}{2}||\nabla(e^j - e^{j-1})||^2) \ \le \ Kh \, .
$$

Here the constant K depends only on G, ν, and the data. Moreover, we even have convergence with respect to the H^2-norm:

$$
\max\{||e^k||_2 \mid k = 1,2,\ldots,N\} \ = \ \circ(1) \ \textit{ as } \ h \longrightarrow 0 \ \textit{ or } \ N \longrightarrow \infty \, .
$$

Proof. From (2.9) and (2.10) we obtain for the defect e^k the identity

$$
(e^k - e^{k-1}) - h\nu P\Delta e^k \ = \ -\nu P\Delta E^k \, . \qquad (2.11)
$$

Multiplying (2.11) scalar in L^2 by $2e^k$ and using (1.5) we obtain

$$
\begin{aligned}
&||e^k||^2 \ - \ ||e^{k-1}||^2 \ + \ ||e^k - e^{k-1}||^2 \ + \ 2h\nu||\nabla e^k||^2 \ = \ 2\nu(\nabla E^k, \nabla e^k) \ \le \\
&2\,(h\nu)^{\frac{1}{2}}||\nabla e^k||\,(h^{-1}\nu)^{\frac{1}{2}}||\nabla E^k|| \ \le \ h\nu||\nabla e^k||^2 + h^{-1}\nu||\nabla E^k||^2 \ = \ S_1 + S_2 .
\end{aligned}
$$

Because of

$$
\begin{aligned}
S_2 &= h^{-1}\nu \ \left\| \int\limits_{(k-1)h}^{kh} \int\limits_{t}^{kh} D_\sigma \nabla v(\sigma) d\sigma dt \right\|^2 \le \nu \int\limits_{(k-1)h}^{kh} \left\| \int\limits_{(k-1)h}^{kh} |D_\sigma \nabla v(\sigma)| d\sigma \right\|^2 dt \\
&\le \nu \, h \left\| \int\limits_{(k-1)h}^{kh} |\nabla v(\sigma)| d\sigma \right\|^2 \le \nu h^2 \int\limits_{(k-1)h}^{kh} ||D_\sigma \nabla v(\sigma)||^2 d\sigma \, ,
\end{aligned}
$$

we find

$$||e^k||^2 - ||e^{k-1}||^2 + ||e^k - e^{k-1}||^2 + h\nu||\nabla e^k||^2 \leq \nu h^2 \int\limits_{(k-1)h}^{kh} ||D_\sigma \nabla v(\sigma)||^2 d\sigma \quad (2.12)$$

for all $k = 1, 2, \ldots, N$. Thus using $||e^0||^2 = 0$ and (2.4), the first estimate is proved. Next let us multiply (2.11) scalar in L^2 by $2(e^k - e^{k-1})$. Here we obtain

$$\begin{aligned}
&\quad 2||e^k - e^{k-1}||^2 + 2h\nu(\nabla e^k, \nabla(e^k - e^{k-1})) \\
&= 2||e^k - e^{k-1}||^2 + h\nu(||\nabla e^k||^2 - ||\nabla e^{k-1}||^2 + ||\nabla(e^k - e^{k-1})||^2) \\
&= 2\nu(\nabla E^k, \nabla(e^k - e^{k-1})) \\
&\leq 2 \cdot \left(\frac{h\nu}{2}\right)^{\frac{1}{2}} ||\nabla(e^k - e^{k-1})|| \cdot (2h^{-1}\nu)^{\frac{1}{2}} ||\nabla E^k|| \\
&\leq \frac{h\nu}{2} ||\nabla(e^k - e^{k-1})||^2 + 2h^{-1}\nu||\nabla E^k||^2 \ = \ S_3 + 2S_2 \,.
\end{aligned}$$

Using the above estimate for S_2 again, we have

$$\begin{aligned}
2||e^k - e^{k-1}||^2 + h\nu(||\nabla e^k||^2 - ||\nabla e^{k-1}||^2) - \frac{h\nu}{2}||\nabla(e^k - e^{k-1})||^2 \\
\leq 2\nu h^2 \int\limits_{(k-1)h}^{kh} ||D_\sigma \nabla v(\sigma)||^2 d\sigma \,,
\end{aligned}$$

hence

$$\begin{aligned}
||\nabla e^k||^2 - ||\nabla e^{k-1}||^2 + 2(h\nu)^{-1}||e^k - e^{k-1}||^2 + \frac{1}{2}||\nabla(e^k - e^{k-1})||^2 \\
\leq 2h \int\limits_{(k-1)h}^{kh} ||D_\sigma \nabla v(\sigma)||^2 d\sigma \,,
\end{aligned}$$

which implies the second estimate. Next we want to prove convergence with respect to the H^2-norm. From (2.11) we conclude

$$P\Delta e^k \ = \ (h\nu)^{-1}(e^k - e^{k-1}) + h^{-1}P\Delta E^k \,,$$

which implies

$$||P\Delta e^k||^2 \ \leq \ 2(h\nu)^{-2}||e^k - e^{k-1}||^2 + 2h^{-2}||P\Delta E^k||^2 \,. \quad (2.13)$$

By (2.3) we find the following estimate for the second term:

$$
\begin{aligned}
2h^{-2}||P\Delta E^k||^2 &\leq 2h^{-2}\left\| \int\limits_{(k-1)h}^{kh} P\Delta(v(t)-v(t_k))dt \right\|^2 \\
&\leq 2 \max_{\substack{\sigma,\tau\in[0,T] \\ |\sigma-\tau|\leq h}} ||P\Delta(v(\sigma)-v(\tau))||^2 \\
&= \circ(1) \;\text{ as }\; h \longrightarrow 0\,.
\end{aligned}
$$

It remains to show that also the first term of (2.13) tends to zero. Using

$$
T^k \;=\; \frac{(e^k-e^{k-1})}{h} \qquad (k=1,2,\ldots,N)
$$

for abbreviation, from (2.11) we obtain the identity

$$
\begin{aligned}
&T^k - T^{k-1} - h\nu P\Delta T^k \\
&= -h^{-1}\nu P\Delta \left\{ \int\limits_{(k-1)h}^{kh} (v(t)-v(t_k))dt - \int\limits_{(k-2)h}^{(k-1)h} (v(t)-v(t_{k-1}))dt \right\} \\
&= -h^{-1}\nu P\Delta G^k\,,
\end{aligned}
$$

where G^k is defined by the above term in brackets. Scalar multiplication in L^2 by $2T^k$ yields as above

$$
||T^k||^2 - ||T^{k-1}||^2 + ||T^k-T^{k-1}||^2 + 2h\nu||\nabla T^k||^2 \;\leq\; h\nu||\nabla T^k||^2 + h^{-3}\nu||\nabla G^k||^2\,, \tag{2.14}
$$

hence

$$
||T^k||^2 - ||T^{k-1}||^2 + ||T^k-T^{k-1}||^2 + h\nu||\nabla T^k||^2 \;\leq\; h^{-3}\nu||\nabla G^k||^2\,.
$$

Because

$$
G^k \;=\; -\int\limits_{(k-1)h}^{kh} \int\limits_{t}^{kh} (D_\sigma v(\sigma) - D_\sigma v(\sigma-h))d\sigma dt\,,
$$

we find the estimate

$$
||\nabla G^k||^2 \;\leq\; h^3 \int\limits_{(k-1)h}^{kh} ||D_\sigma \nabla(v(\sigma)-v(\sigma-h))||^2 d\sigma\,.
$$

Thus from (2.14) we obtain

$$||T^k||^2 - ||T^{k-1}||^2 + ||T^k - T^{k-1}||^2 + h\nu||\nabla T^k||^2 \le \nu \int\limits_{(k-1)h}^{kh} ||D_t \nabla(v(t) - v(t-h))||^2 dt\,,$$

and

$$||T^k||^2 + \sum_{j=2}^{k} (||T^j - T^{j-1}||^2 + \nu h ||\nabla T^j||^2)$$

$$\le ||T^1||^2 + \nu \int\limits_h^T ||D_t \nabla(v(t) - v(t-h))||^2 dt \tag{2.15}$$

$$= \circ(1) \text{ as } h \longrightarrow 0\,,$$

because the integral vanishes as $h \to 0$, and because by (2.12) (note $||e^0|| = 0$)

$$||T^1||^2 \;=\; ||(e^1 - e^0)h^{-1}||^2 \;\le\; \nu \int\limits_0^h ||D_t \nabla v(t)||^2 dt \;=\; \circ(1)\,.$$

Thus (2.15) implies that also the first term of (2.13) tends to zero as $h \to 0$, hence $||P\Delta e^k||^2 = \circ(1)$ as $h \to 0$, and the asserted convergence with respect to the H^2-norm follows by means of Cattabriga's estimate. This proves the theorem.

3 Hydrodynamical Potential Theory

Because every time step $t_k = kh$ ($k = 1,2,\dots,N \in \mathbb{N}$; $h = \frac{T}{N} > 0$) requires the solution of the boundary value problem (**??**), we consider for fixed h,k, and $\lambda = (h\nu)^{-1} > 0$ in G the system

$$(\lambda - \Delta)u + \nabla q \;=\; F\,, \quad \operatorname{div} u \;=\; 0\,, \quad u_{|S} \;=\; 0\,. \tag{3.1}$$

Let us define the formal differential operator of (3.1) by

$$S_\lambda : \begin{pmatrix} u \\ q \end{pmatrix} \longrightarrow S_\lambda \begin{matrix} u \\ q \end{matrix} \;=\; \begin{pmatrix} (\lambda - \Delta)u + \nabla q \\ \nabla \cdot u \end{pmatrix},$$

and let

$$S'_\lambda : \begin{pmatrix} u \\ q \end{pmatrix} \longrightarrow S'_\lambda \begin{matrix} u \\ q \end{matrix} \;=\; \begin{pmatrix} (\lambda - \Delta)u - \nabla q \\ -\nabla \cdot u \end{pmatrix},$$

denote its formally adjoint operator. To construct an explicit solution u,q of (3.1) with methods of potential theory, we first need the singular fundamental tensor $E_\lambda = (E^\lambda_{jk})_{j,k=1,\dots,4}$, i. e. a solution of $S_\lambda E_\lambda = \delta I_4$ in the space of tempered distributions.

Here δ is Dirac's in $\mathbb{R}^3$, I_4 the 4×4 unity matrix, and $S_\lambda E_\lambda = (SE_1^\lambda, SE_3^\lambda, SE_4^\lambda)$ with columns $E_k^\lambda = (E_{jk}^\lambda)_{j=1,\ldots,4}$ for $k=1,\ldots,4$. It is well-known ([?]) that the fundamental tensor $E_\lambda = (E_{jk}^\lambda(x))_{j,k=1,\ldots,4}$ has the following form:

$$
\begin{aligned}
E_{jk}^\lambda(x) &= \frac{1}{4\pi}\left\{\frac{\delta_{jk}}{|x|}e_1(-\sqrt{\lambda}\,|x|)+\frac{x_jx_k}{|x|^3}e_2(-\sqrt{\lambda}\,|x|)\right\} \quad (k,j\neq 4)\\
e_1(\varepsilon) &= \sum_{n=0}^{\infty}\frac{(n+1)^2}{(n+2)!}\varepsilon^n \;=\; \exp(\varepsilon)(1-\varepsilon^{-1}+\varepsilon^{-2})-\varepsilon^{-2}\\
e_2(\varepsilon) &= \sum_{n=0}^{\infty}\frac{1-n^2}{(n+2)!}\varepsilon^n \;=\; \exp(\varepsilon)(-1+3\varepsilon^{-1}-3\varepsilon^{-2})+3\varepsilon^{-2} \qquad (3.2)\\
E_{4k}^\lambda(x) &= E_{k4}^\lambda(x) \;=\; \frac{x_k}{4\pi|x|^3} \quad (k\neq 4)\,,\\
E_{44}^\lambda(x) &= \delta(x)+\frac{\lambda}{4\pi|x|}\,.
\end{aligned}
$$

Using the exponential representation of the functions e_1, e_2 we obtain immediately the behavior of $E_\lambda(x)$ for $x\to 0$ and $x\to\infty$. Setting $r=|x|$ we have for $j,k\neq 4$:

$$
\begin{aligned}
E_{jk}^\lambda(x) &= O(r^{-1}) \quad \text{as} \quad r\longrightarrow 0\\
E_{jk}^\lambda(x) &= O(r^{-3}) \quad \text{as} \quad r\longrightarrow \infty \quad (\lambda>0) \qquad (3.3)\\
E_{4k}^\lambda(x) &= O(r^{-2}) \quad \text{as} \quad r\longrightarrow 0 \;\text{ or }\; r\longrightarrow\infty\,.
\end{aligned}
$$

Note that $E_{jk}^\lambda(\lambda>0)$ decays stronger than $E_{jk}^0(j,k\neq 4)$ as $r\to\infty$.

Now using the right hand side F from (3.1) and the fundamental tensor E_λ, we can construct the hydrodynamical volume potential

$$
(U(x),Q(x)) \;=\; \int_G < \begin{pmatrix}F(y)\\ 0\end{pmatrix}, \quad E_\lambda(x-y) > dy\,, \qquad (3.4)
$$

which satisfies the equations $S_\lambda \begin{smallmatrix}U\\Q\end{smallmatrix} = \binom{F}{0}$ in G due to its construction. Here and in the sequel, for $\xi\in\mathbb{R}^n$ and matrices $A=(A_{ji})\in\mathbb{R}^n\times\mathbb{R}^m$ $(n,m\in\mathbb{N})$ we use

$$
(\xi,A) \;=\; \left(\sum_{j=1}^{n}\xi_jA_{j1},\ldots,\sum_{j=1}^{n}\xi_jA_{jm}\right),
$$

obtaining a row with m components.

In order to represent the solution of (S_λ) by means of potentials we need the hydrodynamical Green's formulae. They are given in terms of the formal differential operators

$$S_\lambda : \begin{pmatrix} u \\ q \end{pmatrix} \longrightarrow S_\lambda \begin{matrix} u \\ q \end{matrix}, \qquad S'_\lambda : \begin{pmatrix} u \\ q \end{pmatrix} \longrightarrow S'_\lambda \begin{matrix} u \\ q \end{matrix}$$

from above, and their corresponding adjoint stress tensors, which are defined by

$$T : \begin{pmatrix} u \\ q \end{pmatrix} \longrightarrow T \begin{matrix} u \\ q \end{matrix} \;=\; (-\nabla u - (\nabla u)^T + qI_3)\,,$$

$$T' : \begin{pmatrix} u \\ q \end{pmatrix} \longrightarrow T' \begin{matrix} u \\ q \end{matrix} \;=\; (-\nabla u - (\nabla u)^T - qI_3)\,.$$

Here $(\nabla u)^T$ is the transposed matrix of $\nabla u = (D_i u_k)_{k,i=1,2,3}$ and I_3 the 3×3 unity matrix.

Let us assume that $u, v \in C^2(G)^3 \cap C^1(\overline{G})^3$ are divergence-free vector fields, that $q, p \in C^1(G) \cap C^0(\overline{G})$, and that $S_\lambda \begin{smallmatrix} u \\ q \end{smallmatrix}$, $S'_\lambda \begin{smallmatrix} v \\ p \end{smallmatrix} \in L^1(G)^3 (\lambda > 0)$. Then we have Green's first identity

$$\begin{aligned} &\int_G (S_\lambda \begin{matrix} u \\ q \end{matrix}, \begin{pmatrix} v \\ p \end{pmatrix})dy \\ &= \int_S (T \begin{matrix} u \\ q \end{matrix} N, v)do_y + \int_G (\lambda u, v)dy + \int_G \frac{1}{2}(\nabla u + (\nabla u)^T,\, \nabla v + (\nabla v)^T)dy\,, \end{aligned} \tag{3.5}$$

and Green's second identity

$$\int_G \left\{ (S_\lambda \begin{matrix} u \\ q \end{matrix}, \begin{pmatrix} v \\ p \end{pmatrix}) - (\begin{pmatrix} u \\ q \end{pmatrix}, S'_\lambda \begin{matrix} v \\ p \end{matrix}) \right\} dy \;=\; \int_S \left\{ (T^u_q N, v) - (u, T'^v_p N) \right\} do_y\,. \tag{3.6}$$

Here we use

$$< \xi, \eta > = \sum_{k=1}^n \xi_k \eta_k \;\text{ for }\; \xi, \eta \in \mathbb{R}^n \;\text{ and }\; < A, B > = \sum_{i,k=1}^n A_{ik} B_{ik}$$

for matrices $A, B \in \mathbb{R}^n \times \mathbb{R}^n$ $(n \in \mathbb{N})$. The vector $N = N(y) \in \mathbb{R}^3$ denotes the exterior normal in $y \in S$ and $T^u_q N$ indicates the usual matrix vector product.

Now applying Green's second identity with a solution $u \in C^2(G)^3 \cap C^1(\overline{G})^3$, $q \in C^1(G) \cap C^0(\overline{G})$ of $S_\lambda \begin{smallmatrix} u \\ q \end{smallmatrix} = \binom{F}{0}$, and with v, p being the columns of the fundamental tensor E_λ, by cutting off the singularity in $x \in G$ we obtain the following representation (compare [3], p. 335) of u and q in $x \in G$ (N denotes the exterior normal on the C^2-boundary S):

$$\begin{aligned} &\int_G (\begin{pmatrix} F(y) \\ 0 \end{pmatrix}, E_\lambda(x-y))dy - (u(x), q(x)) \\ &= \int_S (T^u_q(y)N(y), E^{(r)}_\lambda(x-y))do_y - \int_S (u(y), T'_y E_\lambda(x-y)N(y))do_y\,. \end{aligned} \tag{3.7}$$

Here $E_\lambda^{(r)}$ is the 3×4 matrix obtained from E_λ by eliminating the last row, and the product in the last boundary integral equation is defined as follows: Treating the 4 columns of E_λ with T' yields four 3×3 matrices, which, multiplied by N, give four columns with 3 components, hence a 3×4 matrix. The subscript y in T' means differentiation with respect to y.

The representation formula (3.7) suggests to introduce hydrodynamical boundary layer potentials for general vector valued source densities $\Psi = (\Psi_1, \Psi_2, \Psi_3) \in C(S)^3$. For $x \in \mathbb{R}^3 \backslash S$ we define the single layer potential

$$(E_\lambda \Psi)(x) \;=\; \int\limits_S (\Psi(y), E_\lambda^{(r)}(x-y))do_y$$

and the double layer potential

$$(D_\lambda \Psi)(x) \;=\; \int\limits_S (\Psi(y), T_y' E_\lambda(x-y)N(y))do_y\,.$$

Because $E_\lambda = E_\lambda^T$, the single layer potential can be represented by

$$(E_\lambda \Psi)(x) \;=\; \int\limits_S E_\lambda^{(c)}(x-y)\Psi(y)do_y\,. \tag{3.8}$$

Here the 4×3 matrix $E_\lambda^{(c)}$ is obtained from E_λ by eliminating the last column and $E_\lambda^{(c)}\Psi$ indicates the usual matrix vector product. If no confusion is possible, row representation and column representation will be identified. In order to develop a similar representation for the double layer potential we proceed as follows. Due to $D_{y_i}E_{jk}^\lambda(x-y) = -D_{x_i}E_{jk}^\lambda(x-y)(i,j=1,2,3;\, k=1,\ldots,4)$ and observing the definition of T and T' we have $T_y' E_k^\lambda(x-y) = -T_x E_k^\lambda(x-y)$ where E_k^λ denotes the k^{th} column of $E_\lambda\,(k=1,\ldots,4)$. Defining the 3×4 matrix $(D_\lambda(x,y))^T = -T_x E_\lambda(x-y)N(y)$, we first obtain the row vector

$$(D_\lambda \Psi)(x) \;=\; \int\limits_{\partial G} < \psi(y), (D_\lambda(x,y))^T > do_y$$

and then the column

$$(D_\lambda \Psi)(x) \;=\; \int\limits_{\partial G} D_\lambda(x,y)\Psi(y)do_y\,, \tag{3.9}$$

where the 4×3 matrix $D_\lambda(x,y)$ is defined by

$$D_\lambda(x,y) \;=\; (-T_x E_\lambda(x-y)N(y))^T \;=\; \Big((-T_x E_k^\lambda(x-y))_{ij}N_j(y)\Big)_{ki}\,.$$

Both the single layer potential (3.8) and the double layer potential (3.9) are analytic functions in $\mathbb{R}^3\backslash S$ and satisfy there the homogeneous differential equations

$$S_\lambda \begin{matrix} u \\ q \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By elementary calculations we find (compare [**?**]) that the 4×3 kernel matrix $D_\lambda = (D^\lambda_{ki}(x,y))_{k=1,\dots,4;i=1,2,3}$ of the double layer potential $D_\lambda\Psi$ has the following form: Setting $r = x-y$ and $N = N(y)$ we have

$$\begin{aligned} D^\lambda_{ki}(x,y) &= -\frac{1}{4\pi}\left\{\frac{r_k N_i}{|<r|^3} d_1(-\sqrt{\lambda}|r|) - \left(\frac{N_k r_i}{|r|^3} + \delta_{ki}\frac{r\cdot N}{|r|^3}\right) d_2(-\sqrt{\lambda}|r|) + \right. \\ &\qquad \left. \frac{r_k r_i r\cdot N}{|r|^5}(3 - 3d_1(-\sqrt{\lambda}|r|) + 2d_2(-\sqrt{\lambda|r|}))\right\}, \\ d_1(\varepsilon) &= \sum_{n=2}^{\infty}\frac{2(n^2-1)}{(n+2)!}\varepsilon^n = \exp(\varepsilon)(2 - 6\varepsilon^{-1} + 6\varepsilon^{-2}) - 6\varepsilon^{-2} + 1, \\ d_2(\varepsilon) &= \sum_{n=2}^{\infty}\frac{n(n^2-1)}{(n+2)!}\varepsilon^n = \exp(\varepsilon)(\varepsilon - 3 + 6\varepsilon^{-1} - 6\varepsilon^{-2}), \\ D^\lambda_{4i}(x,y) &= -\frac{1}{4\pi}\left\{6\frac{r_i r\cdot N}{|r|^5} + \frac{\lambda N_i}{|r|} - 2\frac{N_i}{|r|^3}\right\} - N_i\delta(r). \end{aligned} \tag{3.10}$$

The series representation above yields $d_1(0) = d_2(0) = 0$, hence as $\lambda \to 0$ we obtain from (3.6) the well known (see [3], p. 336) double layer kernel matrix for the Stokes equations (S_0):

$$\begin{aligned} D^0_{ki}(x,y) &= -\frac{3}{4\pi}\,\frac{r_k r_i r\cdot N}{|r|^5} \qquad (k,i = 1,2,3), \\ D^0_{4i}(x,y) &= -\frac{1}{2\pi}\left(\frac{3r\cdot N r_i}{|r|^5} - \frac{N_i}{|r|^3}\right) - N_i\delta(r) \qquad (i = 1,2,3). \end{aligned} \tag{3.11}$$

It follows easily that the last summand in d_1 comes from the pressure q. This term determines the decay for $r = |r| = |x-y| \to 0$ and $r \to \infty$. Hence for $k,i \neq 4$ we have $(\lambda > 0)$:

$$\begin{aligned} D^\lambda_{ki}(x,y) &= O(r^{-2}) \quad \text{as} \quad r \longrightarrow 0 \quad \text{or} \quad r \longrightarrow \infty\,(\lambda > 0), \\ D^\lambda_{4i}(x,y) &= O(r^{-3}) \quad \text{as} \quad r \longrightarrow 0, \\ D^\lambda_{4i}(x,y) &= O(r^{-1}) \quad \text{as} \quad r \longrightarrow \infty. \end{aligned} \tag{3.12}$$

In the following we consider the normal stresses of the single layer potential $E_\lambda\Psi$, which are defined in a neighborhood $U \subseteq \mathbb{R}^3$ of S for $x \in U\backslash S$ and $\Psi \in C(S)^3$ by

$$(H_\lambda^*\Psi)(x) \;=\; \int\limits_S T_x(E_\lambda^{(c)}(x-y)\Psi(y))N(\tilde{x})do_y\,.$$

Here the superscript * indicates a column vector with 3 components, and $N(\tilde{x})$ denotes the outward unit normal in $\tilde{x} \in S$, where $\tilde{x}$ is the unique projection of $x \in U\backslash S$ on S. Note that $S \in C^2$ allows the construction of parallel surfaces, which implies the existence of such a neighborhood U. If we use the representation

$$(H_\lambda^*\Psi)(x) \;=\; \int\limits_S H_\lambda(x,y)\Psi(y)do_y \tag{3.13}$$

with some 3×3 matrix $H_\lambda(x,y)$, then

$$H_\lambda(x,y) \;=\; D_\lambda^{(r)}(y,x)^T$$

with $D_\lambda^{(r)}$ is obtained by eliminating the last row of the 4×3 matrix D_λ given above.

The next statements concern the continuity properties of the potentials, if $x \in \mathbb{R}^3\backslash S$ approaches a point $z \in S$. For $x \in \mathbb{R}^3\backslash S$ let

$$(E_\lambda^*\Psi)(x) \;=\; \int\limits_{\partial G} E_\lambda^{(r,c)}(x-y)\Psi(y)do_y\,, \tag{3.14}$$

$$(D_\lambda^*\Psi)(x) \;=\; \int\limits_{\partial G} D_\lambda^{(r)}(x,y)\Psi(y)do_y\,, \tag{3.15}$$

denote the single layer and the double layer potential corresponding to the velocity part of the potentials, respectively. Here $E_\lambda^{(r,c)}$ is the 3×3 matrix obtained from E_λ by eliminating the last row ($\approx r$) and the last column ($\approx c$). We first consider some potentials with special densities.

It is well known (see [3], p. 337) that for the case $\lambda = 0$ we have

$$(D_0^*\beta)(x) \;=\; \int\limits_S D_0(x,y)\beta do_y \;=\; \begin{cases} \beta,\, x \in G\,, \\ \frac{1}{2}\beta,\, x \in S\,, \\ 0,\, x \in \mathbb{R}^3\backslash\overline{G}\,, \end{cases} \tag{3.16}$$

where D_0 is the 3×3 matrix defined in (3.11) and $\beta \in \mathbb{R}^3$ is a constant column vector. For $\lambda > 0$, however,

$$(D_\lambda^*\beta)(x) \;=\; \lambda\int\limits_G E_\lambda^{(r,c)}(x-y)\beta dy \;=\; \begin{cases} \beta,\, (x \in G)\,, \\ \frac{1}{2}\beta,\, (x \in S)\,, \\ 0,\, (x \in \mathbb{R}^3\backslash\overline{G})\,. \end{cases} \tag{3.17}$$

Moreover, if N denotes the outward unit normal field on S, then for the single layer potential $E_\lambda N(\lambda > 0)$ with density N we have

$$(E_\lambda N)(x) \;=\; \int\limits_S E_\lambda^{(c)}(x-y)N(y)do_y \;=\; \begin{cases} -\binom{0}{1}\ (x\in G)\,, \\ -\frac{1}{2}\binom{0}{1}\ (x\in S)\,, \\ \binom{0}{0}\ (x\in \mathbb{R}^3\backslash\overline{G})\,, \end{cases} \tag{3.18}$$

which follows from Green's second identity, and implies $(E_\lambda^* N)(x) = 0$ for all $x \in \mathbb{R}^3$.

Next let us study the continuity properties of potentials with general continuous source densities. Setting

$$w(z) \;=\; \lim_{\substack{x\to z\in S\\ x\in G}} w(x)\,, \qquad\qquad w(z) \;=\; \lim_{\substack{x\to z\in S\\ x\in \mathbb{R}^3\backslash\overline{G}}} w(x)\,,$$

we obtain on the boundary S the important relations

$$(E_\lambda^*\Psi)^i \;=\; E_\lambda^*\Psi \;=\; (E_\lambda^*\Psi)^e\,, \tag{3.19}$$

$$(D_\lambda^*\Psi)^i - D_\lambda^*\Psi \;=\; \frac{1}{2}\Psi \;=\; D_\lambda^*\Psi - (D_\lambda^*\Psi)^e\,, \tag{3.20}$$

$$(H_\lambda^*\Psi)^e - H_\lambda^*\Psi \;=\; \frac{1}{2}\Psi \;=\; H_\lambda^* - (H_\lambda^*\Psi)^i\,, \tag{3.21}$$

where $E_\lambda^*\Psi, D_\lambda^*\Psi$, and $H_\lambda^*\Psi$ are defined by (3.14), (3.15), and (3.13), respectively.

Now let $G^c = \mathbb{R}^3\backslash\overline{G}$ be the complementing exterior domain having the same boundary S as G. We consider the following boundary value problem: For a given boundary $b\in C(S)^3$ find $u\in C^2(G)^3\cap C(\overline{G})^3$, $q\in C^1(G)\cap C(\overline{G})$ satisfying

$$S_\lambda \begin{matrix} u \\ q \end{matrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } G\,, \qquad\qquad u=b \text{ on } S\,. \tag{3.22}$$

We refer to this problem as to the interior hydrodynamic Dirichlet problem. Besides (3.22) we also consider the exterior hydrodynamic Neumann problem

$$S_\lambda \begin{matrix} u \\ q \end{matrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ in } G^c\,, \qquad\qquad T_q^u N \;=\; b \text{ on } S\,, \tag{3.23}$$

being adjoint to (3.22). Using Green's first identity we can easily prove that regular solutions u,q of the exterior Neumann problem are uniquely determined provided that we require for $r=|x|\to\infty(\lambda>0)$

$$u(x) \;=\; O(r^{-2})\,, \qquad \nabla u(x) \;=\; O(r^{-1})\,, \qquad q(x) \;=\; O(r^{-1})\,, \tag{3.24}$$

a condition, which takes into account the special decay properties of the potentials (compare (3.3) and (3.12)).

Concerning the interior Dirichlet problem, u is uniquely determined, while q is uniquely determined up to an additive constant only.

In the following we prove the existence of a solution u,q of the interior Dirichlet problem using the method of boundary integral equations. Let $b \in C(S)^3$ be given with

$$\int_S b \cdot N \, do \;=\; 0\,. \tag{3.25}$$

Choosing in $x \in G$ the ansatz $\binom{u}{q}(x) = (D_\lambda \Psi)(x)$ as double layer potential, due to the jump relations we obtain on S the weakly singular (S is of class C^2) boundary integral equations

$$b \;=\; \frac{1}{2}\,\Psi + (D^* \lambda \Psi) \;\text{ on }\; S\,, \tag{3.26}$$

which is a Fredholm system of the second kind on $C(S)^3$. To solve it we have to consider the corresponding homogeneous adjoint system

$$0 \;=\; \frac{1}{2}\,\Phi + (H_\lambda^* \Phi) \;\text{ on }\; S\,. \tag{3.27}$$

It follows from (3.18) that the normal vector $N \in C(S)^3$ is a solution: Due to $(H_\lambda^* N)(x) = (T(E_\lambda N))(x)N(\tilde{x}) = -N(\tilde{x})$ if $x \in G$ (for $\tilde{x}$ see above (3.13)) and $(H_\lambda^* N)(x) = 0$ if $x \in G^c$, from (3.21) we obtain

$$0 \;=\; \frac{1}{2}\,N + (H_\lambda^* N) \;\text{ on }\; S\,.$$

Moreover, if $\Phi \in C(S)^3$ is any solution of (3.27), then we have $\Phi \in \beta N$ with some constant $\beta \in \mathbb{R}$. To see this, consider the single layer potential $\binom{u}{q} = E_\lambda \Phi$ defined in (3.8). It decays as required in (3.24), and it solves the exterior Neumann problem (3.23) with zero boundary data due to (3.21) and (3.27). Thus we habe $E_\lambda \Phi = \binom{0}{0}$ in G^c from the uniqueness statement, and $E_\lambda^* \Phi = 0$ on S using (3.19). This again implies that $E_\lambda \Phi$ also solves the interior Dirichlet problem with zero boundary data, and the corresponding uniqueness statement yields $E_\lambda \Phi = \binom{0}{\alpha}$ in G, with some constant $\alpha \in \mathbb{R}$. Because $H_\lambda^* \Phi = 0$ in G^c and $H_\lambda^* \Phi = \alpha N$ in G, the assertion follows by (3.21). Now using well known facts of Fredholm's theory on integral equations of second kind in spaces of continuous functions it follows that the condition (3.25) is necessary and sufficient for the existence of a solution $\Psi \in C(S)^3$ of (3.26).

Because (3.27) has a unique nontrivial solution $\Phi = N$, the homogeneous version of (3.26) has a nontrivial solution, too. For numerical purposes, however, it is desirable to deal with uniquely solvable systems. This ca be achieved as follows: Instead of (3.26) consider the boundary integral equations system

$$b \;=\; \frac{1}{2}\,\Psi + (D_\lambda^* \Psi) - (P_N \Psi) \;\text{ on }\; S \tag{3.28}$$

with the one-dimensional operator $P_n : C(S)^3 \to C(S)^3$ given by

$$(P_N\Psi)(x) \;=\; N(x)\int\limits_{\partial G} N\cdot\Psi do\,.$$

Because the normal field N forms a basis of the null space of the operator $\frac{1}{2}I_e + H^*_\lambda$ which is adjoint to $\frac{1}{2}I_3 + D^*_\lambda$, the system (3.28), too. Here the latter follows easily by multiplying (3.28) with N, integrating over S, and noting that

$$\int\limits_{\partial G}(D^*_\lambda\Psi)\cdot N\,do \;=\; \int\limits_{\partial G}\Psi\cdot(H^*_\lambda N)do \;=\; -\frac{1}{2}\int\limits_{\partial G}\Psi\cdot N\,do\,.$$

Thus we have shown

Theorem 2. *Let $b\in C(S)^3$ with (3.25) be given on a C^2-boundary S of a bounded domain $G\subseteq\mathbb{R}^3$, and let $0<\lambda\in\mathbb{R}$. Then the interior hydrodynamic Dirichlet problem (3.22) has a solution $u\in C^2(G)^3\cap C(\overline{G})^3, q\in C^1(G)\cap C(\overline{G})$. Here u is uniquely determined, while q is unique up to an additive constant, only. The solution u,q can be represented in G as a pure double layer potential $\binom{u}{q}(x)=(D_\lambda\Psi)(x)$, where the source density $\Psi\in C(S)^3$ is the unique solution of the second kind Fredholm boundary integral equations system*

$$b \;=\; \frac{1}{2}\,\Psi + (D^*_\lambda\Psi) - (P_N\Psi)\;\; on\;\; S\,.$$

Here $D^*_\lambda\Psi$ is the velocity part of $D_\lambda\Psi$ and $P_N: C(S)^3\to C(S)^3$ is defined by

$$(P_N\Psi)(x) \;=\; N(x)\int\limits_{\partial G} N\cdot\Psi do\,.$$

4 A Boundary Element Method

Summarizing the results from the last two sections we find that the potential representation given in (1.3) defines an approximate solution $(v^k(x),q^k(x))$ of the Stokes equations (1.1) at time $t_k = kh$ $(k=1,2,\ldots,N)$. It depends on the solution Ψ of the boundary integral equations system (1.4), which – for each time step – has the form (3.28). For the discretization of (3.28) we choose a collocation procedure as described in [1], [5]. To be concrete, in the following let us restrict our considerations to the case of the unit ball $G\subseteq\mathbb{R}^3$ with boundary S and let us use the parametrization

$$f: S^\wedge \;=\; [0,1]^2 \longrightarrow S\,, \qquad f(\vartheta,\eta) \;=\; (x_1,x_2,x_3)\in S\,,$$

i. e. $x_1=\sin(\pi\vartheta)\cos(2\pi\eta)$, $x_2=\sin(\pi\vartheta)\sin(2\pi\eta)$, $x_3=\cos(\pi\vartheta)$. For the sake of illustration, in the following we suppress some analytical problems due to the non-uniqueness of the inverse mapping f^{-1}. For $L\in\mathbb{N}$ let $\sigma=(2L)^{-1}$ and define on $S^\wedge$ a so-called collocation grid

$$C_\sigma^\wedge \;=\; \{x^\wedge = (i\sigma, j\sigma) | i,j = 0,\dots,2L\}$$

consisting of $(2L+1)^2$ collocation points and an integration grid

$$J_\sigma^\wedge \;=\; \{((i+0.5)\sigma,\ (j+0.5)\sigma) \mid i,j = 0,\dots,2L-1\}$$

consisting of $(2L)^2$ integration points. For $y^\wedge = ((i+0.5)\sigma,\ (j+0.5)\sigma) \in J_\sigma^\wedge$ let

$$Q_y^\wedge \;=\; \{(\vartheta,\eta) \mid i\sigma < \vartheta < (i+1)\sigma,\ j\sigma < \eta < (j+1)\sigma\}$$

be the square with length σ and center $y^\wedge$. The projections of these sets on S are denoted by

$$C_\sigma \;=\; f(C_\sigma^\wedge), \qquad J_\sigma \;=\; f(J_\sigma^\wedge), \qquad Q_y \;=\; f(Q_y^\wedge)\,.$$

Setting

$$\omega(\tau) \;=\; \begin{cases} \tau+1 \text{ for } \;-1 \le \tau \le 0\,, \\ 1-\tau \text{ for } \quad 0 \le \tau \le 1\,, \\ \quad 0 \quad \text{elsewhere}\,, \end{cases}$$

for every $x^\wedge = (x_1,x_2) \in C_\sigma^\wedge$ let us define a bilinear B-spline

$$\xi^\wedge : S^\wedge \longrightarrow \mathbb{R}\,, \qquad \xi^\wedge(\vartheta,\eta) \;=\; \omega\left(\frac{(\vartheta - x_1)}{\sigma}\right)\omega\left(\frac{(\eta - x_2)}{\sigma}\right)\,.$$

These splines are used for interpolation: the interpolate $P_\sigma^\wedge\Phi^\wedge : S^\wedge \to \mathbb{R}^3$ of some vector function $\Phi^\wedge : S^\wedge \to \mathbb{R}^3$ is defined by

$$(P_\sigma^\wedge\Phi^\wedge)(\vartheta,\eta) \;=\; \sum_{x^\wedge \in C_\sigma^\wedge} \Phi^\wedge(x^\wedge)\xi^\wedge(\vartheta,\eta)\,,$$

and it holds $(P_\sigma^\wedge\Phi^\wedge)(u^\wedge) = \Phi^\wedge(z^\wedge)$ for all $z^\wedge \in C_\sigma^\wedge$. Analogously, we call

$$P_\sigma\Phi \;=\; (P_\sigma^\wedge(\Phi\circ f))\circ f^{-1}$$

the interpolate of $\Phi : S \to \mathbb{R}^3$.

Let us now go back to the boundary integral equations system and look for an approximate solution

$$\Psi_\sigma \;=\; \Psi_\sigma^\wedge \circ f^{-1}\,,$$

where the vector function $\Psi_\sigma^\wedge : S^\wedge \to \mathbb{R}^3$ has the form

$$\Psi_\sigma^\wedge(\vartheta,\eta) \;=\; \sum_{x^\wedge \in C_\sigma^\wedge} \alpha(x^\wedge,\sigma)\xi^\wedge(\vartheta,\eta)\,.$$

Here the unknown coefficients $\alpha(x^\wedge,\sigma) \in \mathbb{R}^3$ have to be determined from the collocation procedure

$$P_\sigma b \;=\; P_\sigma(\frac{1}{2}\Psi_\sigma + (D^*_{\lambda,\sigma}\Psi_\sigma) - (P_{N,\sigma}\Psi_\sigma))\,, \tag{4.1}$$

where (compare (3.15) and (3.28))

$$(D^*_{\lambda,\sigma}\Psi_\sigma)(x) \;=\; \sum_{y\in J_\sigma} D^{(r)}_\lambda(x,y)\Phi_\sigma(y)|Q_y| \quad (x\notin J_\sigma)\,, \tag{4.2}$$

$$(P_{N,\sigma}\Psi_\sigma)(x) \;=\; N(x)\sum_{y\in J_\sigma} N(y)\cdot\Phi(y_\sigma)|Q|_y \quad (x\in S)\,, \tag{4.3}$$

and

$$|Q_y| \;=\; \int\limits_{Q_y} do\,.$$

Hence integration has been replaced by a quadrature formula (midpoint rule). Thus considering (4.1) on the collocation grid only, we obtain a linear algebraic system for $3(2L+1)^2$ unknowns (3 components, $(2L+1)^2$ collocation points) with a non-sparse but diagonal-dominant system matrix, which is invertible for sufficiently small $\omega > 0$. This follows with the usual perturbation theory from the fact that (3.28) is uniquely solvable in $C(S)^3$. Moreover, the following estimates can be obtained in case of boundary values $b \in C(S)^3$ as in [1]:

$$\max_{x\in S}|\Psi(x)-\Psi_\sigma(x)| \;\leq\; c(\lambda)\sigma\ln(\frac{1}{\sigma})\,, \tag{4.4}$$

$$\max_{x\in G_0}|D^*_\lambda\Psi(x)-D^*_{\lambda,\sigma}\Psi_\sigma(x)| \;\leq\; c(\lambda)\sigma\ln(\frac{1}{\sigma})\,. \tag{4.5}$$

Here $(G_\sigma)_{\sigma>0}$ is a family of subregions $G-\sigma$ exhausting G as $\sigma\to 0$.

Extending both grids from the boundary S into the domain G, the volume potentials can be approximated analogously, using the midpoint rule as quadrature formula instead of integration. In the following, we present some test calculations for the non-stationary Stokes equations (1.1) in the $3-d$ unit ball, which have been performed without using any symmetry property of the ball: Let

$$(t,x)\longrightarrow v(t,x) = (t+1)(\exp(-r^2)-\exp(-1))\begin{pmatrix} x_3-x_2\\ x_1-x_3\\ x_1-x_2\end{pmatrix},$$

$$(t,x)\longrightarrow p(t,x) = \text{constant}, \quad r=(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}\,.$$

Then $v,\nabla p$ is the unique solution of a constructed non-stationary Stokes problem (1.1) with

$$\nu \;=\; 1\,, \qquad F \;=\; D_t v - \Delta v + \nabla p\,, \qquad v_0 \;=\; v(0)\,.$$

The following numerical results illustrate the accuracy of our approach. The simulation runs with a time step size $h=0.1$ and a spatial step size $\sigma=\frac{1}{16}$ on a PC with single precision. Let $E(j), j=1,2,3$ denote the mean (in space) relative error (%),

i. e.

$$E(j) \;=\; \frac{100}{L}\sum_{l=1}^{L}\left|\frac{v_j^{appr}(x_l)-v_j^{exe}(x_l)}{v_j^{exe}(x_l)}\right|.$$

Development in time of $\mathbf{E(j)}$

t	$j=1$	$j=2$	$j=3$
0.1	1.216	1.594	2.932
0.2	1.316	1.373	2.721
0.3	1.517	1.354	2.490
0.4	1.615	1.354	2.331
0.5	1.748	1.392	2.229
0.6	1.872	1.444	2.159
0.7	1.931	1.454	2.099
0.8	1.991	1.466	2.049
0.9	2.049	1.478	2.012
1.0	2.108	1.492	1.985

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