Time Discretization of the SST-generalized Navier-Stokes Equations: Positive and Negative Results

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1. Introduction

In the theory of the Navier-Stokes equations, the proofs of some basic known results, like for example the uniqueness of solutions to the stationary Navier-Stokes equations under smallness assumptions on the force (Theorem 22 in sec. 5.1) or the stability of certain difference schemes (Theorems 49 and 50 in sec. 6.1), actually only use a small range of properties and are therefore valid in a more general context. In the present thesis this context is made concrete by means of the concept of SST space. A vector space V over the field $\mathbb{R}$ is called an SST space if it is equipped with two scalar products $(\!(\cdot,\cdot)\!)$ and $(\cdot,\cdot)$ and a trilinear form $(\cdot,\cdot,\cdot)$ such that the norm $\|\cdot\| = (\!(\cdot,\cdot)\!)^{\frac{1}{2}}$ is stronger than the norm $|\cdot| = (\cdot,\cdot)^{\frac{1}{2}}$ and such that the trilinear form is continuous with respect to the norm $\|\cdot\|$ and skew-symmetric in the last two components.[1] The acronym SST stands for scalar product, scalar product, trilinear form. In the special case of the Navier-Stokes equations, V is the set of solenoidal vector fields on an open bounded subset of $\mathbb{R}^3$ with zero boundary values, $(\cdot,\cdot)$ is the L^2 scalar product and $(\!(\cdot,\cdot)\!)$ the Dirichlet scalar product $(\nabla\cdot,\nabla\cdot)$.[2] Other examples of SST spaces include the SST space $V = \mathbb{R}^2$ from Theorem 24 in sec. 5.1.2 and the weighted SST spaces on $\mathbb{Z}\setminus\{0\}$ from Lemma 37 in sec. 5.3.2. The principal relevance of these non-Navier-Stokes SST spaces is that they serve as counterexamples to disprove uniqueness and stability conjectures which are open questions in the special case of the Navier-Stokes equations. Each time such a counterexample is invoked in the present thesis, it proves that the corresponding statement does not hold in SST spaces in general. Of course in each such situation the corresponding statement might nevertheless be true for the special case of the Navier-Stokes equations which is why the counterexample doesn't answer the open question. But it shows that it is impossible to prove the corresponding statement for the Navier-Stokes equations when using only the tools available in SST spaces. These tools are not as weak as one might believe: The above-mentioned basic known uniqueness and stability results, as well as several uniqueness results in sec. 5.3.1 which are to our knowledge new, are proven for general SST spaces.

Chapter 2 is an introduction to SST spaces. After the definition of the term SST space and some related notions and properties in sec. 2.1, it is shown in sec. 2.2 that the concept is a generalization of the functional setting for the Navier-Stokes equations. In so doing we derive explicit optimal constants for Hölder inequalities

[1]For more details see Definition 1 in sec. 2.1.
[2]Precise definitions are given in sec. 2.2.

related to the convective term. To our knowledge these constants have not been determined before.

Chapter 3 provides a list of equations and difference schemes that occur in the present thesis. A table at the end of the chapter gives an overview of the theorems concerning the listed equations and difference schemes sorted by type of result.

The theorems from chapter 4 cover existence and non-existence results. First a general existence theorem that applies to all equations and schemes listed in chapter 3 except for the explicit Euler scheme (Theorem 18) is established for SST spaces that have additional completeness and compactness properties and the trilinear form of which is, roughly speaking, continuous with respect to a seminorm $\|\cdot\|_\infty$ that corresponds to the norm $|\nabla\cdot|_{L^\infty}$ in the special case of the Navier-Stokes equations.[3] The proof is based on the Galerkin method. After that it is shown in Theorem 21 that an existence result for the explicit Euler scheme proven in [Tem84, page 335, Scheme 5.4] for the spatially discretized case does not hold if the SST space's norms are not equivalent, which includes the case of the spatially continuous Navier-Stokes equations. The lack of existence is due to the fact that the Dirichlet norm can in general not be controlled by the L^2 norm.

Chapter 5 is dedicated to uniqueness questions. Its first section concerns the SST-generalized stationary Navier-Stokes equations. It is shown in Theorem 22 that the proof of a known uniqueness result for the stationary Navier-Stokes equations under smallness assumptions on the force (see e. g. [Tem84, page 167, Theorem 1.3]) carries over to the general SST setting. The question whether uniqueness can be established without this smallness assumption seems to be an open question in the special case of the stationary Navier-Stokes equations according to [Tem84, page 168, Remark 1.1]. As far as general SST spaces are concerned, the question has a negative answer: By means of a counterexample it is shown in Theorem 24 that the smallness assumption on the force is indispensable in the general SST setting. After that, in sec. 5.2, the well-known fact that the uniqueness of solutions to linear difference schemes can be proven without smallness assumptions on the data in the case of the Navier-Stokes equations (see e. g. [GR79, page 171, Lemma 2.1]) is shown to generalize to SST spaces. Nonlinear difference schemes are discussed in sec. 5.3. All results are first established for the implicit Euler scheme and subsequently transferred to other nonlinear schemes. To our knowledge, the uniqueness of solutions to nonlinear difference schemes for the Navier-Stokes equations has only been studied in the case of spatial discretizations. In that case the Dirichlet and L^2 norms are equivalent, the consequence being that smallness assumptions on the time step size alone are sufficient for uniqueness [HR90, page 366]. However, when it comes to SST spaces the Dirichlet norm of which can not be controlled by their L^2 norm, there are counterexamples that show that smallness assumptions on the time step size alone are insufficient. More precisely, if the Dirichlet norm of the previous approximation is too large (Theorem 39) or if its L^2 norm is different from zero (Theorem 41), non-

[3] See the assumptions of Theorem 18 for the specific continuity requirements.

uniqueness can occur in SST spaces for arbitrary forces and arbitrary small time step sizes. Yet uniqueness results are not entirely out of reach in the general SST setting: They are established in sec. 5.3.1 under smallness assumptions on the Dirichlet norm of the previous approximation combined with either smallness assumptions on the product of the time step size and the squared L^2 norm of the force, or smallness assumptions on the H^{-1} norm of the force.

Chapter 6 addresses the stability of difference schemes. First the well-known stability proofs for two versions of the implicit Euler scheme and two versions of the Crank-Nicolson scheme (see e. g. [Tem84, page 336ff, Lemma 5.1 and 5.2]) are shown to carry over to general SST spaces. Then in sec. 6.2 we prove that a version of the explicit Euler scheme, a version of the Crank-Nicolson scheme, and the fractional step theta scheme are not stable in the SST setting.[4] For each of these schemes there are examples of SST spaces where the L^2 norm of the first approximation v^1 can be arbitrarily large even if the L^2 norm of the initial value v^0 and the time step size are required to be smaller than ε. According to [MPRT95, page 5], the use of the fractional step theta scheme for the Navier-Stokes equations was first proposed in [BGP87]. A sketch of proof of its second order convergence for $\vartheta = 1 - \frac{1}{2}\sqrt{2}$ when applied to the linear Stokes equations with error constants that grow exponentially in time can be found in [MU93, page 23, Satz 3.1]. A sketch of proof of the same result but with error constants that are time-independent, due to the proof avoiding discrete Grönwall inequalities, is given in [Zan12].[5] The above-mentioned non-stability proof for the fractional step theta scheme (Theorem 53) does not contradict the second order convergence result for the fractional step theta scheme given in [MU93, Kapitel 4] for the nonstationary Navier-Stokes equations since the counterexample used to show non-stability is a non-Navier-Stokes SST space.

Chapter 7 contains two convergence results for versions of the implicit Euler and Crank-Nicolson schemes. They are proven for SST_∞ spaces – a concept more specific than SST spaces but still more general than the Navier-Stokes equations.[6] In Theorem 60 we provide a new proof of a first order convergence result for a version of the implicit Euler scheme that is already known for the special case of the Navier-Stokes equations [GR79, page 179, Theorem 2.2]. Our proof differs from that cited in that it uses different estimations of the convective term.[7] Conditions on the data that are sufficient for the existence of a solution of the nonstationary Navier-Stokes equations that satisfies the regularity assumptions for convergence are given in Theorem 59. As a corollary of the established convergence result we recover a uniqueness result that is known for the special case of the nonstationary Navier-Stokes equations (see Corollary 62). After that the convergence of a version

[4] In the present thesis we use the same version of the fractional step theta scheme as studied in [MU93, page 50], see chapter 3.

[5] In the cited paper the error constants are mistakenly stated to depend on nothing but α, but actually they depend also on the viscosity ν.

[6] The term SST_∞ space is defined in Definition 54.

[7] For more details see page 60.

of the Crank-Nicolson scheme is studied in sec. 7.2. To our knowledge, its second order convergence when applied to the nonstationary Navier-Stokes equations has until now only been shown for spatial discretizations where the Dirichlet and the L^2 norms are equivalent.[8] In Theorem 65 we establish the second order convergence of the said version of the Crank-Nicolson scheme for general SST_∞ spaces. As in the case of the Euler scheme in the preceding section, conditions on the data that guarantee the existence of a solution that satisfies the assumptions of the convergence theorem in the special case of the nonstationary Navier-Stokes equations are provided. Due to the higher regularity requirements, these conditions involve a compatibility condition on the initial acceleration. It is shown via prescription of the initial acceleration that nontrivial data satisfying the compatibility condition actually exist.

[8] See page 66 for a number of known convergence results for Crank-Nicolson type schemes.

2. SST spaces

2.1. Definitions

Definition 1 (SST space)**.** If V is a vector space over $\mathbb{R}$, if $(\!(\cdot,\cdot)\!)$ and $(\cdot,\cdot)$ are scalar products on V, and if $(\cdot,\cdot,\cdot)$ is a trilinear form on V then we call

$$(V, (\!(\cdot,\cdot)\!), (\cdot,\cdot), (\cdot,\cdot,\cdot))$$

an SST space if there are real numbers $c_p > 0$ and $c_t \geqslant 0$ such that for all $u, v, w \in V$

$$|u| \leqslant c_p \|u\| \text{ (Poincaré inequality),} \tag{2.1}$$

$$|(u,v,w)| \leqslant c_t^3 \|u\| \|v\| \|w\| \text{ (continuity), and} \tag{2.2}$$

$$(u,v,v) = 0 \text{ (orthogonality relation),} \tag{2.3}$$

where $\|u\| = (\!(u,u)\!)^{1/2}$ and $|u| = (u,u)^{1/2}$. The acronym SST stands for scalar product, scalar product, trilinear form. We will often call V itself an SST space if there is no doubt about which scalar products and trilinear form we refer to.

Remark 2. We are interested in SST spaces for the following reason: The space of solenoidal vector fields with zero boundary values on an open bounded subset of $\mathbb{R}^3$ equipped with the Dirichlet and L^2 scalar products and the trilinear form $(u \cdot \nabla v, w)$ – which is of central importance in the theory of the Navier-Stokes equations – is an SST space (see sec. 2.2). This is why we refer to $\|\cdot\|$ as the Dirichlet norm, to $|\cdot|$ as the L^2 norm, or use the term Poincaré inequality as a pars pro toto even if the SST space in consideration is not the Navier-Stokes SST space. The property $(u,v,v) = 0$ is called orthogonality relation because in the case of the Navier-Stokes SST space it states that the two vector fields $u \cdot \nabla v$ and v are orthogonal with respect to the L^2 scalar product.

Lemma 3 (Skew-symmetry)**.** *The trilinear form of an SST space is always skew-symmetric in the last two components, i. e.*

$$(u,v,w) + (u,w,v) = 0 \qquad (u,v,w \in V).$$

Proof. For all $u, v, w \in V$ the orthogonality relation 2.3 yields

$$\begin{aligned}(u,v,w) + (u,w,v) &= (u,v,v+w) + (u,w,v+w)\\ &= (u,v+w,v+w) = 0.\end{aligned}$$

□

Definition 4. If V is an SST space, by the dual space V' we understand the space of linear forms on V continuous with respect to the Dirichlet norm $\|\cdot\|$. The norm on V' is defined by

$$\|f\|_{V'} = \sup\left\{|f(v)|\,;\|v\| = 1\right\}.$$

By H' we understand the space of linear forms on V continuous with respect to the norm $|\cdot|$. The norm on H' is defined by

$$|f| = \sup\left\{|f(v)|\,;|v| = 1\right\}.$$

These definitions imply that

$$|f(\varphi)| \leqslant \|f\|_{V'}\|\varphi\|$$

for all $f \in V'$ and $\varphi \in V$ and

$$|f(\varphi)| \leqslant |f|\,|\varphi|$$

for all $f \in H'$ and $\varphi \in V$.

We have $H' \subset V'$ and $\|f\|_{V'} \leqslant c_p\,|f|$ for all $f \in H'$ due to the Poincaré inequality 2.1.

Definition 5. The imbedding $V \hookrightarrow H'$ is defined by

$$V \ni v \mapsto (V \ni \varphi \mapsto (v, \varphi) \in \mathbb{R}) \in H'.$$

It is an isometry from $(V, |\cdot|)$ to $(H', |\cdot|)$.

2.2. The Navier-Stokes SST space

It is proven in this section that the space of solenoidal vector fields with zero boundary values on a bounded domain of $\mathbb{R}^3$ equipped with the Dirichlet and L^2 scalar products and the trilinear form $(u \cdot \nabla v, w)$ is an SST space.

Throughout this section let $n \in \mathbb{N}_{\geqslant 1}$, and let $G \subset \mathbb{R}^n$ be an open, bounded set.

Definition 6 ($C^\infty_{0,\sigma}$ and V)**.** A vector field $u : G \to \mathbb{R}^n$ is called divergence free or solenoidal, if $u \in C^1(G)^n$ with

$$\nabla \cdot u = \sum_i \partial_i u_i = 0.$$

By $C^\infty_{0,\sigma}(G)$ we denote the space of all divergence free C^∞ vector fields $u : G \to \mathbb{R}^n$ that have compact support in G.

By V we denote the closure of $C^\infty_{0,\sigma}(G)$ in $H^1(G)^n$ with respect to the H^1 norm. Note that $V \subset H^1_0(G)^n$ since $C^\infty_{0,\sigma}(G) \subset C^\infty_0(G)^n$.[1]

[1] Here $H^1(G)$ and $H^1_0(G)$ stand for the Sobolev spaces $W^{1,2}(G)$ and $W^{1,2}_0(G)$, respectively, see e. g. [AF03, page 60, Definition 3.2(b,c)].

Definition 7 (Scalar products and Dirichlet norm)**.** The positive semidefinite symmetric bilinear form

$$(\nabla u, \nabla v) := \sum_{i,j} \int_G \partial_j u_i \partial_j v_i$$

is defined on $H^1(G)^n \times H^1(G)^n$. The following theorem shows that $(\nabla\cdot, \nabla\cdot)$ and its associated seminorm $|\nabla\cdot|_{L^2}$, if restricted to $H_0^1(G)^n$, are a scalar product and a norm. The norm $|\nabla\cdot|_{L^2}$ is called the Dirichlet norm. The scalar product

$$(u, v) := \sum_i \int_G u_i v_i$$

is defined on $L^2(G)^n \times L^2(G)^n$, its associated norm is the L^2 norm.

Theorem 8 (Poincaré inequality)**.** *There is a number c_p depending on the width of G such that for every vector field $u \in H_0^1(G)^n$ the inequalities*

$$|u|_{H^1} \leqslant c_p |\nabla u|_{L^2}$$

and

$$|u|_{L^2} \leqslant c_p |\nabla u|_{L^2}$$

hold. Both are referred to as the Poincaré inequality.

Proof. The proof for the scalar case can for example be found in [AF03, page 184, Corollary 6.31 for $m = 1$, $p = 2$]. Note that the definition of the term domain used there doesn't require connectedness [AF03, page 1]. The generalization to vector fields reads

$$|u|_{H^1}^2 = \sum_i |u_i|_{H^1}^2 \leqslant c_p^2 \sum_i |\nabla u_i|_{L^2}^2 = c_p^2 \sum_{i,j} |\partial_j u_i|_{L^2}^2 = c_p^2 |\nabla u|_{L^2}^2 .$$

□

To our knowledge the explicit optimal constants for the Hölder inequalities in the following three lemmas are new.

Lemma 9 (Hölder's inequality for the convective term)**.** *Let $1 \leqslant p, q, r \leqslant \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $u \in L^p(G)^n$, and $v \in W^{1,q}(G)^n$. Then the convective term defined by*

$$u \cdot \nabla v := \sum_i u_i \partial_i v$$

is an element of $L^r(G)^n$ and satisfies

$$|u \cdot \nabla v|_{L^r} \leqslant n^{1-\frac{1}{q}} |u|_{L^p} |\nabla v|_{L^q} .$$

The inequality is sharp in the sense that the constant $n^{1-\frac{1}{q}}$ cannot be improved.

Proof. We only prove the case $\infty \notin \{p,q,r\}$ because the other cases are similar. The exponential triangle inequality A.1 is used in the second and Hölder's inequality in the third line of the estimation

$$
\begin{aligned}
|u\cdot\nabla v|_{L^r} &= \left(\int_G \sum_j \left|\sum_i u_i \partial_i v_j\right|^r\right)^{\frac{1}{r}} \\
&\leqslant \left(\int_G \sum_j n^{r-1} \sum_i |u_i \partial_i v_j|^r\right)^{\frac{1}{r}} \\
&\leqslant n^{1-\frac{1}{r}} \left(\int_G \sum_{i,j} |u_i|^p\right)^{\frac{1}{p}} \left(\int_G \sum_{i,j} |\partial_i v_j|^q\right)^{\frac{1}{q}} \\
&= n^{1-\frac{1}{r}} \left(\int_G n \sum_i |u_i|^p\right)^{\frac{1}{p}} |\nabla v|_{L^q}\,.
\end{aligned}
$$

The example $u_i(x) = 1$, $v_i(x) = \sum_i x_i$ shows that the inequality is sharp. □

Lemma 10 (Hölder's inequality for dot product)**.** *Let* $1 \leqslant p,q,r \leqslant \infty$ *with* $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}$, $u \in L^p(G)^n$, *and* $v \in L^q(G)^n$. *Then the dot product defined by*

$$u \cdot v := \sum_i u_i v_i$$

is an element of $L^r(G)$ *and satisfies*

$$|u\cdot v|_{L^r} \leqslant n^{1-\frac{1}{r}} |u|_{L^p} |v|_{L^q}\,.$$

The inequality is sharp.

Proof. We only proof the case $\infty \notin \{p,q,r\}$ because the other cases are similar. The exponential triangle inequality A.1 is used in the second and Hölder's inequality in the third line of the estimation

$$
\begin{aligned}
|u\cdot v|_{L^r} &= \left(\int_\Omega \left|\sum_i u_i v_i\right|^r\right)^{\frac{1}{r}} \\
&\leqslant \left(\int_\Omega n^{r-1} \sum_i |u_i v_i|^r\right)^{\frac{1}{r}} \\
&\leqslant n^{1-\frac{1}{r}} \left(\int_\Omega \sum_i |u_i|^p\right)^{\frac{1}{p}} \left(\int_\Omega \sum_i |v_i|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

The example $u_i(x) = v_i(x) = 1$ shows that the inequality is sharp. □

Lemma 11 (Hölder's inequality for $(u\cdot\nabla v)\cdot w$)**.** *Let* $1 \leqslant p,q,r,s \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{s}$, $u \in L^p$, $v \in W^{1,q}$, *and* $w \in L^r$. *Then* $(u\cdot\nabla v)\cdot w \in L^s$ *and*

$$\left|(u\cdot\nabla v)\cdot w\right|_{L^s} \leqslant n^{2-\frac{1}{s}-\frac{1}{q}} \left|u\right|_{L^p} \left|\nabla v\right|_{L^q} \left|w\right|_{L^r}.$$

The inequality is sharp.

Proof. Lemma 9 is used for the first and Lemma 10 for the second inequality in the estimation

$$\begin{aligned}\left|(u\cdot\nabla v)\cdot w\right|_{L^s} &\leqslant n^{1-\frac{1}{s}} \left|u\cdot\nabla v\right|_{L^{\left(\frac{1}{p}+\frac{1}{q}\right)^{-1}}} \left|w\right|_{L^r}\\ &\leqslant n^{1-\frac{1}{s}} n^{1-\frac{1}{q}} \left|u\right|_{L^p} \left|\nabla v\right|_{L^q} \left|w\right|_{L^r}.\end{aligned}$$

As sharpness of inequalities is not transitive,[2] we give another example: $u_i(x) = w_i(x) = 1$, $v_i(x) = \sum_i x_i$. □

Lemma 12 (Sobolev imbedding L^4)**.** *If* $n=3$, *then* $H_0^1(G)^3 \subset L^4(G)^3$ *and there is a number* c_s *such that for every* $u \in H_0^1(G)^3$

$$\left|u\right|_{L^4} \leqslant c_s \left|u\right|_{H^1}.$$

Proof. The result for scalar valued functions can be found in [AF03, page 85, Theorem 4.12, part III applied to part I, imbedding (5) for $n=3$, $m=1$, $p=2$, $q=4$]. Note that the definition of the term domain used there doesn't require connectedness [AF03, page 1]. The generalization to the vector valued case reads

$$\left|u\right|_{L^4} = \left(\sum_i \left|u_i\right|_{L^4}^4\right)^{\frac{1}{4}} \leqslant c_S \left(\sum_i \left|u_i\right|_{H^1}^4\right)^{\frac{1}{4}} \leqslant c_s\tilde{c} \left(\sum_i \left|u_i\right|_{H^1}^2\right)^{\frac{1}{2}} = c_s\tilde{c}\left|u\right|_{H^1},$$

where for the second inequality we use that norms on finite dimensional spaces are equivalent and thus there is $\tilde{c} \in \mathbb{R}$ such that $\left|a\right|_{\ell^4} \leqslant \tilde{c}\left|a\right|_{\ell^2}$ for all $a \in \mathbb{R}^3$. □

Lemma 13. *Let* $n=3$ *and* $\kappa \in \mathbb{R}$.[3] *Then the integral in the definition of the trilinear form*

$$(u,v,w) := \kappa \int_G (u\cdot\nabla v)\cdot w \qquad (u,v,w \in V)$$

exists and the trilinear form has continuity property 2.2, i. e.

$$\left|(u,v,w)\right| \leqslant c_t^3 \left|\nabla u\right|_{L^2} \left|\nabla v\right|_{L^2} \left|\nabla w\right|_{L^2}$$

holds for all $u,v,w \in V$, *where* $c_t^3 = \sqrt{3}\left|\kappa\right| c_s^2 c_p^2$.

[2] For example, if $|\cdot|$ denotes the euclician norm on $\mathbb{R}^2$ and $A:\mathbb{R}^2 \to \mathbb{R}^2, (x_0,x_1) \mapsto (2x_0,x_1)$, each of the inequalities $|x| \leqslant |Ax|$ and $|Ax| \leqslant 2|x|$ is sharp but the inequality $|x| \leqslant 2|x|$ is not.

[3] See Remark 16 on the parameter κ below.

Proof. The second, the third, and the fourth inequality in the estimation

$$\begin{aligned}|(u,v,w)| &\leqslant |\kappa|\,|(u\cdot\nabla v)\cdot w|_{L^1}\\ &\leqslant 3^{\frac{1}{2}}\,|\kappa|\,|u|_{L^4}\,|\nabla v|_{L^2}\,|w|_{L^4}\\ &\leqslant 3^{\frac{1}{2}}\,|\kappa|\,c_s^2\,|u|_{H^1}\,|\nabla v|_{L^2}\,|w|_{H^1}\\ &\leqslant 3^{\frac{1}{2}}\,|\kappa|\,c_s^2c_p^2\,|\nabla u|_{L^2}\,|\nabla v|_{L^2}\,|\nabla w|_{L^2}\,.\end{aligned}$$

are based on Hölder's inequality from Lemma 11 with $n=3$, $p=r=4$, $q=2$, and $s=1$, on the Sobolev inequality from Lemma 12, and on the Poincaré inequality from Theorem 8, respectively. □

Lemma 14 (Orthogonality relation)**.** *Let $n=3$ and $\kappa\in\mathbb{R}$. Then the trilinear form defined in Lemma 13 satisfies orthogonality relation 2.3, i. e.*

$$(u,v,v)=0 \qquad (u,v\in V)\,.$$

Proof. See [Tem84, page 163, Lemma 1.3] and note that the assumption $v\in L^3(G)^3$ made there is satisfied because the Sobolev imbedding $H_0^1\subset L^q$ used in the proof of Lemma 12 with $q=4$ also holds with $q=3$ [AF03, page 85, Theorem 4.12, part III applied to part I, imbedding (5) for $n=3$, $m=1$, $p=2$, $q=3$]. □

Definition 15 (Navier-Stokes SST space)**.** Let $n=3$ and $\kappa\in\mathbb{R}$. The space V from Definition 6 equipped with the scalar products $(\!(\cdot,\cdot)\!)=(\nabla\cdot,\nabla\cdot)_{L^2}$ and $(\cdot,\cdot)=(\cdot,\cdot)_{L^2}$ from Definition 7 and the trilinear form $(\cdot,\cdot,\cdot)$ from Lemma 13 is called the Navier-Stokes SST space. It is an SST space because of the Poincaré inequality from Theorem 8, the continuity established in Lemma 13, and the orthogonality relation from Lemma 14.

Remark 16 (on the parameter κ)**.** The nonstationary Navier-Stokes equations as stated in their weak form in chapter 7, Definition 56 can also be stated in an apparently more general way with viscosity $\nu>0$ in the form

$$\partial_t(u,\varphi)+\kappa(u\cdot\nabla u,\varphi)+\nu(\!(u,\varphi)\!)=f(\varphi) \qquad (\varphi\in V)\,. \tag{2.4}$$

However, if $\kappa\neq 0$, solving equation 2.4 amounts to solving the same equation with $\kappa=\nu=1$ because the equations have the following scaling property: Whenever $u:\mathbb{R}\to V$ and $f:\mathbb{R}\to V'$, if $\tilde u$ and $\tilde f$ are defined by

$$\begin{aligned}\tilde u:\mathbb{R} &\to V & \tilde f:\mathbb{R} &\to V'\\ \tau &\mapsto \kappa\nu^{-1}u(\nu^{-1}\tau) & \tau &\mapsto \kappa\nu^{-2}f(\nu^{-1}\tau)\,,\end{aligned}$$

u is a solution of equation 2.4 with force f if and only if $\tilde u$ is a solution of the same equation with force $\tilde f$ but with $\kappa=\nu=1$. Nevertheless there is a benefit in not omitting the parameter κ: Every result obtained for the Navier-Stokes equations with $\kappa\in\mathbb{R}$ also applies to the linear Stokes equations by setting $\kappa=0$. The same could be done by allowing the viscosity ν to vanish which corresponds to the Euler equations. But since pretty much all of the results in the present thesis rely on the regularizing effect of the viscous term, the case $\nu = 0$ would have to be excluded in the assumptions of most of the results anyway, hence we can just as well assume that $\nu = 1$.

3. List of equations and difference schemes

This chapter lists the equations and difference schemes discussed in the present thesis. All equations and schemes as well as the SST-generalized nonstationary Navier-Stokes equations (see Definition 56 in chapter 7) are stated in a weak and thus pressure-free formulation. It is shown in [Tem84, pages 21-23, 160-161, 252-253, 280-281] that in the special case of the Navier-Stokes SST space the pressure-free formulations are equivalent to their corresponding formulations with pressure in both the stationary and nonstationary cases.[1]

Throughout the chapter let V be an SST space.

By the **stationary Navier-Stokes equations** we understand the weakly formulated equation

$$(v, v, \varphi) + ((v, \varphi)) = f(\varphi) \qquad (\varphi \in V), \tag{3.1}$$

where $f \in V'$ is given and $v \in V$ searched-for. However we will refer to equation 3.1 as the **SST-generalized stationary Navier-Stokes equations** whenever it is not clear from the context that the considered SST space is not necessarily the Navier-Stokes SST space.

By the **explicit Euler scheme** we understand the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \left(v^{k-1}, v^{k-1}, \varphi\right) + \left(\left(v^{k-1}, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.2}$$

by the **almost explicit Euler scheme** the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \left(v^{k-1}, v^{k-1}, \varphi\right) + \left(\left(v^{k}, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.3}$$

by the **almost implicit Euler scheme** the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \left(v^{k-1}, v^{k}, \varphi\right) + \left(\left(v^{k}, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.4}$$

by the **implicit Euler scheme** the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \left(v^{k}, v^{k}, \varphi\right) + \left(\left(v^{k}, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.5}$$

[1] Note in this context that we assume space dimension $n = 3$ in Definition 15.

by the **sum Crank-Nicolson scheme** the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \frac{1}{2}\left(v^{k-1}, v^{k-1}, \varphi\right) + \frac{1}{2}\left(v^k, v^k, \varphi\right) + \frac{1}{2}\left(\left(v^{k-1} + v^k, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.6}$$

by the **linear Crank-Nicolson scheme** the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \frac{1}{2}\left(v^{k-1}, v^{k-1} + v^k, \varphi\right) + \frac{1}{2}\left(\left(v^{k-1} + v^k, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.7}$$

and by the **product Crank-Nicolson scheme** the weakly formulated scheme

$$\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \frac{1}{4}\left(v^{k-1} + v^k, v^{k-1} + v^k, \varphi\right) + \frac{1}{2}\left(\left(v^{k-1} + v^k, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V), \tag{3.8}$$

where in the case of each scheme the step size $h > 0$, the initial value $v^0 \in V$, and a sequence $(f^k)_{k\geqslant 1}$ of evaluations of the force with $f^k \in V'$ are given, and a sequence $(v^k)_{k\geqslant 1}$ of solutions with $v^k \in V$ is searched-for.

Note that the four names chosen for the Euler schemes can be inappropriate in a context where additional Euler schemes are considered: If, for example, the scheme $\frac{1}{h}\left(v^k - v^{k-1}, \varphi\right) + \left(v^k, v^{k-1}, \varphi\right) + \left(\left(v^k, \varphi\right)\right) = f^k(\varphi)$ were also in the list of schemes, it would merit the name almost implicit Euler scheme to the same extent as scheme 3.4 does. Therefore, in such a case, more specific names would have to be found.

By the **fractional step theta scheme** we understand the scheme each step of which consists of the three weakly formulated substeps

$$\frac{1}{\vartheta h}\left(v^{k-1+\vartheta} - v^{k-1}, \varphi\right) + \left(v^{k-1}, v^{k-1}, \varphi\right) + \left(\left((1-\alpha)\, v^{k-1} + \alpha v^{k-1+\vartheta}, \varphi\right)\right) = f^{k-1+\vartheta}(\varphi) \qquad (\varphi \in V), \tag{3.9}$$

$$\frac{1}{(1-2\vartheta)\, h}\left(v^{k-\vartheta} - v^{k-1+\vartheta}, \varphi\right) + \left(v^{k-\vartheta}, v^{k-\vartheta}, \varphi\right) + \left(\left(\alpha v^{k-1+\vartheta} + (1-\alpha)\, v^{k-\vartheta}, \varphi\right)\right) = f^{k-\vartheta}(\varphi) \qquad (\varphi \in V), \tag{3.10}$$

and

$$\frac{1}{\vartheta h}\left(v^k - v^{k-\vartheta}, \varphi\right) + \left(v^{k-\vartheta}, v^{k-\vartheta}, \varphi\right) + \left(\left((1-\alpha)\, v^{k-\vartheta} + \alpha v^k, \varphi\right)\right) = f^k(\varphi) \qquad (\varphi \in V) \tag{3.11}$$

of size ϑh, $(1-2\vartheta)\,h$, and ϑh respectively, where the parameters $0 < \vartheta < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$, the step size $h > 0$, the initial value $v^0 \in V$, and a sequence $f^{\vartheta}, f^{1-\vartheta}, f^1, f^{1+\vartheta}, f^{2-\vartheta}, \ldots \in V'$ of evaluations of the force are given, and a sequence $v^{\vartheta}, v^{1-\vartheta}, v^1, v^{1+\vartheta}, v^{2-\vartheta}, \ldots \in V$ of solutions is searched-for. The parameter ϑ is usually chosen as $\vartheta = 1 - \frac{1}{2}\sqrt{2}$ so as to obtain second order convergence, see [MU93, MPRT95, Zan12].[2]

The following table provides quick reference to results relative to the equations and difference schemes in the present thesis. The numbers in the table are the numbers of the theorems where the corresponding result is proven.

	Existence	Non-existence	Uniqueness	Non-uniqueness	Stability	Non-stability	Convergence
Stationary NSE	19		22	24			
Explicit Euler		21	26				
Almost explicit Euler	19		26			51	
Almost implicit Euler	19		26		49		60
Implicit Euler	19		29, 30, 32, 33	39, 41	49		
Sum Crank-Nicolson	19		35	45		52	
Linear Crank-Nicolson	19		26		50		
Product Crank-Nicolson	19		34	43	50		65
Fractional step theta	19		26, 36	47		53	

[2] The error constants in [Zan12] are mistakenly stated to depend on nothing but α although they also depend on the viscosity ν.

4. Existence-related results

4.1. Existence

In this section we prove a general existence theorem based on the Galerkin method and apply it to all the equations and schemes from chapter 3 except for the explicit Euler scheme 3.2. The existence result is well-known for the special case of the Navier-Stokes SST space and $\lambda = \eta = 0$ and can for example be found in [Tem84, page 164, Theorem 1.2]. Then we show that the Navier-Stokes SST space satisfies the assumptions made in the existence theorem. The section starts with a lemma based on the Brouwer fixed point theorem.

Lemma 17. *Let X be a finite-dimensional vector space over $\mathbb{R}$ with scalar product and associated norm noted by $(\!(\cdot,\cdot)\!)$ and $\|\cdot\|$. If $P : X \to X$ is continuous and if there is a number $r > 0$ such that*

$$(\!(P(v), v)\!) > 0$$

for all $v \in X$ with $\|v\| = r$, then there is a vector $u \in X$ with $P(u) = 0$.

Proof. As it can be found in [Tem84, page 164, Lemma 1.4], the proof is omitted. □

Theorem 18 (General existence)**.** *Let V be an SST space and let $C \subset V$ be a subspace dense in V with respect to the Dirichlet norm $\|\cdot\|$. Let $\|\cdot\|_\infty$ be a seminorm on C such that there is a number $c_{H\infty H}$ with*

$$|(u, v, w)| \leqslant c_{H\infty H} |u| \|v\|_\infty |w| \tag{4.1}$$

for all $u, w \in V$ and all $v \in C$. Moreover suppose that $(V, (\!(\cdot,\cdot)\!))$ is a separable Hilbert space and that the imbedding $(V, \|\cdot\|) \hookrightarrow (H', |\cdot|)$ from Definition 5 is compact. Let $g \in V'$, $w \in V$, $\lambda > -c_p^{-2}$, and $\mu, \eta \in \mathbb{R}$. Then the weakly formulated equation

$$\lambda (v, \varphi) + \mu (v, v, \varphi) + \eta (w, v, \varphi) + (\!(v, \varphi)\!) = g(\varphi) \qquad (\varphi \in V) \tag{4.2}$$

has at least one solution $v \in V$ the Dirichlet norm of which satisfies

$$\|v\| \leqslant \left(\min\left\{1, 1 + \lambda c_p^2\right\}\right)^{-1} \|g\|_{V'}.$$

Proof. We consider the form

$$\begin{aligned}&V \times V \to \mathbb{R}\\ &(v,\varphi) \mapsto R^v\varphi := \lambda(v,\varphi) + \mu(v,v,\varphi) + \eta(w,v,\varphi) + ((v,\varphi)) - g(\varphi).\end{aligned}$$

The continuity of $(v,\varphi) \mapsto R^v\varphi$ on $V \times V$ with respect to the Dirichlet norm $\|\cdot\|$ is a consequence of the Cauchy-Schwarz-Bunyakovsky and the Poincaré inequality for the term $\lambda(v,\varphi)$, of the trilinear form's continuity with respect to $\|\cdot\|$, of the Cauchy-Schwarz-Bunyakovsky inequality for the term $((v,\varphi))$, and of the definition of V' for the term $g(\varphi)$.

Due to the assumed separability of V and the density of C in V there is a sequence

$$w_0, w_1, w_2, \dots \in C$$

such that $\{w_i; i \in \mathbb{N}\}$ is dense in V with respect to $\|\cdot\|$. For every $m \in \mathbb{N}$,

$$X_m := \operatorname{Span}\{w_0, \dots, w_{m-1}\}$$

equipped with the restriction of the scalar product $((\cdot,\cdot))$ to $X_m \times X_m$ is a Hilbert space of dimension at most m. For every $v \in X_m$ the linear form

$$\begin{aligned}R_m^v : X_m &\to \mathbb{R}\\ \varphi &\mapsto R^v\varphi\end{aligned}$$

is continuous since it is the restriction of $V \ni \varphi \mapsto R^v\varphi$ to X_m.[1] The Riesz representation theorem implies the unique existence of an element of X_m that we note by $P_m(v)$, such that $((P_m(v),\varphi)) = R_m^v\varphi$ for all $\varphi \in X_m$. The mapping

$$P_m : X_m \to X_m,\ v \mapsto P_m(v)$$

thereby defined is continuous since $X_m \ni v \mapsto R_m^v \in X_m'$ and the Riesz mapping are continuous. In the sequel we can assume that $g \neq 0$ because if $g = 0$, zero is a solution of equation 4.2. The next step is the application of Lemma 17 to the finite-dimensional space X_m and the continuous map P_m. For every $v \in X_m$ we have

$$((P_m(v),v)) \geqslant \left(\min\left\{1, 1+\lambda c_p^2\right\}\|v\| - \|g\|_{V'}\right)\|v\| \tag{4.3}$$

because

$$\begin{aligned}((P_m(v),v)) &= R_m^v v = R^v v\\ &= \lambda|v|^2 + \|v\|^2 - g(v)\\ &\geqslant \min\{0,\lambda\}\, c_p^2\|v\|^2 + \|v\|^2 - \|g\|_{V'}\|v\|.\end{aligned}$$

[1] One could of course just as well point out that the form R_m^v is linear on a finite-dimensional space in order to establish its continuity.

If we fix some

$$r > \left(\min\left\{1, 1 + \lambda c_p^2\right\}\right)^{-1} \|g\|_{V'} > 0,$$

it follows from inequality 4.3 that every $v \in X_m$ with $\|v\| = r$ satisfies $(\!(P_m(v), v)\!) > 0$. Hence due to Lemma 17 there is $u_m \in X_m$ with $P_m(u_m) = 0$. Choosing $v = u_m$ in inequality 4.3 we obtain

$$\|u_m\| \leqslant \left(\min\left\{1, 1 + \lambda c_p^2\right\}\right)^{-1} \|g\|_{V'}.$$

The sequence $(u_m)_m$ obtained in this way is bounded in the Hilbert space V and therefore possesses a subsequence we also denote by $(u_m)_m$ that $(\!(\cdot,\cdot)\!)$-weakly converges to some $u_\infty \in V$. The estimation of the norms $\|u_m\|$ carries over to the weak limit:

$$\|u_\infty\| \leqslant \left(\min\left\{1, 1 + \lambda c_p^2\right\}\right)^{-1} \|g\|_{V'}.$$

Since the imbedding $(V, \|\cdot\|) \hookrightarrow (H', |\cdot|)$ is compact and an isometry from $(V, |\cdot|)$ to $(H', |\cdot|)$ (see Definition 5), the sequence $(u_m)_m$ converges to u_∞ in the norm $|\cdot|$.

We now show that u_∞ is a solution of equation 4.2. For that we have to show that $R^{u_\infty}\varphi = 0$ for every $\varphi \in V$. Let $\varphi \in V$. It is sufficient to show that $|R^{u_\infty}\varphi| < \delta$ for every $\delta > 0$. Let $\delta > 0$. Since $\{w_i; i \in \mathbb{N}\}$ is dense in V and $x \mapsto R^{u_\infty}x$ is continuous on V, there is some $k \in \mathbb{N}$ such that

$$|R^{u_\infty}\varphi - R^{u_\infty}w_k| < \frac{\delta}{2}.$$

For this number $k \in \mathbb{N}$, we would like to show that

$$R^{u_m}w_k = \lambda(u_m, w_k) + \mu(u_m, u_m, w_k) + \eta(w, u_m, w_k) + (\!(u_m, w_k)\!) - g(w_k)$$

converges to $R^{u_\infty}w_k$ as $m \to \infty$. This is easy to see for the first term of the right hand side (with Cauchy-Schwarz-Bunyakovsky and because $u_m \xrightarrow{m\to\infty} u_\infty$ with respect to the norm $|\cdot|$) and for the penultimate term of the right hand side (because of the $(\!(\cdot,\cdot)\!)$-weak convergence $u_m \rightharpoonup u_\infty$). If we knew that $u_m \xrightarrow{m\to\infty} u_\infty$ with respect to the norm $\|\cdot\|$ (which we don't), this would also be easy to see for the trilinear terms due to the continuity of the trilinear form with respect to $\|\cdot\|$. Instead, we argue as follows:

$$\begin{aligned}
&|(u_\infty, u_\infty, w_k) - (u_m, u_m, w_k)| \\
&= |(u_m, w_k, u_m) - (u_\infty, w_k, u_\infty)| \\
&\leqslant |(u_m, w_k, u_m) - (u_m, w_k, u_\infty)| + |(u_m, w_k, u_\infty) - (u_\infty, w_k, u_\infty)| \\
&= |(u_m, w_k, u_m - u_\infty)| + |(u_m - u_\infty, w_k, u_\infty)| \\
&\leqslant c_{H\infty H}\left(|u_m| + |u_\infty|\right)\|w_k\|_\infty |u_m - u_\infty| \xrightarrow{m\to\infty} 0
\end{aligned}$$

and

$$\begin{aligned}&|(w,u_\infty,w_k)-(w,u_m,w_k)|\\&=|(w,w_k,u_\infty-u_m)|\\&\leqslant c_{H\infty H}\,|w|\,\|w_k\|_\infty\,|u_\infty-u_m|\xrightarrow{m\to\infty}0.\end{aligned}$$

As desired we now know that $R^{u_m}w_k\xrightarrow{m\to\infty}R^{u_\infty}w_k$ and this is why there is a number $M\in\mathbb{N}$ such that both

$$|R^{u_\infty}w_k-R^{u_M}w_k|<\frac{\delta}{2}$$

and

$$M>k.$$

The latter implies $w_k\in X_M$ and therefore $R^{u_M}w_k=((P_M(u_M),w_k))=0$. We conclude

$$|R^{u_\infty}\varphi|\leqslant\underbrace{|R^{u_\infty}\varphi-R^{u_\infty}w_k|}_{<\frac{\delta}{2}}+\underbrace{|R^{u_\infty}w_k-R^{u_M}w_k|}_{<\frac{\delta}{2}}+\underbrace{|R^{u_M}w_k|}_{=0}.$$

□

Theorem 18 applies to all equations and schemes from chapter 3 except for the explicit Euler scheme 3.2:

Theorem 19 (Existence)**.** *Let V be an SST space that satisfies the assumptions of Theorem 18 (general existence). Then each of*

- *the stationary Navier-Stokes equations 3.1,*
- *the almost explicit Euler scheme 3.3,*
- *the almost implicit Euler scheme 3.4,*
- *the implicit Euler scheme 3.5,*
- *the sum Crank-Nicolson scheme 3.6,*
- *the linear Crank-Nicolson scheme 3.7,*
- *the product Crank-Nicolson scheme 3.8, and*
- *the three substeps 3.9, 3.10, and 3.11 of the fractional step theta scheme*

have at least one solution $v\in V$ (resp. $v^k\in V$, $v^{k-1+\vartheta}\in V$, $v^{k-\vartheta}\in V$) for given force $f\in V'$ (resp. $f^k\in V'$, $f^{k-1+\vartheta}\in V'$, $f^{k-\vartheta}\in V'$), previous approximation $v^{k-1}\in V$ (resp. $v^{k-1+\vartheta}\in V$, $v^{k-\vartheta}\in V$), step size $h>0$, and parameters $0<\vartheta<\frac{1}{2}$ and $\frac{1}{2}<\alpha<1$.

Proof. Theorem 18 (general existence) applies to

- the stationary Navier-Stokes equations 3.1 setting $\lambda = 0$, $\mu = 1$, $\eta = 0$, and $g = f$,
- the almost explicit Euler scheme 3.3 setting $\lambda = \frac{1}{h}$, $\mu = \eta = 0$, and
$$g(\varphi) = f^k(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right) - \left(v^{k-1}, v^{k-1}, \varphi\right),$$
- the almost implicit Euler scheme 3.4 setting $\lambda = \frac{1}{h}$, $\mu = 0$, $\eta = 1$, $w = v^{k-1}$, and
$$g(\varphi) = f^k(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right),$$
- the implicit Euler scheme 3.5 setting $\lambda = \frac{1}{h}$, $\mu = 1$, $\eta = 0$, and
$$g(\varphi) = f^k(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right),$$
- the sum Crank-Nicolson scheme 3.6 setting $\lambda = \frac{2}{h}$, $\mu = 1$, $\eta = 0$, and
$$g(\varphi) = 2f^k(\varphi) + \frac{2}{h}\left(v^{k-1}, \varphi\right) - \left(v^{k-1}, v^{k-1}, \varphi\right) - \left(\left(v^{k-1}, \varphi\right)\right),$$
- the linear Crank-Nicolson scheme 3.7 setting $\lambda = \frac{2}{h}$, $\mu = 0$, $\eta = 1$, $w = v^{k-1}$, and
$$g(\varphi) = 2f^k(\varphi) + \frac{2}{h}\left(v^{k-1}, \varphi\right) - \left(v^{k-1}, v^{k-1}, \varphi\right) - \left(\left(v^{k-1}, \varphi\right)\right),$$
- the product Crank-Nicolson scheme 3.8 setting $\lambda = \frac{2}{h}$, $\mu = 1$, $\eta = 0$, and
$$g(\varphi) = f^k(\varphi) + \frac{2}{h}\left(v^{k-1}, \varphi\right)$$
(and obtaining $v^k = 2v - v^{k-1}$ as a solution where v is a solution to equation 4.2),
- the first substep 3.9 of the fractional step theta scheme setting $\lambda = \frac{1}{\alpha\vartheta h}$, $\mu = \eta = 0$, and
$$g(\varphi) = \frac{1}{\alpha} f^{k-1+\vartheta}(\varphi) + \frac{1}{\alpha\vartheta h}\left(v^{k-1}, \varphi\right) - \frac{1}{\alpha}\left(v^{k-1}, v^{k-1}, \varphi\right) - \frac{1-\alpha}{\alpha}\left(\left(v^{k-1}, \varphi\right)\right),$$
- the middle substep 3.10 of the fractional step theta scheme setting $\lambda = \frac{1}{(1-\alpha)(1-2\vartheta)h}$, $\mu = \frac{1}{1-\alpha}$, $\eta = 0$, and
$$g(\varphi) = \frac{1}{1-\alpha} f^{k-\vartheta}(\varphi) + \frac{1}{(1-\alpha)(1-2\vartheta)h}\left(v^{k-1+\vartheta}, \varphi\right) - \frac{\alpha}{1-\alpha}\left(\left(v^{k-1+\vartheta}, \varphi\right)\right),$$
and

- the last substep 3.11 of the fractional step theta scheme setting $\lambda = \frac{1}{\alpha\vartheta h}$, $\mu = \eta = 0$, and

$$g(\varphi) = \frac{1}{\alpha} f^k(\varphi) + \frac{1}{\alpha\vartheta h}\left(v^{k-\vartheta}, \varphi\right) - \frac{1}{\alpha}\left(v^{k-\vartheta}, v^{k-\vartheta}, \varphi\right) - \frac{1-\alpha}{\alpha}\left(\!\left(v^{k-\vartheta}, \varphi\right)\!\right)$$

because the Cauchy-Schwarz-Bunyakovsky inequality, the Poincaré inequality, and the continuity of the trilinear form with respect to the Dirichlet norm yield that in each of the enumerated cases the chosen right hand side g is in V'. □

Theorem 20. *The Navier-Stokes SST space from Definition 15 satisfies the additional assumptions made in Theorem 18 (general existence) or more specifically there is a subspace $C \subset V$ dense in V with respect to the Dirichlet norm $\|\cdot\|$, there is a seminorm $\|\cdot\|_\infty$ on C and a number $c_{H\infty H}$ such that inequality 4.1 holds for all $u, w \in V$ and all $v \in C$, the space $(V, (\!(\cdot,\cdot)\!))$ is a separable Hilbert space, and the imbedding $(V, \|\cdot\|) \hookrightarrow (H', |\cdot|)$ from Definition 5 is compact.*

Proof. Let C be the space $C^\infty_{0,\sigma}(G)$ from Definition 6. By definition it is dense in V with respect to the H^1 norm. The H^1 and the Dirichlet norms being equivalent on $V \subset H^1_0(G)^3$, the chosen space is also dense in V with respect to the Dirichlet norm. Choose

$$\|u\|_\infty := |\nabla u|_{L^\infty} = \operatorname*{ess\,sup}_{x\in G} \max_{i,j} |\partial_i u_j(x)| \qquad (u \in C)$$

as a seminorm on C.[2] Lemma 11 with $p = r = 2$, $q = \infty$, and $s = 1$ is used in the second inequality of the estimation

$$\begin{aligned} |(u, v, w)| &\leqslant |(u \cdot \nabla v) \cdot w|_{L^1} \\ &\leqslant 3\,|u|_{L^2}\,|\nabla v|_{L^\infty}\,|w|_{L^2}\,, \end{aligned}$$

valid for all $u, w \in V$ and $v \in C$. Hence inequality 4.1 holds with $c_{H\infty H} = 3$.

The Sobolev space $H^1(G)^3$ is separable with respect to the H^1 norm, see for example [AF03, page 61, Theorem 3.6]. Hence its subspace V is separable with respect to the same norm as well. As the Dirichlet and the H^1 norms are equivalent on $V \subset H^1_0(G)^3$, the space V is also separable with respect to the Dirichlet norm. The Sobolev space $H^1(G)^3$ is complete with respect to the H^1 norm, see [AF03, page 60, Theorem 3.3]. Being closed in $H^1(G)^3$ with respect to the H^1 norm, V is complete with respect to the same norm as well. The Dirichlet and the H^1 norms are equivalent on $V \subset H^1_0(G)^3$, thus V is also complete with respect to the Dirichlet norm. Altogether $(V, (\!(\cdot,\cdot)\!))$ is a separable Hilbert space.

In order to prove that the imbedding $(V, \|\cdot\|) \hookrightarrow (H', |\cdot|)$ is compact, suppose $(\tau_k)_{k\geqslant 0}$ is a sequence of vector fields $\tau_k \in V$ that is bounded with respect to $\|\cdot\|$. As a consequence of the Poincaré inequality in Theorem 8, the sequence is bounded in $H^1_0(G)^3$

[2] In fact, $\|\cdot\|_\infty = |\nabla\cdot|_{L^\infty}$ is even a norm on C but the proof of Theorem 18 also works if $\|\cdot\|_\infty$ is only a seminorm.

with respect to the H^1 norm. The imbedding $H_0^1(G) \hookrightarrow L^2(G)$ is compact, see [AF03, page 168, Theorem 6.3, part IV applied to imbedding (3) for $n = k = 3$, $\Omega = \Omega_0 = G$, $j = 0$, $m = 1$, $p = q = 2$] and note that the definition of the term domain used there doesn't require connectedness [AF03, page 1]. Moreover, note that the assumption that G be bounded can be replaced by weaker conditions that are also sufficient for the compactness of the imbedding [AF03, page 175, paragraph 6.14]. As a consequence of the compactness, there is a subsequence of $(\tau_k)_{k\geqslant 0}$ that is convergent with respect to the L^2 norm. In particular it is a Cauchy sequence with respect to the L^2 norm. The image of the subsequence under the imbedding $V \hookrightarrow H'$ from Definition 5 is a Cauchy sequence in $(H', |\cdot|)$ because the imbedding is an isometry from $(V, |\cdot|)$ to $(H', |\cdot|)$. As a dual space, $(H', |\cdot|)$ is complete. Therefore the image of the subsequence is convergent. This proves the compactness of the imbedding $(V, \|\cdot\|) \hookrightarrow (H', |\cdot|)$. □

4.2. Non-existence

The following theorem shows that the explicit Euler scheme lacks existence of solutions in the case of the Navier-Stokes SST space and more generally for every SST space the Dirichlet norm of which can not be controlled by its L^2 norm. However, existence can be established if the norms are equivalent, as proven in [Tem84, page 335, Scheme 5.4] for spatial discretizations of the Navier-Stokes SST space.

Theorem 21. *Let V be an SST space the norms $\|\cdot\|$ and $|\cdot|$ of which are not equivalent. Then for every $h > 0$ and $v^{k-1} \in V$ there is some $f^k \in V'$ such that the explicit Euler scheme 3.2 has no solution $v^k \in V$.*

Proof. Let $h > 0$ and $v^{k-1} \in V$. So as to prove the theorem by contradiction, suppose that for every $f^k \in V'$ the explicit Euler scheme 3.2 has a solution $v^k \in V$. First we show that under this assumption the imbedding

$$\begin{aligned} I : V &\to H' \\ v &\mapsto (V \ni \varphi \mapsto (v, \varphi)) \end{aligned}$$

from Definition 5 is onto and that not only $H' \subset V'$ (which holds for every SST space) but actually $H' = V'$. Let $g \in V'$. Then the linear form $f^k : V \to \mathbb{R}$ defined by

$$f^k(\varphi) = g(\varphi) - \frac{1}{h}\left(v^{k-1}, \varphi\right) + \left(v^{k-1}, v^{k-1}, \varphi\right) + \left(\left(v^{k-1}, \varphi\right)\right)$$

is in V' because of the Cauchy-Schwarz-Bunyakovsky and the Poincaré inequality and the continuity of the trilinear form with respect to $\|\cdot\|$. Hence, due to the reductio assumption, there is a solution $v^k \in V$ to the explicit Euler scheme 3.2 with step size h and data v^{k-1}, f^k. This yields

$$\begin{aligned} g(\varphi) &= f^k(\varphi) + \frac{1}{h}\left(v^{k-1},\varphi\right) - \left(v^{k-1},v^{k-1},\varphi\right) - \left(\!\left(v^{k-1},\varphi\right)\!\right) \\ &= \frac{1}{h}\left(v^k,\varphi\right) = \left(I\left(\frac{1}{h}v^k\right)\right)(\varphi), \end{aligned}$$

which implies both the surjectivity of I and $H' = V'$. The space $(V, |\cdot|)$ is a Banach space because it is isometrically isomorphic to $(H', |\cdot|)$ via I and because dual spaces are always complete. For every $e \in V$ the linear operator

$$T_e : V \to \mathbb{R}, \varphi \mapsto (\!(e,\varphi)\!)$$

is continuous from $(V, |\cdot|)$ to $\mathbb{R}$ because $T_e \in V'$ due to the Cauchy-Schwarz-Bunyakovsky inequality and because $H' = V'$. The collection

$$F := \{T_e; e \in V, |e| = 1\}$$

of continuous linear operators from $(V, |\cdot|)$ to $\mathbb{R}$ is pointwise bounded because for every point $\varphi \in V$ and every $T_e \in F$

$$|T_e(\varphi)| = |T_\varphi(e)| \leqslant |T_\varphi|\,|e| = |T_\varphi|\,,$$

where $|T_\varphi|$ denotes the norm of T_φ in $(H', |\cdot|)$. Meeting the requirements of the uniform boundedness principle (see e. g. [Rud73, page 44, Theorem 2.6]), the collection F is uniformly bounded, i. e. there is a number $C \in \mathbb{R}$ such that

$$|T_e| \leqslant C$$

for all $T_e \in F$. This allows for the estimation

$$\|v\|^2 = |v|\left(\!\left(\frac{v}{|v|},v\right)\!\right) = |v|\,T_{\frac{v}{|v|}}(v) \leqslant |v|\left|T_{\frac{v}{|v|}}\right||v| \leqslant C\,|v|^2$$

valid for all $v \in V \setminus \{0\}$, which, combined with the Poincaré inequality 2.1, implies the equivalence of the norms $\|\cdot\|$ and $|\cdot|$. □

5. Uniqueness-related results

5.1. The SST-generalized stationary Navier-Stokes equations

This section contains uniqueness and non-uniqueness results for the SST-generalized stationary Navier-Stokes equations.

5.1.1. Uniqueness

The following result is well-known for the special case of the Navier-Stokes SST space and can for example be found in [Tem84, page 167, Theorem 1.3]. The proof applies, as we will see, also in the general SST setting.

Theorem 22. *Let V be an SST space with $c_t > 0$ and $f \in V'$ with norm*

$$\|f\|_{V'} < c_t^{-3}.$$

Then the stationary Navier-Stokes equations 3.1, i. e. the weakly formulated equation

$$(v, v, \varphi) + ((v, \varphi)) = f(\varphi) \qquad (\varphi \in V),$$

have at most one solution $v \in V$.

Proof. We first establish an a priori estimate on all solutions of equation 3.1. Let v be such a solution. Choosing $\varphi = v$ we obtain $\|v\|^2 = f(v)$. From the estimate $|f(v)| \leqslant \|f\|_{V'} \|v\|$ we can now deduce the a priori estimate $\|v\| \leqslant \|f\|_{V'}$.

Suppose u and v are both solutions of equation 3.1. Then we obtain

$$\begin{aligned} 0 &= f(u-v) - f(u-v) \\ &= (u, u, u-v) + ((u, u-v)) - (v, v, u-v) - ((v, u-v)) \\ &= (u, v, u) - (v, v, u) + \|u-v\|^2 \\ &= (u-v, v, u-v) + \|u-v\|^2 . \end{aligned}$$

From this we deduce

$$\|u-v\|^2 \leqslant c_t^3 \|u-v\|^2 \|v\| .$$

If we now suppose that u and v are different, the a priori estimate and the assumption that $\|f\|_{V'} < c_t^{-3}$ yield the contradiction

$$1 \leqslant c_t^3 \|f\|_{V'} < 1.$$

□

5.1.2. Non-uniqueness

We can now ask the question whether or not uniqueness of solutions to the SST-generalized stationary Navier-Stokes equations 3.1 can also be proved for larger values of $\|f\|_{V'}$. This seems to be an open question in the special case of the Navier-Stokes SST space according to [Tem84, page 168, Remark 1.1]. In this subsection we show that there is no hope to prove uniqueness inside the SST setting if the smallness assumptions on the data are too weak.

Conjecture 23. *Let V be an SST space with $c_t > 0$. Then there is a number $N_f > c_t^{-3}$ such that for every $f \in V'$ with $\|f\|_{V'} = N_f$, the stationary Navier-Stokes equations 3.1 have at most one solution $v \in V$.*

As the following theorem shows, Conjecture 23 is in general false. This means that a proof for the special case of the Navier-Stokes SST space, if such exists, must inevitably make use of more than only the properties of SST spaces.

Theorem 24. *Conjecture 23 is false. There are counterexamples for all $c_p, c_t > 0$.*

Proof. Let $c_p, c_t > 0$. Consider $V = \mathbb{R}^2$ with the scalar products $((u, v)) = u_0 v_0 + u_1 v_1$ and $(u, v) = c_p^2 ((u, v))$ and the trilinear form

$$(u, v, w) = c_t^3 u_0 (v_0 w_1 - v_1 w_0).$$

The only non-obvious property when checking that V is an SST space might be the continuity of the trilinear form with respect to the Dirichlet norm. To show this continuity, we essentially use the Cauchy-Schwarz-Bunyakovsky inequality in $\mathbb{R}^2$:

$$\begin{aligned} |(u, v, w)| &\leqslant c_t^3 |u_0| \left|\left(\left(\begin{pmatrix} v_0 \\ -v_1 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \right)\right)\right| \\ &\leqslant c_t^3 \left(u_0^2 + u_1^2\right)^{\frac{1}{2}} \left|\left(\left(\begin{pmatrix} v_0 \\ -v_1 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_0 \end{pmatrix} \right)\right)\right| \\ &\leqslant c_t^3 \left(u_0^2 + u_1^2\right)^{\frac{1}{2}} \left(v_0^2 + v_1^2\right)^{\frac{1}{2}} \left(w_1^2 + w_0^2\right)^{\frac{1}{2}} = c_t^3 \|u\| \|v\| \|w\| . \end{aligned}$$

Let now $N_f > c_t^{-3}$. Set $f : V \to \mathbb{R}, v \mapsto N_f v_1$. Then $f \in V'$ with $\|f\|_{V'} = N_f$. A short calculation shows that the vectors

$$u = \begin{pmatrix} 0 \\ N_f \end{pmatrix}, \; v = \begin{pmatrix} c_t^{-3} \left(c_t^3 N_f - 1\right)^{\frac{1}{2}} \\ c_t^{-3} \end{pmatrix}, \text{ and } w = \begin{pmatrix} -c_t^{-3} \left(c_t^3 N_f - 1\right)^{\frac{1}{2}} \\ c_t^{-3} \end{pmatrix}$$

are three different solutions to equation 3.1 which contradicts the conjectured uniqueness. □

5.2. Uniqueness for linear difference schemes

It is well-known that the uniqueness of solutions to linear difference schemes can be proven without smallness assumptions on the data in the case of the Navier-Stokes SST space (see e. g. [GR79, page 171, Lemma 2.1]). The same is true for general SST spaces:

Theorem 25. *Let V be an SST space, $g : V \to \mathbb{R}$, $w \in V$, $\lambda > 0$, $\eta \in \mathbb{R}$, and $\mu \geqslant 0$. Then the weakly formulated equation*

$$\lambda\,(v, \varphi) + \eta\,(w, v, \varphi) + \mu\,((v, \varphi)) = g\,(\varphi) \qquad (\varphi \in V)$$

has at most one solution $v \in V$.

Proof. Suppose that $v, \tilde{v} \in V$ are two solutions. Then

$$\lambda\,(v - \tilde{v}, \varphi) + \eta\,(w, v - \tilde{v}, \varphi) + \mu\,((v - \tilde{v}, \varphi)) = 0$$

for all $\varphi \in V$. Choosing $\varphi = v - \tilde{v}$ yields

$$\lambda\,|v - \tilde{v}|^2 + \mu\,\|v - \tilde{v}\|^2 = 0.$$

The assumptions $\lambda > 0$ and $\mu \geqslant 0$ imply $v = \tilde{v}$. □

Theorem 25 applies to all linear schemes:

Theorem 26. *Let V be an SST space. Then each of*

- *the explicit Euler scheme 3.2,*
- *the almost explicit Euler scheme 3.3,*
- *the almost implicit Euler scheme 3.4,*
- *the linear Crank-Nicolson scheme 3.7, and*
- *the first substep 3.9 and the last substep 3.11 of the fractional step theta scheme*

have at most one solution $v^k \in V$ (resp. $v^{k-1+\vartheta} \in V$) for given force $f^k \in V'$ (resp. $f^{k-1+\vartheta} \in V'$), previous approximation $v^{k-1} \in V$ (resp. $v^{k-\vartheta} \in V$), step size $h > 0$, and parameters $0 < \vartheta < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$.

Proof. Theorem 25 applies to

- the explicit Euler scheme 3.2 setting $\lambda = \frac{1}{h}$, $\eta = \mu = 0$, and

$$g\,(\varphi) = f^k\,(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right) - \left(v^{k-1}, v^{k-1}, \varphi\right) - \left(\left(v^{k-1}, \varphi\right)\right),$$

- the almost explicit Euler scheme 3.3 setting $\lambda = \frac{1}{h}$, $\eta = 0$, $\mu = 1$, and

$$g(\varphi) = f^k(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right) - \left(v^{k-1}, v^{k-1}, \varphi\right)$$

- the almost implicit Euler scheme 3.4 setting $\lambda = \frac{1}{h}$, $\eta = \mu = 1$, $w = v^{k-1}$, and

$$g(\varphi) = f^k(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right),$$

- the linear Crank-Nicolson scheme 3.7 setting $\lambda = \frac{1}{h}$, $\eta = \mu = \frac{1}{2}$, $w = v^{k-1}$, and

$$g(\varphi) = f^k(\varphi) + \frac{1}{h}\left(v^{k-1}, \varphi\right) - \frac{1}{2}\left(v^{k-1}, v^{k-1}, \varphi\right) - \frac{1}{2}\left(\left(v^{k-1}, \varphi\right)\right),$$

- the first substep 3.9 of the fractional step theta scheme setting $\lambda = \frac{1}{\vartheta h}$, $\eta = 0$, $\mu = \alpha$, and

$$g(\varphi) = f^{k-1+\vartheta}(\varphi) + \frac{1}{\vartheta h}\left(v^{k-1}, \varphi\right) - \left(v^{k-1}, v^{k-1}, \varphi\right) - (1-\alpha)\left(\left(v^{k-1}, \varphi\right)\right),$$

  and

- the last substep 3.11 of the fractional step theta scheme setting $\lambda = \frac{1}{\vartheta h}$, $\eta = 0$, $\mu = \alpha$, and

$$g(\varphi) = f^k(\varphi) + \frac{1}{\vartheta h}\left(v^{k-\vartheta}, \varphi\right) - \left(v^{k-\vartheta}, v^{k-\vartheta}, \varphi\right) - (1-\alpha)\left(\left(v^{k-\vartheta}, \varphi\right)\right).$$

□

5.3. Nonlinear difference schemes

In this section we consider uniqueness questions in the context of nonlinear difference schemes. If the underlying SST space's norms $\|\cdot\|$ and $|\cdot|$ are equivalent – which is the case if spatial discretizations of the Navier-Stokes SST space are used – the choice of a small step size in time is sufficient for uniqueness as noted in [HR90, page 366] for the case of the product Crank-Nicolson scheme. However, in the case of the spatially continuous Navier-Stokes SST space, where the Dirichlet norm can not be controlled by the L^2 norm, establishing uniqueness for nonlinear difference schemes under smallness assumptions on the step size alone is either impossible or can only be achieved by means of properties that are not available in general SST spaces, see the Theorems 39 and 41 and their transfer to other nonlinear schemes below. The implicit Euler scheme 3.5 seems to be the simplest scheme where both uniqueness and non-uniqueness phenomena can be observed. As we study uniqueness questions in a single time step, it is convenient to use the superscript-free notation

$$\frac{1}{h}(v-w,\varphi)+(v,v,\varphi)+(\!(v,\varphi)\!)=f(\varphi) \qquad (\varphi\in V) \tag{5.1}$$

for the implicit Euler scheme 3.5, where $h>0$ is the step size, $w\in V$ the previous approximation, $v\in V$ the searched-for approximation (noted respectively by v^{k-1} and v^k when using the customary index notation), and $f\in V'$ the evaluation of the force. The results obtained for the implicit Euler scheme 5.1 are subsequently transferred to more complex schemes.

5.3.1. Uniqueness

Lemma 27. *Let V be an SST space with $c_t>0$, let $h>0$, $w\in V$, and $f\in V'$. Suppose that all solutions $v\in V$ of the implicit Euler scheme 5.1 satisfy the a priori estimate*

$$\|v\|\leqslant c_t^{-3}. \tag{5.2}$$

Then the scheme has at most one solution.

Proof. Suppose u and v are two solutions to scheme 5.1. Then we have

$$\begin{aligned}0&=f(u-v)-f(u-v)\\&=\frac{1}{h}(u-w,u-v)+(u,u,u-v)+(\!(u,u-v)\!)\\&\quad-\frac{1}{h}(v-w,u-v)-(v,v,u-v)-(\!(v,u-v)\!)\\&=\frac{1}{h}|u-v|^2+(u,v,u)-(v,v,u)+\|u-v\|^2\\&=\frac{1}{h}|u-v|^2+(u-v,v,u-v)+\|u-v\|^2.\end{aligned}$$

This implies

$$\frac{1}{h}|u-v|^2+\|u-v\|^2\leqslant c_t^3\|u-v\|^2\|v\|.$$

With a priori estimate 5.2 we obtain

$$\frac{1}{h}|u-v|^2+\|u-v\|^2\leqslant\|u-v\|^2,$$

which implies $u=v$. $\square$

In the proofs of Theorem 29 and Theorem 30 we use the following lemma.

Lemma 28. *Let V be an SST space, $h > 0$, $w \in V$, and $f \in H'$. Then every solution $v \in V$ of the implicit Euler scheme 5.1 satisfies*

$$\left(1 - c_t^3 \|w\|\right) \left(\|v\|^2 + \|v - w\|^2\right) \leqslant \|w\|^2 + \frac{h}{2} |f|^2 .$$

Proof. Let $v \in V$ be a solution of scheme 5.1. Testing with $\varphi = 2(v - w)$ and using the identity $2\,((v, v - w)) = \|v\|^2 - \|w\|^2 + \|v - w\|^2$ we obtain

$$\frac{2}{h} |v - w|^2 + 2(v, v, v - w) + \|v\|^2 - \|w\|^2 + \|v - w\|^2 = 2f(v - w) .$$

We use Young's inequality A.4 in both the estimation of the right hand side

$$|2f(v - w)| \leqslant 2|f|\,|v - w| \leqslant \frac{h}{2} |f|^2 + \frac{2}{h} |v - w|^2$$

and the trilinear term

$$\begin{aligned} |2(v, v, v - w)| = |2(v, v, w)| &= |2(v, v - w, w)| \\ &\leqslant 2c_t^3 \|v\|\,\|v - w\|\,\|w\| \\ &\leqslant c_t^3 \|w\| \left(\|v\|^2 + \|v - w\|^2\right) . \end{aligned}$$

All in all we obtain

$$\|v\|^2 + \|v - w\|^2 \leqslant \|w\|^2 + \frac{h}{2} |f|^2 + c_t^3 \|w\| \left(\|v\|^2 + \|v - w\|^2\right) ,$$

which simplifies to the claimed inequality. □

The following theorems supply sufficient conditions on the data h, w, and f for a priori estimate 5.2 to be satisfied.

Theorem 29. *Let V be an SST space with $c_t > 0$, let $h > 0$, $w \in V$, and $f \in H'$ such that*

$$\|w\| \leqslant \frac{1}{2} c_t^{-3}$$

and

$$h\,|f|^2 \leqslant \frac{1}{2} c_t^{-6}.$$

Then the implicit Euler scheme 5.1 has at most one solution $v \in V$.

Remark. No matter how large $|f|$ is, the smallness assumption $h\,|f|^2 \leqslant \frac{1}{2} c_t^{-6}$ can always be satisfied by choosing a small step size h. Unfortunately this is not the case for the smallness assumption $\|w\| \leqslant \frac{1}{2} c_t^{-3}$. We will see in sec. 5.3.2 that this difficulty can not in general be overcome in SST spaces.

Proof. We prove the theorem by establishing a priori estimate 5.2. Suppose $v \in V$ is a solution of the implicit Euler scheme 5.1. Due to Lemma 28 we have

$$\left(1 - c_t^3 \|w\|\right)\left(\|v\|^2 + \|v-w\|^2\right) \leqslant \|w\|^2 + \frac{h}{2}|f|^2 .$$

The smallness assumptions $\|w\| \leqslant \frac{1}{2}c_t^{-3}$ and $h|f|^2 \leqslant \frac{1}{2}c_t^{-6}$ imply

$$\begin{aligned}\frac{1}{2}\left(\|v\|^2 + \|v-w\|^2\right) &\leqslant \|w\|^2 + \frac{h}{2}|f|^2 \\ &\leqslant \frac{1}{4}c_t^{-6} + \frac{1}{4}c_t^{-6} = \frac{1}{2}c_t^{-6}\end{aligned}$$

and therefore $\|v\| \leqslant c_t^{-3}$, i. e. a priori estimate 5.2. Now Lemma 27 applies. □

At the price of having a more technical proof, the following theorem improves Theorem 29 to the extent that the smallness assumption on the previous approximation w is less restrictive.

Theorem 30. *Let V be an SST space with $c_t > 0$, let $w \in V$ with norm*

$$\|w\| < \frac{1}{2}\left(3 - \sqrt{3}\right)c_t^{-3}, \tag{5.3}$$

and $f \in H'$. Then the smallness assumption

$$h|f|^2 \leqslant \left(1 - c_t^3\|w\|\right)\left(3c_t^{-6} - 2\|w\|^2\right) - 2\|w\|^2 \tag{5.4}$$

can always be satisfied by choosing a sufficiently small step size $h > 0$. If both 5.3 and 5.4 hold, the implicit Euler scheme 5.1 has at most one solution $v \in V$.

Proof. A straight forward estimation using exclusively smallness assumption 5.3 shows that the upper bound in smallness assumption 5.4 is positive. Therefore a sufficiently small step size $h > 0$ can be found no matter how large $|f|$ is. This shows the first claim of the theorem. To show uniqueness we proceed by establishing a priori estimate 5.2. Let $v \in V$ be a solution of scheme 5.1. Due to Lemma 28 we have

$$\left(1 - c_t^3 \|w\|\right)\left(\|v\|^2 + \|v-w\|^2\right) \leqslant \|w\|^2 + \frac{h}{2}|f|^2 .$$

The first parenthesis $(1 - c_t^3\|w\|)$ is positive because $c_t^3\|w\| < \frac{1}{2}\left(3-\sqrt{3}\right) < 1$. The second parenthesis $\left(\|v\|^2 + \|v-w\|^2\right)$ can, because of

$$\frac{1}{2}\|v\|^2 = \frac{1}{2}\|(v-w)+w\|^2 \leqslant \|v-w\|^2 + \|w\|^2 ,$$

be estimated from below by $\left(\frac{3}{2}\|v\|^2 - \|w\|^2\right)$. This leads to

$$\frac{3}{2}\|v\|^2 \leqslant \|w\|^2 + \left(1 - c_t^3\|w\|\right)^{-1}\left(\|w\|^2 + \frac{h}{2}|f|^2\right).$$

Bearing in mind that $(1 - c_t^3\|w\|)$ is positive and using smallness assumption 5.4, we obtain

$$\begin{aligned}\|v\|^2 &\leqslant \frac{2}{3}\|w\|^2 + \frac{2}{3}\left(1 - c_t^3\|w\|\right)^{-1} \\ &\qquad \left(\|w\|^2 + \frac{1}{2}\left(\left(1 - c_t^3\|w\|\right)\left(3c_t^{-6} - 2\|w\|^2\right) - 2\|w\|^2\right)\right) \\ &= \frac{2}{3}\|w\|^2 + \frac{2}{3}\left(1 - c_t^3\|w\|\right)^{-1}\frac{1}{2}\left(1 - c_t^3\|w\|\right)\left(3c_t^{-6} - 2\|w\|^2\right) \\ &= \frac{2}{3}\|w\|^2 + \frac{1}{3}\left(3c_t^{-6} - 2\|w\|^2\right) \\ &= c_t^{-6}.\end{aligned}$$

A priori estimate 5.2 holds, Lemma 27 applies, and there is at most one solution. □

Theorems 29 and 30 assume that $f \in H$. In what follows we establish uniqueness results for the more general case $f \in V'$. The following lemma serves in the proofs of the Theorems 32 and 33.

Lemma 31. *Let V be an SST space, $h > 0$, $w \in V$, $f \in V'$, and $0 < \lambda < 1$. Then every solution $v \in V$ of the implicit Euler scheme 5.1 satisfies*

$$\frac{2}{h}|v - w|^2 + \left(1 - \lambda^{-\frac{1}{2}}c_t^3\|w\|\right)\left(\|v\|^2 + \lambda\|v - w\|^2\right) \leqslant \|w\|^2 + (1 - \lambda)^{-1}\|f\|_{V'}^2.$$

Proof. Let $v \in V$ be a solution of scheme 5.1 and let $0 < \lambda < 1$. Testing with $\varphi = 2(v - w)$ and using the identity $2\left(\!\left(v, v - w\right)\!\right) = \|v\|^2 - \|w\|^2 + \|v - w\|^2$ we obtain

$$\frac{2}{h}|v - w|^2 + 2(v, v, v - w) + \|v\|^2 - \|w\|^2 + \|v - w\|^2 = 2f(v - w).$$

We use Young's inequality A.4 in both the estimation of the right hand side

$$|2f(v - w)| \leqslant 2\|f\|_{V'}\|v - w\| \leqslant (1 - \lambda)^{-1}\|f\|_{V'}^2 + (1 - \lambda)\|v - w\|^2$$

and the trilinear term

$$\begin{aligned}|2(v, v, v - w)| &= |2(v, v, w)| = |2(v, v - w, w)| \\ &\leqslant 2c_t^3\|v\|\|v - w\|\|w\| \\ &\leqslant c_t^3\|w\|\left(\lambda^{-\frac{1}{2}}\|v\|^2 + \lambda^{\frac{1}{2}}\|v - w\|^2\right) \\ &= \lambda^{-\frac{1}{2}}c_t^3\|w\|\left(\|v\|^2 + \lambda\|v - w\|^2\right).\end{aligned}$$

All in all we obtain

$$\begin{aligned}
&\frac{2}{h}\left|v-w\right|^2+\left\|v\right\|^2-\left\|w\right\|^2+\left\|v-w\right\|^2\\
&\leqslant (1-\lambda)^{-1}\left\|f\right\|_{V'}^2+(1-\lambda)\left\|v-w\right\|^2+\lambda^{-\frac{1}{2}}c_t^3\left\|w\right\|\left(\left\|v\right\|^2+\lambda\left\|v-w\right\|^2\right),
\end{aligned}$$

which first simplifies to

$$\begin{aligned}
&\frac{2}{h}\left|v-w\right|^2+\left(\left\|v\right\|^2+\lambda\left\|v-w\right\|^2\right)\\
&\leqslant \left\|w\right\|^2+(1-\lambda)^{-1}\left\|f\right\|_{V'}^2+\lambda^{-\frac{1}{2}}c_t^3\left\|w\right\|\left(\left\|v\right\|^2+\lambda\left\|v-w\right\|^2\right)
\end{aligned}$$

and then to the claimed inequality. □

Theorem 32. *Let V be an SST space with $c_t>0$, let $h>0$, $w\in V$, and $f\in V'$ such that*

$$\left\|w\right\|\leqslant\frac{1}{3}c_t^{-3}$$

and

$$\left\|f\right\|_{V'}\leqslant\frac{1}{6}\sqrt{6}c_t^{-3}.$$

Then the implicit Euler scheme 5.1 has at most one solution $v\in V$.

Remark. In contrast to the Theorems 29 and 30, arbitrary large norms of f can not be compensated by small step sizes h here.

Proof. Like before the proof is based on Lemma 27. Suppose $v\in V$ is a solution of the implicit Euler scheme 5.1. Lemma 31 with $\lambda=\frac{1}{4}$ yields

$$\left(1-2c_t^3\left\|w\right\|\right)\left(\left\|v\right\|^2+\frac{1}{4}\left\|v-w\right\|^2\right)\leqslant\left\|w\right\|^2+\frac{4}{3}\left\|f\right\|_{V'}^2.$$

The smallness assumptions $\left\|w\right\|\leqslant\frac{1}{3}c_t^{-3}$ and $\left\|f\right\|_{V'}\leqslant\frac{1}{6}\sqrt{6}c_t^{-3}$ imply

$$\begin{aligned}
\frac{1}{3}\left(\left\|v\right\|^2+\frac{1}{4}\left\|v-w\right\|^2\right)&\leqslant\left\|w\right\|^2+\frac{4}{3}\left\|f\right\|_{V'}^2\\
&\leqslant\frac{1}{9}c_t^{-6}+\frac{2}{9}c_t^{-6}=\frac{1}{3}c_t^{-6}
\end{aligned}$$

and therefore $\left\|v\right\|\leqslant c_t^{-3}$, i.e. a priori estimate 5.2. Now Lemma 27 applies. □

The smallness assumptions on $\left\|w\right\|$ as well as on $\left\|f\right\|_{V'}$ in the following theorem are superior to those in Theorem 32. Therefore the only reason to prefer Theorem 32 to Theorem 33 is that it has a shorter proof.

Theorem 33. *Let V be an SST space with $c_t > 0$, let $h > 0$, $w \in V$, and $f \in V'$ such that*

$$\|w\| \leqslant \frac{2}{5} c_t^{-3}$$

and

$$\|f\|_{V'} \leqslant \frac{13}{45} \sqrt{2} c_t^{-3}.$$

Then the implicit Euler scheme 5.1 has at most one solution $v \in V$.

Proof. So as to establish an a priori estimate, let $v \in V$ be a solution of scheme 5.1. Lemma 31 with $\lambda = \frac{4}{9}$ implies

$$\left(1 - \frac{3}{2} c_t^3 \|w\|\right) \left(\|v\|^2 + \frac{4}{9} \|v - w\|^2\right) \leqslant \|w\|^2 + \frac{9}{5} \|f\|_{V'}^2 .$$

Due to the inequality $\frac{1}{2} \|v\|^2 = \frac{1}{2} \|(v - w) + w\|^2 \leqslant \|v - w\|^2 + \|w\|^2$ we have

$$\left(1 - \frac{3}{2} c_t^3 \|w\|\right) \left(\frac{11}{9} \|v\|^2 - \frac{4}{9} \|w\|^2\right) \leqslant \|w\|^2 + \frac{9}{5} \|f\|_{V'}^2 .$$

The assumption $\|w\| \leqslant \frac{2}{5} c_t^{-3}$ ensures that the first parenthesis is bounded from below by $\frac{2}{5}$, thus

$$\frac{11}{9} \|v\|^2 - \frac{4}{9} \|w\|^2 \leqslant \frac{5}{2} \|w\|^2 + \frac{9}{2} \|f\|_{V'}^2 .$$

Finally the assumption $\|f\|_{V'} \leqslant \frac{13}{45} \sqrt{2} c_t^{-3}$ yields $\|v\|^2 \leqslant c_t^{-6}$, and Lemma 27 implies the claimed uniqueness. □

In what follows, some of the uniqueness results obtained for the implicit Euler scheme 5.1 are transferred to the other nonlinear schemes. To begin with we transfer Theorem 30 to the product Crank-Nicolson scheme:

Theorem 34. *Let V be an SST space with $c_t > 0$, let $w \in V$ with norm*

$$\|w\| < \frac{1}{2} \left(3 - \sqrt{3}\right) c_t^{-3}, \tag{5.5}$$

and $f \in H'$. Then the smallness assumption

$$h |f|^2 \leqslant 2 \left(1 - c_t^3 \|w\|\right) \left(3 c_t^{-6} - 2 \|w\|^2\right) - 4 \|w\|^2 \tag{5.6}$$

can always be satisfied by choosing a sufficiently small step size $h > 0$. If both 5.5 and 5.6 hold, the product Crank-Nicolson scheme 3.8 in superscript-free notation

$$\frac{1}{h} (v - w, \varphi) + \frac{1}{4} (w + v, w + v, \varphi) + \frac{1}{2} ((w + v, \varphi)) = f(\varphi) \qquad (\varphi \in V)$$

has at most one solution $v \in V$.

Proof. As the upper bounds in the smallness assumptions 5.4 and 5.6 differ only by a factor of two, the existence of a sufficiently small step size $h > 0$ as stated in Theorem 34 follows from Theorem 30.

In order to prove the claimed uniqueness, suppose that the assumptions of Theorem 34 are satisfied and that there are two different solutions v^0 and v^1 to the product Crank-Nicolson scheme in superscript-free notation with step size h and data w, f. The calculation

$$\begin{aligned}&\frac{2}{h}\left(\frac{1}{2}\left(w+v^j\right)-w,\varphi\right)+\left(\frac{1}{2}\left(w+v^j\right),\frac{1}{2}\left(w+v^j\right),\varphi\right)+\left(\left(\frac{1}{2}\left(w+v^j\right),\varphi\right)\right)\\&=\frac{1}{h}\left(v^j-w,\varphi\right)+\frac{1}{4}\left(w+v^j,w+v^j,\varphi\right)+\frac{1}{2}\left(\left(w+v^j,\varphi\right)\right)=f\left(\varphi\right)\end{aligned}$$

valid for both $j \in \{0,1\}$ shows that $\frac{1}{2}\left(w+v^0\right)$ and $\frac{1}{2}\left(w+v^1\right)$ are two different solutions to the implicit Euler scheme 5.1 with step size $\frac{1}{2}h$ and data w, f. This contradicts the fact that the step size $\frac{1}{2}h$ and the data w, f satisfy the assumptions of Theorem 30. □

Next we transfer Theorem 33 to the sum Crank-Nicolson scheme:

Theorem 35. *Let V be an SST space with $c_t > 0$, let $h > 0$, $w \in V$, and $f \in V'$ such that*

$$2\left\|f\right\|_{V'}+c_t^3\left\|w\right\|^2+\left\|w\right\|\leqslant\frac{2}{5}c_t^{-3}. \tag{5.7}$$

Then the sum Crank-Nicolson scheme 3.6 in superscript-free notation

$$\frac{1}{h}\left(v-w,\varphi\right)+\frac{1}{2}\left(w,w,\varphi\right)+\frac{1}{2}\left(v,v,\varphi\right)+\frac{1}{2}\left(\left(w+v,\varphi\right)\right)=f\left(\varphi\right)\qquad(\varphi\in V)$$

has at most one solution $v \in V$.

Proof. Suppose that the assumptions of Theorem 35 are satisfied and that v^0 and v^1 are two different solutions to the sum Crank-Nicolson scheme in superscript-free notation with step size h and data w, f. The linear form

$$g : V \to \mathbb{R}, \varphi \mapsto 2f\left(\varphi\right)-\left(w,w,\varphi\right)-\left(\left(w,\varphi\right)\right)$$

is an element of V' with norm

$$\left\|g\right\|_{V'}\leqslant 2\left\|f\right\|_{V'}+c_t^3\left\|w\right\|^2+\left\|w\right\|$$

due to the continuity of the trilinear form and the Cauchy-Schwarz-Bunyakovsky inequality. Now smallness assumption 5.7 and the fact that $\frac{2}{5} < \frac{13}{45}\sqrt{2}$ yield that g satisfies the smallness assumption on the force in Theorem 33. Aside from that, smallness assumption 5.7 also implies that w satisfies the smallness assumption on the previous approximation in Theorem 33. Therefore the implicit Euler scheme 5.1

with data w, g has at most one solution regardless of the step size. This is in contradiction to the calculation

$$\begin{aligned}
&\frac{2}{h}\left(v^j - w, \varphi\right) + \left(v^j, v^j, \varphi\right) + \left(\!\left(v^j, \varphi\right)\!\right) \\
&= 2\left(\frac{1}{h}\left(v^j - w, \varphi\right) + \frac{1}{2}\left(w, w, \varphi\right) + \frac{1}{2}\left(v^j, v^j, \varphi\right) + \frac{1}{2}\left(\!\left(w + v^j, \varphi\right)\!\right)\right) \\
&\qquad - \left(w, w, \varphi\right) - \left(\!\left(w, \varphi\right)\!\right) \\
&= 2f\left(\varphi\right) - \left(w, w, \varphi\right) - \left(\!\left(w, \varphi\right)\!\right) = g\left(\varphi\right)
\end{aligned}$$

valid for both $j \in \{0, 1\}$ that shows that v^0 and v^1 are two different solutions to the implicit Euler scheme 5.1 with step size $\frac{1}{2}h$ and data w, g. □

Finally we transfer Theorem 33 also to the middle time step 3.10 of the fractional step theta scheme:

Theorem 36. *Let V be an SST space with $c_t > 0$, let $0 < \vartheta < \frac{1}{2}$, $\frac{1}{2} < \alpha < 1$, $h > 0$, $w \in V$, and $f \in V'$ such that*

$$\left(1 - \alpha\right)^{-2}\left(\|f\|_{V'} + \alpha\|w\|\right) \leqslant \frac{2}{5}c_t^{-3}. \tag{5.8}$$

Then the middle time step 3.10 of the fractional step theta scheme in superscript-free notation

$$\frac{1}{\left(1 - 2\vartheta\right)h}\left(v - w, \varphi\right) + \left(v, v, \varphi\right) + \left(\!\left(\alpha w + \left(1 - \alpha\right)v, \varphi\right)\!\right) = f\left(\varphi\right) \qquad \left(\varphi \in V\right)$$

has at most one solution $v \in V$.

Proof. Suppose that v^0 and v^1 are two different solutions to the middle time step 3.10 in superscript-free notation with step size h and data w, f, even though the assumptions of Theorem 36 are satisfied. The Cauchy-Schwarz-Bunyakovsky inequality implies that the linear form

$$g : V \to \mathbb{R}, \varphi \mapsto \left(1 - \alpha\right)^{-2}\left(f\left(\varphi\right) - \alpha\left(\!\left(w, \varphi\right)\!\right)\right)$$

is an element of V' with norm

$$\|g\|_{V'} \leqslant \left(1 - \alpha\right)^{-2}\left(\|f\|_{V'} + \alpha\|w\|\right).$$

As a consequence of smallness assumption 5.8, g satisfies $\|g\|_{V'} \leqslant \frac{2}{5}c_t^{-3}$. Like in the proof of Theorem 35, the fact that $\frac{2}{5} < \frac{13}{45}\sqrt{2}$ now implies that g satisfies the smallness assumption on the force from Theorem 33. Furthermore, $\alpha > \frac{1}{2}$ and smallness assumption 5.8 allow for the estimation

$$\left\|\left(1 - \alpha\right)^{-1}w\right\| \leqslant \frac{\alpha}{\left(1 - \alpha\right)^2}\|w\| \leqslant \frac{2}{5}c_t^{-3}$$

which states that $(1-\alpha)^{-1} w$ satisfies the smallness assumption on the previous approximation from Theorem 33. Therefore the implicit Euler scheme 5.1 with data $(1-\alpha)^{-1} w$ and g has at most one solution no matter what step size is chosen. But the calculation

$$\begin{aligned}
&\frac{1}{(1-\alpha)(1-2\vartheta) h}\left((1-\alpha)^{-1} v^j-(1-\alpha)^{-1} w, \varphi\right) \\
&\quad+\left((1-\alpha)^{-1} v^j,(1-\alpha)^{-1} v^j, \varphi\right)+\left(\left((1-\alpha)^{-1} v^j, \varphi\right)\right) \\
&=\frac{1}{(1-\alpha)^2}\left[\frac{1}{(1-2\vartheta) h}\left(v^j-w, \varphi\right)+\left(v^j, v^j, \varphi\right)+\left(\left(\alpha w+(1-\alpha) v^j, \varphi\right)\right)\right. \\
&\qquad\qquad \left.-\alpha((w, \varphi))\right] \\
&=(1-\alpha)^{-2}(f(\varphi)-\alpha((w, \varphi)))=g(\varphi)
\end{aligned}$$

valid for both $j \in\{0,1\}$ shows that $(1-\alpha)^{-1} v^0$ and $(1-\alpha)^{-1} v^1$ are two different solutions to the implicit Euler scheme 5.1 with step size $(1-\alpha)(1-2\vartheta) h$ and data $(1-\alpha)^{-1} w$ and g. $\square$

5.3.2. Non-uniqueness

The uniqueness results established in sec. 5.3.1 raise the question to what extent the smallness assumptions can be relaxed without losing uniqueness. In what follows we show that there is no hope to prove uniqueness for the implicit Euler scheme 5.1 inside the SST setting if the smallness assumptions are too weak. Afterwards we transfer the results to more complex schemes.

Lemma 37 (Weighted SST space on $\mathbb{Z}\backslash\{0\}$)**.** *Let $c_p>0$, $c_t \geqslant 0$, and*

$$\delta, \lambda: \mathbb{Z} \backslash\{0\} \rightarrow \mathbb{R}_{>0}$$

with $\delta \geqslant \lambda$. Then

$$V:=\left\{v: \mathbb{Z} \backslash\{0\} \rightarrow \mathbb{R} ; \sum_{i \neq 0}\left|\delta_i v_i\right|^2<\infty\right\},$$

$$((u, v)):=\sum_{i \neq 0} \delta_i^2 u_i v_i \qquad (u, v \in V),$$

$$(u, v):=c_p^2 \sum_{i \neq 0} \lambda_i^2 u_i v_i \qquad (u, v \in V),$$

and

$$(u, v, w) := c_t^3 \sum_{n \geqslant 1} \delta_n^2 \delta_{-n} u_n \left(v_n w_{-n} - v_{-n} w_n\right) \qquad (u, v, w \in V)$$

defines an SST space.

Proof. The counting measure weighted with δ makes $\mathbb{Z} \setminus \{0\}$ a measure space. V is the space of square integrable functions with respect to this measure and $((\cdot, \cdot))$ is the associated scalar product. Hence V is a vector space and the series in the definition of $((u, v))$ converges absolutely for all $u, v \in V$. The absolute convergence of the series in the definition of (u, v) and the validity of the Poincaré inequality are a consequence of the assumption $\delta \geqslant \lambda$. The observation that $(u, v, v) = 0$ for all $u, v \in V$ is immediate. The continuity of the trilinear form and the absolute convergence of the series in its definition result from the estimation

$$\begin{aligned}
|(u, v, w)| &\leqslant c_t^3 \sum_{n \geqslant 1} |\delta_n u_n| \left(|\delta_n v_n \delta_{-n} w_{-n}| + |\delta_{-n} v_{-n} \delta_n w_n|\right) \\
&\leqslant c_t^3 \|u\| \sum_{n \geqslant 1} \left(|\delta_n v_n \delta_{-n} w_{-n}| + |\delta_{-n} v_{-n} \delta_n w_n|\right) \\
&\leqslant c_t^3 \|u\| \sum_{n \geqslant 1} \left(|\delta_n v_n|^2 + |\delta_{-n} v_{-n}|^2\right)^{\frac{1}{2}} \left(|\delta_{-n} w_{-n}|^2 + |\delta_n w_n|^2\right)^{\frac{1}{2}} \\
&\leqslant c_t^3 \|u\| \left(\sum_{n \geqslant 1} \left(|\delta_n v_n|^2 + |\delta_{-n} v_{-n}|^2\right)\right)^{\frac{1}{2}} \left(\sum_{n \geqslant 1} \left(|\delta_{-n} w_{-n}|^2 + |\delta_n w_n|^2\right)\right)^{\frac{1}{2}} \\
&= c_t^3 \|u\| \|v\| \|w\|,
\end{aligned}$$

where the last two inequalities are the Cauchy-Schwarz-Bunyakovsky inequality in $\mathbb{R}^2$ and $\ell^2(\mathbb{N}_{>0})$, respectively. □

Conjecture 38. *Let V be an SST space with $c_t > 0$. Then there are numbers $N_w > c_t^{-3}$, $N_f \geqslant 0$, and $h_0 > 0$ such that for every $w \in V$ with $\|w\| = N_w$, $f \in V'$ with $\|f\|_{V'} = N_f$, and $0 < h \leqslant h_0$, the implicit Euler scheme 5.1 has at most one solution $v \in V$.*

The following theorem shows that Conjecture 38 is in general false. Of course this doesn't admit the conclusion that it is false for the special case of the Navier-Stokes SST space. But it shows that if there exists a proof of Conjecture 38 for the special case of the Navier-Stokes SST space, such a proof would have to use techniques beyond those available in SST spaces.

Theorem 39. *Conjecture 38 is false. There are counterexamples for all $c_p, c_t > 0$.*

Proof. Let $c_p, c_t > 0$. Consider the weighted SST space on $\mathbb{Z} \setminus \{0\}$ from Lemma 37 with weights $\delta_n = n, \delta_{-n} = \lambda_n = \lambda_{-n} = 1$ $(n \in \mathbb{N}_{>0})$, i. e.

$$V = \left\{ v : \mathbb{Z} \setminus \{0\} \to \mathbb{R}; \sum_{n \geqslant 1} \left(|n v_n|^2 + |v_{-n}|^2\right) < \infty \right\}$$

with

$$(\!(u,v)\!) = \sum_{n\geqslant 1} \left(n^2 u_n v_n + u_{-n} v_{-n} \right),$$

$$(u,v) = c_p^2 \sum_{i\neq 0} u_i v_i,$$

and

$$(u,v,w) = c_t^3 \sum_{n\geqslant 1} n^2 u_n \left(v_n w_{-n} - v_{-n} w_n \right)$$

for all $u, v, w \in V$. Under the reductio assumption that Conjecture 38 holds, there are numbers $N_w > c_t^{-3}$, $N_f \geqslant 0$, and $h_0 > 0$ such that for every $w \in V$ with $\|w\| = N_w$, every $f \in V'$ with $\|f\|_{V'} = N_f$, and every $0 < h \leqslant h_0$, the implicit Euler scheme 5.1 has at most one solution $v \in V$. It is now, because of $N_w > c_t^{-3}$, possible to choose a step size $h > 0$ with

$$h < \min \left\{ h_0, c_p^2 \left(c_t^3 N_w - 1 \right) \right\}$$

and after that a number $m \in \mathbb{N}_{>0}$ with

$$m^2 > \frac{c_p^2 \left(1 + \frac{1}{h} c_p^2 \right)}{c_p^2 \left(c_t^3 N_w - 1 \right) - h}.$$

Setting

$$w_i := \delta_{i,-m} N_w \qquad (i \in \mathbb{Z} \setminus \{0\})$$

(δ with two indices denotes the Kronecker delta and not the weight δ) and

$$f : V \to \mathbb{R}, \varphi \mapsto N_f \varphi_{-m}$$

defines $w \in V$ and $f \in V'$ with $\|w\| = N_w$ and $\|f\|_{V'} = N_f$. The proof proceeds with the specification of three different solutions to the implicit Euler scheme 5.1. The choice of h and m ensures that the radicand in the definition of

$$\pi_0 := 0,$$

$$\pi_1 := h^{-\frac{1}{2}} c_t^{-3} m^{-2} \sqrt{\left(h c_t^3 N_f + c_p^2 \left(c_t^3 N_w - 1 \right) - h \right) m^2 - c_p^2 \left(1 + \frac{1}{h} c_p^2 \right)}, \text{ and}$$

$$\pi_2 := -\pi_1$$

is positive. For every $j \in \{0, 1, 2\}$ let

$$\nu_j := \left(1 + \frac{1}{h} c_p^2 \right)^{-1} \left(N_f + \frac{1}{h} c_p^2 N_w - c_t^3 m^2 \pi_j^2 \right)$$

and with this

$$v_i^j := \delta_{i,m}\pi_j + \delta_{i,-m}\nu_j \qquad (i \in \mathbb{Z}\setminus\{0\})\,.$$

The elements v^0, v^1, and v^2 hereby defined are solutions to the implicit Euler scheme 5.1 because for every $j \in \{0,1,2\}$ and every $\varphi \in V$

$$\begin{aligned}
&\frac{1}{h}\left(v^j - w, \varphi\right) + \left(v^j, v^j, \varphi\right) + \left(\left(v^j, \varphi\right)\right) \\
&= \frac{1}{h}c_p^2 \sum_{i\neq 0}\left(v_i^j - w_i\right)\varphi_i + c_t^3 \sum_{n\geqslant 1} n^2 v_n^j \left(v_n^j \varphi_{-n} - v_{-n}^j \varphi_n\right) \\
&\quad + \sum_{n\geqslant 1}\left(n^2 v_n^j \varphi_n + v_{-n}^j \varphi_{-n}\right) \\
&= \underbrace{\frac{1}{h}c_p^2 \pi_j \varphi_m - c_t^3 m^2 \pi_j \nu_j \varphi_m + m^2 \pi_j \varphi_m}_{=:\,\Phi_m} \\
&\quad + \underbrace{\frac{1}{h}c_p^2\left(\nu_j - N_w\right)\varphi_{-m} + c_t^3 m^2 \pi_j^2 \varphi_{-m} + \nu_j \varphi_{-m}}_{=:\,\Phi_{-m}}\,,
\end{aligned}$$

$$\begin{aligned}
\Phi_m &= \left(\frac{1}{h}c_p^2\pi_j - c_t^3 m^2 \pi_j \left(1 + \frac{1}{h}c_p^2\right)^{-1}\left(N_f + \frac{1}{h}c_p^2 N_w - c_t^3 m^2 \pi_j^2\right) + m^2\pi_j\right)\varphi_m \\
&= \pi_j\left(1 + \frac{1}{h}c_p^2\right)^{-1}\Bigg(\left(1 + \frac{1}{h}c_p^2\right)\frac{1}{h}c_p^2 - c_t^3 m^2\left(N_f + \frac{1}{h}c_p^2 N_w - c_t^3 m^2 \pi_j^2\right) \\
&\qquad + \left(1 + \frac{1}{h}c_p^2\right)m^2\Bigg)\varphi_m \\
&= \pi_j\left(1 + \frac{1}{h}c_p^2\right)^{-1}\Bigg(\frac{1}{h}c_p^2\left(1 + \frac{1}{h}c_p^2\right) - \left(c_t^3 N_f + \frac{1}{h}c_p^2\left(c_t^3 N_w - 1\right) - 1\right)m^2 \\
&\qquad + c_t^6 m^4 \pi_j^2\Bigg)\varphi_m \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{-m} &= \left(\left(\frac{1}{h}c_p^2 + 1\right)\nu_j - \frac{1}{h}c_p^2 N_w + c_t^3 m^2 \pi_j^2\right)\varphi_{-m} \\
&= \left(\left(N_f + \frac{1}{h}c_p^2 N_w - c_t^3 m^2 \pi_j^2\right) - \frac{1}{h}c_p^2 N_w + c_t^3 m^2 \pi_j^2\right)\varphi_{-m} \\
&= N_f \varphi_{-m} \\
&= f(\varphi)\,.
\end{aligned}$$

The existence of three different solutions contradicts uniqueness. $\square$

As Theorem 39 shows, uniqueness can be violated if the Dirichlet norm $\|\cdot\|$ of the previous approximation w exceeds c_t^{-3}. When it comes to the norm $|\cdot|$, the situation is even worse: Theorem 41 shows that uniqueness can be violated for arbitrary small but non vanishing $|w|$.

Conjecture 40. *Let V be an SST space with $c_t > 0$. Then there are numbers $N_w > 0$, $N_f \geqslant 0$, and $h_0 > 0$ such that for every $w \in V$ with $|w| = N_w$, $f \in V'$ with $\|f\|_{V'} = N_f$, and $0 < h \leqslant h_0$, the implicit Euler scheme 5.1 has at most one solution $v \in V$.*

As the following theorem shows, Conjecture 40 is in general false.

Theorem 41. *Conjecture 40 is false. There are counterexamples for all $c_p, c_t > 0$.*

Proof. Let $c_p, c_t > 0$. Consider the weighted SST space on $\mathbb{Z}\setminus\{0\}$ from Lemma 37 with weights $\delta_n = 2n, \delta_{-n} = n, \lambda_n = \lambda_{-n} = 1$ ($n \in \mathbb{N}_{>0}$), i. e.

$$V = \left\{ v : \mathbb{Z}\setminus\{0\} \to \mathbb{R}; \sum_{n \geqslant 1} \left(|2nv_n|^2 + |nv_{-n}|^2 \right) < \infty \right\}$$

with

$$((u, v)) = \sum_{n \geqslant 1} \left(4n^2 u_n v_n + n^2 u_{-n} v_{-n} \right),$$

$$(u, v) = c_p^2 \sum_{i \neq 0} u_i v_i,$$

and

$$(u, v, w) = c_t^3 \sum_{n \geqslant 1} 4n^3 u_n \left(v_n w_{-n} - v_{-n} w_n \right)$$

for all $u, v, w \in V$. If Conjecture 40 were true, there would be numbers $N_w > 0$, $N_f \geqslant 0$, and $h_0 > 0$ such that for every $w \in V$ with $|w| = N_w$, every $f \in V'$ with $\|f\|_{V'} = N_f$, and every $0 < h \leqslant h_0$, scheme 5.1 would have at most one solution $v \in V$. Under these assumptions a step size $0 < h \leqslant h_0$ satisfying both

$$h \leqslant \left(1 - \frac{1}{\sqrt{2}}\right)^2 c_p^2 \tag{5.9}$$

and

$$h \leqslant \frac{1}{18} c_p^2 c_t^6 N_w^2 \tag{5.10}$$

can be chosen. Inequality 5.9 implies

$$h^{-\frac{1}{2}}c_p - (2h)^{-\frac{1}{2}}c_p = h^{-\frac{1}{2}}c_p\left(1 - \frac{1}{\sqrt{2}}\right) \geqslant 1.$$

Hence there is a number $m \in \mathbb{N}_{>0}$ with

$$(2h)^{-\frac{1}{2}}c_p < m \leqslant h^{-\frac{1}{2}}c_p. \tag{5.11}$$

The previous approximation $w \in V$ and force $f \in V'$ defined by

$$w_i := \delta_{i,-m}N_w \qquad (i \in \mathbb{Z}\setminus\{0\})$$

(note that δ with two indices denotes the Kronecker delta and not the weight δ) and

$$f : V \to \mathbb{R}, \varphi \mapsto N_f m\varphi_{-m}$$

satisfy $|w| = N_w$ and $\|f\|_{V'} = N_f$. The number

$$B := c_t^3 m^2 h N_f + c_p^2\left(c_t^3 m N_w - 1\right) - hm^2 \tag{5.12}$$

satisfies $B > c_p^2$ which can be seen using first the nonnegativity of N_f, then both inequalities 5.11, and after that inequality 5.10 as follows:

$$\begin{aligned} B &\geqslant c_p^2\left(c_t^3 m N_w - 1\right) - hm^2 \\ &> c_p^2\left(c_t^3 (2h)^{-\frac{1}{2}} c_p N_w - 1\right) - c_p^2 \\ &\geqslant c_p^2. \end{aligned}$$

The positivity of the number

$$P := 4hm^2B - c_p^2\left(c_p^2 + hm^2\right) \tag{5.13}$$

results from the inequality $B > c_p^2$ established a moment ago and from inequality 5.11 as follows:

$$\begin{aligned} P &> c_p^2\left(3hm^2 - c_p^2\right) \\ &> \frac{1}{2}c_p^4. \end{aligned}$$

Hence setting

$$\begin{aligned} \pi_0 &:= 0, \\ \pi_1 &:= \frac{1}{4}h^{-1}c_t^{-3}m^{-3}\sqrt{P}, \text{ and} \\ \pi_2 &:= -\pi_1 \end{aligned}$$

defines three different numbers. For every $j \in \{0,1,2\}$ set

$$\nu_j := \left(c_p^2 + hm^2\right)^{-1} \left(c_p^2 N_w + hmN_f - 4hc_t^3 m^3 \pi_j^2\right).$$

The three elements v^0, v^1, and v^2 defined by

$$v_i^j := \delta_{i,m}\pi_j + \delta_{i,-m}\nu_j \qquad (i \in \mathbb{Z}\setminus\{0\}, j \in \{0,1,2\})$$

are solutions to the implicit Euler scheme 5.1 because for every $j \in \{0,1,2\}$ and every $\varphi \in V$ it holds

$$\begin{aligned}
&\frac{1}{h}\left(v^j - w, \varphi\right) + \left(v^j, v^j, \varphi\right) + \left(\left(v^j, \varphi\right)\right) \\
&= \frac{1}{h} c_p^2 \sum_{i \neq 0} \left(v_i^j - w_i\right)\varphi_i + 4c_t^3 \sum_{n \geqslant 1} n^3 v_n^j \left(v_n^j \varphi_{-n} - v_{-n}^j \varphi_n\right) \\
&\quad + \sum_{n \geqslant 1} \left(4n^2 v_n^j \varphi_n + n^2 v_{-n}^j \varphi_{-n}\right) \\
&= \underbrace{\frac{1}{h} c_p^2 \pi_j \varphi_m - 4c_t^3 m^3 \pi_j \nu_j \varphi_m + 4m^2 \pi_j \varphi_m}_{=:\, \Phi_m} \\
&\quad + \underbrace{\frac{1}{h} c_p^2 \left(\nu_j - N_w\right)\varphi_{-m} + 4c_t^3 m^3 \pi_j^2 \varphi_{-m} + m^2 \nu_j \varphi_{-m}}_{=:\, \Phi_{-m}}
\end{aligned}$$

with (using equations 5.12 and 5.13 in the last two lines)

$$\begin{aligned}
\Phi_m &= \left(\frac{1}{h} c_p^2 - 4c_t^3 m^3 \left(c_p^2 + hm^2\right)^{-1} \left(c_p^2 N_w + hmN_f - 4hc_t^3 m^3 \pi_j^2\right) + 4m^2\right) \pi_j \varphi_m \\
&= \left(c_p^2 + hm^2\right)^{-1} \left(\frac{1}{h} c_p^2 \left(c_p^2 + hm^2\right) - 4c_t^3 m^3 \left(c_p^2 N_w + hmN_f - 4hc_t^3 m^3 \pi_j^2\right)\right. \\
&\qquad \left. + 4m^2 \left(c_p^2 + hm^2\right)\right) \pi_j \varphi_m \\
&= \left(c_p^2 + hm^2\right)^{-1} \left(\frac{1}{h} c_p^2 \left(c_p^2 + hm^2\right)\right. \\
&\qquad \left. - 4m^2 \left(m c_p^2 c_t^3 N_w + hm^2 c_t^3 N_f - c_p^2 - hm^2\right) + 16hc_t^6 m^6 \pi_j^2\right) \pi_j \varphi_m \\
&= \left(c_p^2 + hm^2\right)^{-1} \left(\frac{1}{h} c_p^2 \left(c_p^2 + hm^2\right) - 4m^2 B + 16hc_t^6 m^6 \pi_j^2\right) \pi_j \varphi_m \\
&= \left(c_p^2 + hm^2\right)^{-1} \left(-\frac{1}{h} P + 16hc_t^6 m^6 \pi_j^2\right) \pi_j \varphi_m = 0
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{-m} &= \left(\left(\frac{1}{h} c_p^2 + m^2\right)\nu_j - \frac{1}{h} c_p^2 N_w + 4c_t^3 m^3 \pi_j^2\right) \varphi_{-m} \\
&= \left(\frac{1}{h}\left(c_p^2 N_w + hmN_f - 4hc_t^3 m^3 \pi_j^2\right) - \frac{1}{h} c_p^2 N_w + 4c_t^3 m^3 \pi_j^2\right) \varphi_{-m} \\
&= mN_f \varphi_{-m} \\
&= f(\varphi).
\end{aligned}$$

The existence of three different solutions contradicts uniqueness. □

In what follows, Theorem 39 is transferred to the other nonlinear schemes. We begin with the product Crank-Nicolson scheme:

Conjecture 42. *Let V be an SST space with $c_t > 0$. Then there are numbers $N_w > c_t^{-3}$, $N_f \geqslant 0$, and $h_0 > 0$ such that for every $w \in V$ with $\|w\| = N_w$, $f \in V'$ with $\|f\|_{V'} = N_f$, and $0 < h \leqslant h_0$, the product Crank-Nicolson scheme 3.8 in superscript-free notation*

$$\frac{1}{h}(v-w,\varphi)+\frac{1}{4}(w+v,w+v,\varphi)+\frac{1}{2}((w+v,\varphi))=f(\varphi) \qquad (\varphi\in V)$$

has at most one solution $v \in V$.

Theorem 43. *Conjecture 42 is false. There are counterexamples for all $c_p, c_t > 0$.*

Proof. Let $c_p, c_t > 0$. In consequence of Theorem 39 there is an SST space V that is a counterexample to Conjecture 38. We prove that V is also a counterexample to Conjecture 42. Let $N_w > c_t^{-3}$, $N_f \geqslant 0$, and $h_0 > 0$. As V is a counterexample to Conjecture 38, there are $w \in V$ with $\|w\| = N_w$, $f \in V'$ with $\|f\|_{V'} = N_f$, $0 < h \leqslant \frac{1}{2}h_0$, and two different solutions $v^0, v^1 \in V$ to the implicit Euler scheme 5.1 with step size h and data w, f. Now $2v^0 - w$ and $2v^1 - w$ are two different solutions to the product Crank-Nicolson scheme 3.8 in superscript-free notation with step size $2h \leqslant h_0$ and data w, f as the following calculation with $j \in \{0, 1\}$ shows:

$$\begin{aligned}
&\frac{1}{2h}\left(\left(2v^j-w\right)-w,\varphi\right)+\frac{1}{4}\left(w+\left(2v^j-w\right),w+\left(2v^j-w\right),\varphi\right)\\
&\qquad+\frac{1}{2}\left(\left(w+\left(2v^j-w\right),\varphi\right)\right)\\
&=\frac{1}{h}\left(v^j-w,\varphi\right)+\left(v^j,v^j,\varphi\right)+\left(\left(v^j,\varphi\right)\right)=f(\varphi).
\end{aligned}$$

The existence of two different solutions contradicts uniqueness. □

Next, we transfer Theorem 39 to the sum Crank-Nicolson scheme:

Conjecture 44. *Let V be an SST space with $c_t > 0$. Then there are numbers $N_w > c_t^{-3}$ and $h_0 > 0$ such that for every $w \in V$ with $\|w\| = N_w$, $f \in V'$ with $\|f\|_{V'} \leqslant \frac{1}{2}c_t^3 N_w^2 + \frac{1}{2}N_w$, and $0 < h \leqslant h_0$, the sum Crank-Nicolson scheme 3.6 in superscript-free notation*

$$\frac{1}{h}(v-w,\varphi)+\frac{1}{2}(w,w,\varphi)+\frac{1}{2}(v,v,\varphi)+\frac{1}{2}((w+v,\varphi))=f(\varphi) \qquad (\varphi\in V)$$

has at most one solution $v \in V$.

Theorem 45. *Conjecture 44 is false. There are counterexamples for all $c_p, c_t > 0$.*

Proof. Let $c_p, c_t > 0$. By Theorem 39 there is an SST space V that is a counterexample to Conjecture 38. We prove that V is also a counterexample to Conjecture 44. Let $N_w > c_t^{-3}$ and $h_0 > 0$. As V is a counterexample to Conjecture 38, there are $w \in V$ with $\|w\| = N_w$, $f \in V'$ with $\|f\|_{V'} = 0$, $0 < h \leqslant \frac{1}{2}h_0$, and two different solutions $v^0, v^1 \in V$ to the implicit Euler scheme 5.1 with step size h and data w, f. The linear functional

$$g : V \to \mathbb{R}, \varphi \mapsto \frac{1}{2}(w, w, \varphi) + \frac{1}{2}((w, \varphi))$$

is in V' with

$$\|g\|_{V'} \leqslant \frac{1}{2}c_t^3 N_w^2 + \frac{1}{2}N_w$$

because of the continuity of the trilinear form and the Cauchy-Schwarz-Bunyakovsky inequality. As the following calculation with $j \in \{0, 1\}$ shows, v^0 and v^1 are two different solutions to the sum Crank-Nicolson scheme 3.6 in superscript-free notation with step size $2h \leqslant h_0$ and data w, g:

$$\begin{aligned}&\frac{1}{2h}\left(v^j - w, \varphi\right) + \frac{1}{2}(w, w, \varphi) + \frac{1}{2}\left(v^j, v^j, \varphi\right) + \frac{1}{2}\left(\left(w + v^j, \varphi\right)\right)\\ &= \frac{1}{2}f(\varphi) + \frac{1}{2}(w, w, \varphi) + \frac{1}{2}((w, \varphi)) = g(\varphi).\end{aligned}$$

The existence of two different solutions contradicts uniqueness. □

Finally we transfer Theorem 39 to the middle time step of the fractional step theta scheme as well:

Conjecture 46. *Let V be an SST space with $c_t > 0$, let $0 < \vartheta < \frac{1}{2}$, and $\frac{1}{2} < \alpha < 1$. Then there are numbers $N_w > (1 - \alpha)c_t^{-3}$ and $h_0 > 0$ such that for every $w \in V$ with $\|w\| = N_w$, $f \in V'$ with $\|f\|_{V'} = \alpha N_w$, and $0 < h \leqslant h_0$, the middle time step 3.10 of the fractional step theta scheme in superscript-free notation*

$$\frac{1}{(1 - 2\vartheta)h}(v - w, \varphi) + (v, v, \varphi) + ((\alpha w + (1 - \alpha)v, \varphi)) = f(\varphi) \qquad (\varphi \in V)$$

has at most one solution $v \in V$.

Theorem 47. *Conjecture 46 is false. There are counterexamples for all $c_p, c_t > 0$, $0 < \vartheta < \frac{1}{2}$, and $\frac{1}{2} < \alpha < 1$.*

Proof. Let $c_p, c_t > 0$, $0 < \vartheta < \frac{1}{2}$, and $\frac{1}{2} < \alpha < 1$. Due to Theorem 39 there is an SST space V that is a counterexample to Conjecture 38. We prove that V is also a counterexample to Conjecture 46. Let $N_w > (1 - \alpha)c_t^{-3}$ and $h_0 > 0$. As V is a counterexample to Conjecture 38 and because of $(1 - \alpha)^{-1} N_w > c_t^{-3}$, there are $w \in V$ with $\|w\| = (1 - \alpha)^{-1} N_w$, $f \in V'$ with $\|f\|_{V'} = 0$, $0 < h \leqslant (1 - \alpha)(1 - 2\vartheta)h_0$, and two different solutions $v^0, v^1 \in V$ to the implicit Euler scheme 5.1 with step size h and data w, f. The linear functional

$$g : V \to \mathbb{R}, \varphi \mapsto \alpha\,(1-\alpha)\,((w,\varphi))$$

is in V' with

$$\|g\|_{V'} = \alpha\,(1-\alpha)\,\|w\| = \alpha N_w.$$

The calculation

$$\begin{aligned}
&\frac{(1-\alpha)\,(1-2\vartheta)}{(1-2\vartheta)\,h}\Big((1-\alpha)\,v^j - (1-\alpha)\,w, \varphi\Big) + \Big((1-\alpha)\,v^j, (1-\alpha)\,v^j, \varphi\Big) \\
&\quad + \Big(\Big(\alpha\,(1-\alpha)\,w + (1-\alpha)^2\,v^j, \varphi\Big)\Big) \\
&= (1-\alpha)^2\left[\frac{1}{h}\Big(v^j - w, \varphi\Big) + \Big(v^j, v^j, \varphi\Big) + \Big(\Big(v^j, \varphi\Big)\Big)\right] + \alpha\,(1-\alpha)\,((w,\varphi)) \\
&= (1-\alpha)^2\,f\,(\varphi) + \alpha\,(1-\alpha)\,((w,\varphi)) = g\,(\varphi)
\end{aligned}$$

valid for both $j \in \{0,1\}$ shows that $(1-\alpha)\,v^0$ and $(1-\alpha)\,v^1$ are two different solutions to the middle time step 3.10 of the fractional step theta scheme in superscript-free notation with step size $(1-\alpha)^{-1}\,(1-2\vartheta)^{-1}\,h \leqslant h_0$ and data $(1-\alpha)\,w$ and g where $\|(1-\alpha)\,w\| = N_w$ and $\|g\|_{V'} = \alpha N_w$. The existence of two different solutions contradicts uniqueness. $\square$

6. Stability-related results

In the following definition, with “difference scheme” we mean the difference schemes defined in chapter 3.

Definition 48 (V-stability)**.** Let V be an SST space. A difference scheme is said to be V-stable if for every $S > 0$ there is $D > 0$ such that for every step size $h > 0$, every index $n \in \mathbb{N}_{\geqslant 1}$, every initial value $v^0 \in V$, and every sequence of evaluations of the force $f^1, f^2, \ldots, f^n \in V'$ (resp. $f^\vartheta, f^{1-\vartheta}, f^1, f^{1+\vartheta}, \ldots, f^{n-\vartheta}, f^n \in V'$ in the case of the fractional step theta scheme 3.9-3.11) satisfying

$$\left|v^0\right|^2 + \sum_{k=1}^{n} h \left\|f^k\right\|_{V'}^2 \leqslant D$$

(resp. $|v^0| + \sum_{k=1}^{n} \vartheta h \left\|f^{k-1+\vartheta}\right\|_{V'} + (1-2\vartheta)\, h \left\|f^{k-\vartheta}\right\|_{V'} + \vartheta h \left\|f^k\right\|_{V'} \leqslant D$), every sequence of solutions to the scheme $v^1, v^2, \ldots, v^n \in V$ (resp. $v^\vartheta, v^{1-\vartheta}, v^1, v^{1+\vartheta}, \ldots, v^{n-\vartheta}, v^n \in V$) satisfies

$$|v^n|^2 \leqslant S.$$

The following sec. 6.1 treats difference schemes that are V-stable for every SST space V, whereas sec. 6.2 treats schemes that lack this property.

6.1. Stable schemes

The stability results in this section are known for the special case of the Navier-Stokes SST space [Tem84, page 336ff, Lemma 5.1 and 5.2]. It is shown in this section that their proofs apply in general SST spaces as well.

6.1.1. The almost implicit and implicit Euler schemes

Theorem 49. *Let V be an SST space. Then for every initial value $v^0 \in V$, every step size $h > 0$, every $n \in \mathbb{N}_{\geqslant 1}$, every sequence of evaluations of the force $f^1, \ldots, f^n \in V'$, and every sequence $v^1, \ldots, v^n \in V$ that solves the almost implicit Euler scheme 3.4 (resp. the implicit Euler scheme 3.5), the a priori estimate*

$$|v^n|^2 + \sum_{k=1}^{n} \left|v^k - v^{k-1}\right|^2 + \sum_{k=1}^{n} h \left\|v^k\right\|^2 \leqslant \left|v^0\right|^2 + \sum_{k=1}^{n} h \left\|f^k\right\|_{V'}^2 .$$

holds. In particular the almost implicit and the implicit Euler schemes are V-stable for every SST space V.

Proof. Suppose the assumptions of the theorem are satisfied. Then for every $k \in \{1,\ldots,n\}$, using $\varphi = 2hv^k$ as a test function, the orthogonality relation yields

$$\left(v^k - v^{k-1}, 2v^k\right) + 2h\left(\left(v^k, v^k\right)\right) = 2hf^k\left(v^k\right)$$

for the almost implicit Euler scheme as well as for the implicit Euler scheme. The identity $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ and Young's inequality A.4 then yield

$$\begin{aligned}
&\left|v^k\right|^2 - \left|v^{k-1}\right|^2 + \left|v^k - v^{k-1}\right|^2 + 2h\left\|v^k\right\|^2 \\
&= 2hf^k\left(v^k\right) \leqslant 2h\left\|f^k\right\|_{V'}\left\|v^k\right\| = 2\left(h^{\frac{1}{2}}\left\|f^k\right\|_{V'}\right)\left(h^{\frac{1}{2}}\left\|v^k\right\|\right) \\
&\leqslant h\left\|f^k\right\|_{V'}^2 + h\left\|v^k\right\|^2
\end{aligned}$$

which simplifies to

$$\left|v^k\right|^2 - \left|v^{k-1}\right|^2 + \left|v^k - v^{k-1}\right|^2 + h\left\|v^k\right\|^2 \leqslant h\left\|f^k\right\|_{V'}^2.$$

When adding these inequalities for all $k \in \{1,\ldots,n\}$, telescopic canceling yields the claimed a priori estimate. □

6.1.2. The linear and product Crank-Nicolson schemes

Theorem 50. *Let V be an SST space. Then for every initial value $v^0 \in V$, every step size $h > 0$, every $n \in \mathbb{N}_{\geqslant 1}$, every sequence of evaluations of the force $f^1,\ldots,f^n \in V'$, and every sequence $v^1,\ldots,v^n \in V$ that solves the linear Crank-Nicolson scheme 3.7 (resp. the product Crank-Nicolson scheme 3.8), the a priori estimate*

$$|v^n|^2 + \frac{1}{4}\sum_{k=1}^{n} h\left\|v^{k-1} + v^k\right\|^2 \leqslant \left|v^0\right|^2 + \sum_{k=1}^{n} h\left\|f^k\right\|_{V'}^2.$$

holds. In particular the linear and the product Crank-Nicolson schemes are V-stable for every SST space V.

Proof. Let the assumptions of the theorem be satisfied. Using $\varphi = h\left(v^{k-1} + v^k\right)$ as a test function in the k-th time step, the orthogonality relation yields

$$\left(v^k - v^{k-1}, v^{k-1} + v^k\right) + \frac{1}{2}h\left(\left(v^{k-1} + v^k, v^{k-1} + v^k\right)\right) = hf^k\left(v^{k-1} + v^k\right)$$

for the linear Crank-Nicolson scheme as well as for the product Crank-Nicolson scheme. The identity $(a+b, a-b) = |a|^2 - |b|^2$ and Young's inequality A.4 then yield

$$\begin{aligned}
&\left|v^k\right|^2 - \left|v^{k-1}\right|^2 + \frac{1}{2}h\left\|v^{k-1}+v^k\right\|^2 \\
&= hf^k\left(v^{k-1}+v^k\right) \leqslant h\left\|f^k\right\|_{V'}\left\|v^{k-1}+v^k\right\| = 2\left(h^{\frac{1}{2}}\left\|f^k\right\|_{V'}\right)\left(\frac{1}{2}h^{\frac{1}{2}}\left\|v^{k-1}+v^k\right\|\right) \\
&\leqslant h\left\|f^k\right\|_{V'}^2 + \frac{1}{4}h\left\|v^{k-1}+v^k\right\|^2
\end{aligned}$$

which simplifies to

$$\left|v^k\right|^2 - \left|v^{k-1}\right|^2 + \frac{1}{4}h\left\|v^{k-1}+v^k\right\|^2 \leqslant h\left\|f^k\right\|_{V'}^2.$$

When adding these inequalities for all $k \in \{1, \ldots, n\}$, telescopic canceling yields the claimed a priori estimate. ☐

6.2. Non-stable schemes

It is proven in this section that the almost explicit Euler, the sum Crank-Nicolson, and the fractional step theta schemes are not V-stable for a family of weighted SST spaces on $\mathbb{Z}\setminus\{0\}$. Of course, this does not necessarily prevent the said schemes from beeing V-stable for the special case that V be the Navier-Stokes SST space.[1] But it shows that a proof of V-stability for the Navier-Stokes SST space, if such exists, would have to rely on properties that are not available in general SST spaces.

6.2.1. The almost explicit Euler scheme

Theorem 51. *For every $c_p, c_t > 0$ there is an SST space V such that for every $\varepsilon > 0$, $S \in \mathbb{R}$, and step size $h > 0$ there is an initial value $v^0 \in V$ with*

$$\left|v^0\right| \leqslant \varepsilon$$

and a solution v^1 to the first step of the almost explicit Euler scheme 3.3 with force $f^1 = 0$ such that

$$\left|v^1\right| \geqslant S.$$

In particular the almost explicit Euler scheme is not V-stable.

[1] See Definition 15 for the Navier-Stokes SST space.

Proof. Let $c_p, c_t > 0$. Consider the weighted SST space on $\mathbb{Z}\setminus\{0\}$ from Lemma 37 with weights $\delta_n = n, \delta_{-n} = \lambda_n = \lambda_{-n} = 1$ ($n \in \mathbb{N}_{\geqslant 1}$), i. e.

$$V = \left\{ v : \mathbb{Z}\setminus\{0\} \to \mathbb{R}; \sum_{n\geqslant 1} \left(|nv_n|^2 + |v_{-n}|^2 \right) < \infty \right\}$$

with

$$((u, v)) = \sum_{n\geqslant 1} \left(n^2 u_n v_n + u_{-n} v_{-n} \right),$$

$$(u, v) = c_p^2 \sum_{i\neq 0} u_i v_i,$$

and

$$(u, v, w) = c_t^3 \sum_{n\geqslant 1} n^2 u_n \left(v_n w_{-n} - v_{-n} w_n \right)$$

for all $u, v, w \in V$. Let $\varepsilon > 0$, $S \in \mathbb{R}$, and $h > 0$. There is $m \in \mathbb{N}_{\geqslant 1}$ with

$$\left(\frac{1}{h} c_p^2 + 1 \right)^{-1} c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S.$$

As initial value choose $v^0 \in V$ defined by

$$v_i^0 = \delta_{im} c_p^{-1} \varepsilon \qquad (i \in \mathbb{Z}\setminus\{0\})$$

where δ_{im} denotes the Kronecker delta, not the weight δ. The chosen initial value satisfies

$$\left| v^0 \right| = \varepsilon.$$

Now define $v^1 \in V$ by setting

$$v_i^1 = \delta_{im} \left(\frac{1}{h} c_p^2 + m^2 \right)^{-1} \frac{1}{h} c_p \varepsilon - \delta_{i,-m} \left(\frac{1}{h} c_p^2 + 1 \right)^{-1} c_p^{-2} c_t^3 m^2 \varepsilon^2 \qquad (i \in \mathbb{Z}\setminus\{0\}).$$

The calculation

$$\begin{aligned}
&\frac{1}{h} \left(v^1 - v^0, \varphi \right) + \left(v^0, v^0, \varphi \right) + \left(\left(v^1, \varphi \right) \right) \\
&= \left(\frac{1}{h} c_p^2 \left(v_m^1 - c_p^{-1} \varepsilon \right) + 0 + m^2 v_m^1 \right) \varphi_m \\
&\qquad + \left(\frac{1}{h} c_p^2 \left(v_{-m}^1 - 0 \right) + c_t^3 m^2 c_p^{-2} \varepsilon^2 + v_{-m}^1 \right) \varphi_{-m} \\
&= \left(\left(\frac{1}{h} c_p^2 + m^2 \right) v_m^1 - \frac{1}{h} c_p \varepsilon \right) \varphi_m + \left(\left(\frac{1}{h} c_p^2 + 1 \right) v_{-m}^1 + c_p^{-2} c_t^3 m^2 \varepsilon^2 \right) \varphi_{-m} \\
&= 0
\end{aligned}$$

valid for all $\varphi \in V$ shows that v^1 is a solution to the first step of the almost explicit Euler scheme 3.3 with step size $h > 0$, initial value v^0 and force $f^1 = 0$. Furthermore v^1 satisfies

$$\left|v^1\right| \geqslant c_p \left|v^1_{-m}\right| = \left(\frac{1}{h}c_p^2 + 1\right)^{-1} c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S$$

due to the choice of m. □

6.2.2. The sum Crank-Nicolson scheme

Theorem 52. *For every $c_p, c_t > 0$ there is an SST space V such that for every $\varepsilon > 0$, $S \in \mathbb{R}$, and step size $h > 0$ there is an initial value $v^0 \in V$ with*

$$\left|v^0\right| \leqslant \varepsilon$$

and a solution v^1 to the first step of the sum Crank-Nicolson scheme 3.6 with force $f^1 = 0$ such that

$$\left|v^1\right| \geqslant S.$$

In particular the sum Crank-Nicolson scheme is not V-stable.

Proof. Let $c_p, c_t > 0$. Consider the weighted SST space on $\mathbb{Z}\setminus\{0\}$ from Lemma 37 with weights $\delta_n = n, \delta_{-n} = \lambda_n = \lambda_{-n} = 1$ $(n \in \mathbb{N}_{\geqslant 1})$, i. e. the same SST space as in the proof of Theorem 51. Let $\varepsilon > 0$, $S \in \mathbb{R}$, and $h > 0$. There is $m \in \mathbb{N}_{\geqslant 1}$ with

$$\left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)^{-1} \frac{1}{2} c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S.$$

As initial value choose $v^0 \in V$ defined by

$$v_i^0 = \delta_{im} c_p^{-1} \varepsilon \qquad (i \in \mathbb{Z}\setminus\{0\})$$

where δ_{im} denotes the Kronecker delta, not the weight δ. The chosen initial value satisfies

$$\left|v^0\right| = \varepsilon.$$

There is a number $\pi \in \mathbb{R}$ such that

$$\frac{1}{h}c_p^2\left(\pi - v_m^0\right) + \frac{1}{2}c_t^3 m^2 \pi \left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)^{-1} \frac{1}{2} c_t^3 m^2 \left(\left(v_m^0\right)^2 + \pi^2\right) + \frac{1}{2} m^2 \left(v_m^0 + \pi\right) = 0$$

because the expression is a third order polynomial in π. The element $v^1 \in V$ defined by

$$v_i^1 = \delta_{im}\pi - \delta_{i,-m} \left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)^{-1} \frac{1}{2} c_t^3 m^2 \left(\left(v_m^0\right)^2 + \pi^2\right)$$

is a solution to the first step of the sum Crank-Nicolson scheme 3.6 with step size h, initial value v^0 and force $f^1 = 0$ because for every $\varphi \in V$

$$\begin{aligned}
&\frac{1}{h}\left(v^1 - v^0, \varphi\right) + \frac{1}{2}\left(v^0, v^0, \varphi\right) + \frac{1}{2}\left(v^1, v^1, \varphi\right) + \frac{1}{2}\left(\left(v^0 + v^1, \varphi\right)\right) \\
&= \left(\frac{1}{h}c_p^2\left(\pi - v_m^0\right) + \frac{1}{2}c_t^3 m^2 \pi \left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)^{-1} \frac{1}{2}c_t^3 m^2 \left(\left(v_m^0\right)^2 + \pi^2\right)\right. \\
&\qquad\qquad \left. + \frac{1}{2}m^2\left(v_m^0 + \pi\right)\right)\varphi_m \\
&\qquad + \left(\frac{1}{h}c_p^2\left(v_{-m}^1 - 0\right) + \frac{1}{2}c_t^3 m^2\left(\left(v_m^0\right)^2 + \pi^2\right) + \frac{1}{2}\left(0 + v_{-m}^1\right)\right)\varphi_{-m} \\
&= \left(\left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)v_{-m}^1 + \frac{1}{2}c_t^3 m^2\left(\left(v_m^0\right)^2 + \pi^2\right)\right)\varphi_{-m} = 0.
\end{aligned}$$

The solution $v^1 \in V$ satisfies

$$\begin{aligned}
\left|v^1\right| &= c_p\left(\left(v_m^1\right)^2 + \left(v_{-m}^1\right)^2\right)^{\frac{1}{2}} \geqslant c_p\left|v_{-m}^1\right| \\
&= c_p\left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)^{-1}\frac{1}{2}c_t^3 m^2\left(c_p^{-2}\varepsilon^2 + \pi^2\right) \\
&\geqslant \left(\frac{1}{h}c_p^2 + \frac{1}{2}\right)^{-1}\frac{1}{2}c_p^{-1}c_t^3 m^2 \varepsilon^2 \geqslant S.
\end{aligned}$$

□

6.2.3. The fractional step theta scheme

Theorem 53. *For every $c_p, c_t > 0$ there is an SST space V such that for every pair of parameters $0 < \vartheta < \frac{1}{2}$, $\frac{1}{2} < \alpha < 1$, every $\varepsilon > 0$, $S \in \mathbb{R}$, and step size*

$$0 < h \leqslant c_p^2,$$

there is an initial value $v^0 \in V$ with

$$\left|v^0\right| \leqslant \varepsilon$$

and there are solutions v^ϑ, $v^{1-\vartheta}$, and v^1 to the first, the middle, and the last substep 3.9, 3.10, and 3.11 of the first step of the fractional step theta scheme with force $f^\vartheta = f^{1-\vartheta} = f^1 = 0$ such that

$$\left|v^\vartheta\right| \geqslant S, \qquad \left|v^{1-\vartheta}\right| \geqslant S, \qquad \left|v^1\right| \geqslant S.$$

In particular the fractional step theta scheme is not V-stable.

Proof. Let $c_p, c_t > 0$. Consider the weighted SST space on $\mathbb{Z} \setminus \{0\}$ from Lemma 37 with weights $\delta_n = n, \delta_{-n} = \lambda_n = \lambda_{-n} = 1$ ($n \in \mathbb{N}_{\geqslant 1}$), i. e. the same SST space as in the proofs of Theorem 51 and Theorem 52. Let $0 < \vartheta < \frac{1}{2}$, $\frac{1}{2} < \alpha < 1$, $\varepsilon > 0$, $S \in \mathbb{R}$, and $0 < h \leqslant c_p^2$. The numbers

$$\sigma_+ := \left(\frac{1}{\vartheta h} c_p^2 + \alpha\right)^{-1}$$

and

$$\tau_+ := \left(\frac{1}{(1-2\vartheta)\, h} c_p^2 + (1-\alpha)\right)^{-1}$$

are positive. The number

$$\tau_- := \frac{1}{(1-2\vartheta)\, h} c_p^2 - \alpha$$

is positive due to the assumptions $h \leqslant c_p^2$, $\vartheta > 0$, and $\alpha < 1$. The number

$$\sigma_- := \frac{1}{\vartheta h} c_p^2 - (1-\alpha)$$

is larger that $\frac{3}{2}$ due to the assumptions $h \leqslant c_p^2$, $\vartheta < \frac{1}{2}$, and $\alpha > \frac{1}{2}$. Hence there is $m \in \mathbb{N}_{\geqslant 1}$ such that

$$\begin{aligned}
&\sigma_+ c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S, \\
&\tau_+ \tau_- \sigma_+ c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S, \text{ and} \\
&\sigma_+^2 \sigma_- \tau_+ \tau_- c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S.
\end{aligned}$$

As initial value choose $v^0 \in V$ defined by

$$v_i^0 = \delta_{im} c_p^{-1} \varepsilon \qquad (i \in \mathbb{Z} \setminus \{0\})$$

where δ_{im} denotes the Kronecker delta, not the weight δ. The chosen initial value satisfies

$$\left| v^0 \right| = \varepsilon.$$

Now first define $v^\vartheta \in V$ by setting

$$v_i^\vartheta = \delta_{im} \left(\frac{1}{\vartheta h} c_p^2 + \alpha m^2\right)^{-1} \left(\frac{1}{\vartheta h} c_p^2 - (1-\alpha)\, m^2\right) v_m^0 - \delta_{i,-m} \sigma_+ c_p^{-2} c_t^3 m^2 \varepsilon^2.$$

The fact that

$$v_{-m}^\vartheta < 0 \tag{6.1}$$

will be used later. As for every $\varphi \in V$

$$\begin{aligned}
&\frac{1}{\vartheta h}\left(v^{\vartheta}-v^{0},\varphi\right)+\left(v^{0},v^{0},\varphi\right)+\left(\left(\left(1-\alpha\right)v^{0}+\alpha v^{\vartheta},\varphi\right)\right)\\
&=\left(\frac{1}{\vartheta h}c_p^2\left(v_m^{\vartheta}-v_m^{0}\right)+0+\left(1-\alpha\right)m^2v_m^0+\alpha m^2 v_m^{\vartheta}\right)\varphi_m\\
&\quad+\left(\frac{1}{\vartheta h}c_p^2\left(v_{-m}^{\vartheta}-0\right)+c_t^3m^2\left(v_m^0\right)^2+0+\alpha v_{-m}^{\vartheta}\right)\varphi_{-m}\\
&=\left(\left(\frac{1}{\vartheta h}c_p^2+\alpha m^2\right)v_m^{\vartheta}-\left(\frac{1}{\vartheta h}c_p^2-\left(1-\alpha\right)m^2\right)v_m^0\right)\varphi_m\\
&\quad+\Bigg(\underbrace{\left(\frac{1}{\vartheta h}c_p^2+\alpha\right)}_{=\sigma_+^{-1}}v_{-m}^{\vartheta}+c_p^{-2}c_t^3m^2\varepsilon^2\Bigg)\varphi_{-m}\\
&=0,
\end{aligned}$$

v^{ϑ} is a solution to the first substep 3.9 of the first step of the fractional step theta scheme with initial value v^0 and force $f^{\vartheta}=0$. The norm of v^{ϑ} satisfies

$$\left|v^{\vartheta}\right|=c_p\left(\left(v_m^{\vartheta}\right)^2+\left(v_{-m}^{\vartheta}\right)^2\right)^{\frac{1}{2}}\geqslant c_p\left|v_{-m}^{\vartheta}\right|=\sigma_+c_p^{-1}c_t^3m^2\varepsilon^2\geqslant S.$$

There is a number $\pi\in\mathbb{R}$ such that

$$\frac{1}{\left(1-2\vartheta\right)h}c_p^2\left(\pi-v_m^{\vartheta}\right)-c_t^3m^2\pi\tau_+\left(\tau_-v_{-m}^{\vartheta}-c_t^3m^2\pi^2\right)+\alpha m^2v_m^{\vartheta}+\left(1-\alpha\right)m^2\pi=0$$

because the expression is a third order polynomial in π. Now define $v^{1-\vartheta}\in V$ by setting

$$v_i^{1-\vartheta}=\delta_{im}\pi+\delta_{i,-m}\tau_+\left(\tau_-v_{-m}^{\vartheta}-c_t^3m^2\pi^2\right)\qquad\left(i\in\mathbb{Z}\setminus\{0\}\right).$$

The fact that

$$v_{-m}^{1-\vartheta}<0 \tag{6.2}$$

is a consequence of 6.1 and will be used later. Inequality 6.1 is also used in the first line of the estimation

$$\begin{aligned}
\left|v_{-m}^{1-\vartheta}\right|&=\tau_+\left(\tau_-\left|v_{-m}^{\vartheta}\right|+c_t^3m^2\pi^2\right)\geqslant\tau_+\tau_-\left|v_{-m}^{\vartheta}\right|\\
&=\tau_+\tau_-\sigma_+c_p^{-2}c_t^3m^2\varepsilon^2.
\end{aligned}\tag{6.3}$$

Since for every $\varphi \in V$

$$\begin{aligned}
&\frac{1}{(1-2\vartheta)h}\left(v^{1-\vartheta}-v^{\vartheta},\varphi\right)+\left(v^{1-\vartheta},v^{1-\vartheta},\varphi\right)+\left(\left(\alpha v^{\vartheta}+(1-\alpha)\,v^{1-\vartheta},\varphi\right)\right)\\
&=\underbrace{\left(\frac{1}{(1-2\vartheta)h}c_p^2\left(\pi-v_m^{\vartheta}\right)-c_t^3m^2\pi v_{-m}^{1-\vartheta}+\alpha m^2v_m^{\vartheta}+(1-\alpha)\,m^2\pi\right)}_{=0\text{ due to the choice of }\pi}\varphi_m\\
&\quad+\left(\frac{1}{(1-2\vartheta)h}c_p^2\left(v_{-m}^{1-\vartheta}-v_{-m}^{\vartheta}\right)+c_t^3m^2\pi^2+\alpha v_{-m}^{\vartheta}+(1-\alpha)\,v_{-m}^{1-\vartheta}\right)\varphi_{-m}\\
&=\left(\left(\frac{1}{(1-2\vartheta)h}c_p^2+(1-\alpha)\right)v_{-m}^{1-\vartheta}-\left(\frac{1}{(1-2\vartheta)h}c_p^2-\alpha\right)v_{-m}^{\vartheta}+c_t^3m^2\pi^2\right)\varphi_{-m}\\
&=\left(\tau_+^{-1}v_{-m}^{1-\vartheta}-\tau_-v_{-m}^{\vartheta}+c_t^3m^2\pi^2\right)\varphi_{-m}\\
&=0,
\end{aligned}$$

$v^{1-\vartheta}$ is a solution of the middle substep 3.10 of the first step of the fractional step theta scheme with previous approximation v^{ϑ} and force $f^{1-\vartheta}=0$. The norm of $v^{1-\vartheta}$ satisfies

$$\begin{aligned}
\left|v^{1-\vartheta}\right|&=c_p\left(\left(v_m^{1-\vartheta}\right)^2+\left(v_{-m}^{1-\vartheta}\right)^2\right)^{\frac{1}{2}}\geqslant c_p\left|v_{-m}^{1-\vartheta}\right|\\
&\geqslant\tau_+\tau_-\sigma_+c_p^{-1}c_t^3m^2\varepsilon^2\geqslant S,
\end{aligned}$$

where 6.3 was used in the second line. Finally define $v^1\in V$ by setting

$$\begin{aligned}
v_i^1&=\delta_{im}\left(\frac{1}{\vartheta h}c_p^2+\alpha m^2\right)^{-1}\left(\frac{1}{\vartheta h}c_p^2-(1-\alpha)\,m^2+c_t^3m^2v_{-m}^{1-\vartheta}\right)v_m^{1-\vartheta}\\
&\quad+\delta_{i,-m}\sigma_+\left(\sigma_-v_{-m}^{1-\vartheta}-c_t^3m^2\left(v_m^{1-\vartheta}\right)^2\right)\qquad(i\in\mathbb{Z}\setminus\{0\})\,.
\end{aligned}$$

Then v^1 is a solution of the last substep 3.11 of the first step of the fractional step theta scheme with previous approximation $v^{1-\vartheta}$ and force $f^1=0$ because for every $\varphi\in V$

$$\begin{aligned}
&\frac{1}{\vartheta h}\left(v^1-v^{1-\vartheta},\varphi\right)+\left(v^{1-\vartheta},v^{1-\vartheta},\varphi\right)+\left(\left((1-\alpha)\,v^{1-\vartheta}+\alpha v^1,\varphi\right)\right)\\
&=\left(\frac{1}{\vartheta h}c_p^2\left(v_m^1-v_m^{1-\vartheta}\right)-c_t^3m^2v_m^{1-\vartheta}v_{-m}^{1-\vartheta}+(1-\alpha)\,m^2v_m^{1-\vartheta}+\alpha m^2v_m^1\right)\varphi_m\\
&\quad+\left(\frac{1}{\vartheta h}c_p^2\left(v_{-m}^1-v_{-m}^{1-\vartheta}\right)+c_t^3m^2\left(v_m^{1-\vartheta}\right)^2+(1-\alpha)\,v_{-m}^{1-\vartheta}+\alpha v_{-m}^1\right)\varphi_{-m}\\
&=\left(\left(\frac{1}{\vartheta h}c_p^2+\alpha m^2\right)v_m^1-\left(\frac{1}{\vartheta h}c_p^2-(1-\alpha)\,m^2+c_t^3m^2v_{-m}^{1-\vartheta}\right)v_m^{1-\vartheta}\right)\varphi_m\\
&\quad+\left(\underbrace{\left(\frac{1}{\vartheta h}c_p^2+\alpha\right)}_{=\sigma_+^{-1}}v_{-m}^1-\underbrace{\left(\frac{1}{\vartheta h}c_p^2-(1-\alpha)\right)}_{=\sigma_-}v_{-m}^{1-\vartheta}+c_t^3m^2\left(v_m^{1-\vartheta}\right)^2\right)\varphi_{-m}\\
&=0.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\left|v^1\right| &= c_p \left(\left(v_m^1\right)^2 + \left(v_{-m}^1\right)^2 \right)^{\frac{1}{2}} \geqslant c_p \left|v_{-m}^1\right| \\
&= c_p \sigma_+ \left(\sigma_- \left|v_{-m}^{1-\vartheta}\right| + c_t^3 m^2 \left(v_m^{1-\vartheta}\right)^2 \right) \geqslant c_p \sigma_+ \sigma_- \left|v_{-m}^{1-\vartheta}\right| \\
&\geqslant c_p \sigma_+ \sigma_- \tau_+ \tau_- \sigma_+ c_p^{-2} c_t^3 m^2 \varepsilon^2 \\
&= \sigma_+^2 \sigma_- \tau_+ \tau_- c_p^{-1} c_t^3 m^2 \varepsilon^2 \geqslant S,
\end{aligned}$$

where 6.2 was used in the second line and 6.3 in the third line. □

7. Convergence results

The convergence proofs for the almost implicit Euler and the product Crank-Nicolson schemes in this chapter are valid for a class of SST spaces called SST_∞ spaces defined in Definition 54 below. The additionally required properties are related to Bochner integrability and estimates involving a further norm $|\cdot|_\infty$ that corresponds to the L^∞ norm in the case of the Navier-Stokes SST space.[1]

Definition 54 (SST_∞ space)**.** Let V be an SST space that is a Hilbert space with respect to the scalar product $((\cdot,\cdot))$. Let L^∞ be a vector space over $\mathbb{R}$ such that the vector space operations of V and L^∞ coincide on the non-empty intersection $V \cap L^\infty$. Let $|\cdot|_\infty$ be a norm on L^∞ that makes L^∞ a Banach space. Suppose that there are numbers c_∞, $c_{\infty VH}$, and $c_{HV\infty}$ such that for all $u \in V \cap L^\infty$ and $v, w \in V$

$$
\begin{aligned}
&|u| \leqslant c_\infty |u|_\infty , \\
&|(u, v, w)| \leqslant c_{\infty VH} |u|_\infty \|v\| |w| , \text{ and} \\
&|(w, v, u)| \leqslant c_{HV\infty} |w| \|v\| |u|_\infty .
\end{aligned}
$$

Then $(V, ((\cdot,\cdot)), (\cdot,\cdot), (\cdot,\cdot,\cdot), L^\infty, |\cdot|_\infty)$ is called an SST_∞ space. We will also use the short notation (V, L^∞).

The following theorem is the reason why the convergence theorems 60 and 65 are valid for the Navier-Stokes SST space in particular.

Theorem 55. *The Navier-Stokes SST space from Definition 15 and the space $L^\infty(G)^3$ of essentially bounded vector fields equipped with the norm*

$$
|u|_\infty := |u|_{L^\infty} = \operatorname*{ess\,sup}_{x \in G} \max_j |u_j(x)| \qquad (u \in C)
$$

form an SST_∞ space.

Proof. It has been shown in sec. 2.2 that the Navier-Stokes SST space is an SST space and in Theorem 20 that $(V, ((\cdot,\cdot)))$ is a Hilbert space. As V and $L^\infty(G)^3$ are function spaces on the same domain G, the vector space operations coincide on their intersection. The identically zero function lies in both spaces. The norm $|\cdot|_{L^\infty}$

[1]For Bochner integrability see sec. A.2.

makes $L^\infty(G)^3$ a Banach space, see for example [AF03, page 29, Theorem 2.16]. Let $u \in V \cap L^\infty(G)^3$ and $v, w \in V$. The estimation

$$|u|^2 = \sum_j \int_G |u_j|^2 \leqslant \sum_j \int_G |u|_\infty^2 = 3\lambda(G)\,|u|_\infty^2$$

shows that $|u| \leqslant c_\infty |u|_\infty$ holds with $c_\infty = (3\lambda(G))^{\frac{1}{2}}$ where $\lambda(G)$ denotes the Lebesgue measure of G. Hölder's inequality from Lemma 11 is used with $n = 3$, $p = \infty$, $q = r = 2$, and $s = 1$ for the second inequality of the estimation

$$\begin{aligned} |(u, v, w)| &\leqslant |(u \cdot \nabla v) \cdot w|_{L^1} \\ &\leqslant 3^{\frac{1}{2}} |u|_{L^\infty} |\nabla v|_{L^2} |w|_{L^2} \end{aligned}$$

and with $n = 3$, $p = q = 2$, $r = \infty$, and $s = 1$ for the second inequality of the estimation

$$\begin{aligned} |(u, v, w)| &\leqslant |(u \cdot \nabla v) \cdot w|_{L^1} \\ &\leqslant 3^{\frac{1}{2}} |u|_{L^2} |\nabla v|_{L^2} |w|_{L^\infty} . \end{aligned}$$

This shows that $|(u, v, w)| \leqslant c_{\infty VH} |u|_\infty \|v\| |w|$ and $|(w, v, u)| \leqslant c_{HV\infty} |w| \|v\| |u|_\infty$ hold with $c_{\infty VH} = c_{HV\infty} = 3^{\frac{1}{2}}$. ☐

Definition 56 (SST-generalized nonstationary Navier-Stokes equations)**.** Let V be an SST space, $T > 0$, $u^0 \in V$ and $f : [0, T] \to V'$. Then $u : [0, T] \to V$ is called a solution of the SST-generalized nonstationary Navier-Stokes equations with initial value u^0 and force f if u is differentiable (considered as a V'-valued function via the imbedding $V \hookrightarrow H' \subset V'$, see Definition 5) with respect to the norm $\|\cdot\|_{V'}$, if $u(0) = u^0$, and if the weakly formulated equation

$$u_t(\varphi) + (u, u, \varphi) + ((u, \varphi)) = f(\varphi) \qquad (\varphi \in V) \tag{7.1}$$

holds for every $\tau \in [0, T]$.[2]

Remark 57 (Weak solution)**.** Note that if V is the Navier-Stokes SST space from sec. 2.2, the notion of solution defined above is related to the notion of weak solution of the nonstationary Navier-Stokes equations as defined in [Tem84, page 280, Problem 3.1] as follows: If $u^0 \in V$ and $f \in L^2(0, T, V')$, then $u \in L^2(0, T, V)$ is a solution as defined in Definition 56 if and only if it is a weak solution of the nonstationary Navier-Stokes equations.

In the following definition the term "difference scheme" means the difference schemes defined in chapter 3.

[2]The argument of time-dependent functions is omitted so as not to overload the notation whenever possible. For example, in Definition 56, u_t stands for $u_t(\tau)$ and u for $u(\tau)$ in the weakly formulated equation but of course not in the equation $u(0) = u^0$.

Definition 58 (Superscript notation, approximation error)**.** Let V be an SST space, $T > 0$, and let $u : [0,T] \to V$ be a solution to the SST-generalized nonstationary Navier-Stokes equations with initial value $u^0 \in V$ and force $f : [0,T] \to V'$. Let $N \in \mathbb{N}_{\geqslant 1}$, $h := \frac{T}{N}$, and let $v^0, v^1, \ldots, v^N$ be a sequence of solutions to some difference scheme with initial value $v^0 = u^0$ and some sequence of evaluations of the force $f^1, f^2, \ldots, f^N$. Then for $k \in \{0,1,\ldots,N\}$ the notations

$$u^k := u(kh) \qquad \text{and} \qquad u_t^k := u_t(kh)$$

are used to denote the values of the function u and its derivative u_t at time $\tau = kh$. The corresponding notation is used for higher derivatives $u_{tt}, u_{ttt}, \ldots$ if they exist. For $k \in \{0,1,\ldots,N\}$ the notation

$$t_k := kh$$

is used and

$$e^k := v^k - u^k \in V$$

denotes the approximation error. Note that although the notations v^k and u^k are identical, there is a fundamental difference: v^k is the value of $\{0,1,\ldots,N\} \ni i \mapsto v^i$ at $i = k$ and u^k is the value of $[0,T] \ni \tau \mapsto u(\tau)$ at $\tau = t_k$.

7.1. The almost implicit Euler scheme

This section contains a proof of the first order convergence of the approximations of the almost implicit Euler scheme towards solutions of the SST-generalized nonstationary Navier-Stokes equations provided the solutions are sufficiently regular. More precisely the regularity assumptions on the solution in Theorem 60 below are[3]

$$u \in L^\infty(0,T,L^\infty) \text{ and } u_t \in L^2(0,T,V). \tag{7.2}$$

The following Theorem 59 specifies conditions on the data u^0 and f that ensure the existence of such a solution in the case of the Navier-Stokes SST space.

Theorem 59. *Let V be the Navier-Stokes SST space from sec. 2.2 and suppose $G \subset \mathbb{R}^3$ has a C^4 boundary. Let $u^0 \in V \cap H^2(G)^3$, $T' > 0$, $f \in C([0,T'],H)$, and $f_t \in L^2(0,T',V')$. Then there is some $0 < T \leqslant T'$ such that the nonstationary Navier-Stokes equations with initial value u^0 and force f have one and only one solution $u \in C\left([0,T], V \cap H^2(G)^3\right)$ with $u_t \in L^2(0,T,V) \cap C([0,T],H)$.*[4]

[3]For the spaces $L^p(0,T,G)$ and $C^k([0,T],G)$ see sec. A.2.

[4]Note that $u \in C\left([0,T], V \cap H^2(G)^3\right)$ implies $u \in C\left([0,T], L^\infty(G)^3\right)$ due to the Sobolev imbedding $H^2(G) \hookrightarrow C_B(G)$ (see [AF03, page 85, Theorem 4.12 for $n = k = 3$, $j = 0$, $m = 2$, $p = 2$]) and hence $u \in L^\infty\left(0,T,L^\infty(G)^3\right)$.

Proof. The result is obtained from [Tem82] as follows: Statement (3.8) on page 86 of [Tem82] implies the unique existence of a solution $u \in C([0,T], V)$ for some $0 < T \leqslant T'$. The regularity assumptions of Theorem 3.1 for $m = 2$ on page 86 of [Tem82] are satisfied. The compatibility condition $-\mathcal{A}u^0 - B(u^0, u^0) + f^0 \in H$ (statement (3.11) on page 87 of [Tem82]) holds because $\mathcal{A}u^0 \in H$ due to statement (1.13) on page 75 of [Tem82], because $B(u^0, u^0) \in H \cap H^1(G)^3$ due to statement (3.5) on page 86 of [Tem82], and because $f^0 \in H$. Therefore the sufficiency part of the cited Theorem 3.1 yields $u \in C\left([0,T], V \cap H^2(G)^3\right)$ with $u_t \in C([0,T], H)$. The property $u_t \in L^2(0,T,V)$ is not part of the assertions of the cited Theorem 3.1 but it is established in its proof [Tem82, page 88, statement (3.19)]. □

A proof of the first order convergence of the almost implicit Euler scheme for the nonstationary linear Stokes equations can be found in [Var92, page 44, Theorem 3.2].[5] It is based on energy methods and holds for solutions with regularity

$$u \in C\left([0,T], V \cap H^2(G)^3\right) \text{ and } u_t \in C([0,T], H) \cap L^2\left(0,T,H^1(G)^3\right).$$

The same convergence order is proven for the case of the nonstationary Navier-Stokes equations in [Rau95, page 1083, Theorem 2] with semigroup methods for solutions with regularity

$$u \in C\left([0,T], V \cap H^{2+2\zeta}(G)^3\right)$$

where $\zeta \in \left(0, \frac{1}{4}\right)$ and $f = 0$.[6] When applied to the Navier-Stokes SST space, Theorem 60 below assumes less regularity since already $u \in C\left([0,T], V \cap H^2(G)^3\right)$ is sufficient for its regularity assumptions 7.2 to hold as a consequence of Theorem 59. The first order convergence of the almost explicit Euler scheme for the nonstationary Navier-Stokes equations is also proven in [GR79, page 179, Theorem 2.2] with energy methods under the regularity assumptions

$$u_t \in L^2(0,T,V) \text{ and } u_{tt} \in L^2(0,T,V'). \tag{7.3}$$

The proof makes use of Hölder's inequality with $\frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$, the inequality $|\cdot|_{L^3}^2 \leqslant |\cdot|_{L^6} |\cdot|_{L^2}$, and Sobolev imbeddings for the estimation of the trilinear form whereas the proof of Theorem 60 uses Hölder's inequality with $\frac{1}{2} + \frac{1}{2} + \frac{1}{\infty} = 1$ and $\frac{1}{\infty} + \frac{1}{2} + \frac{1}{2} = 1$ for the estimation of the trilinear form. Every solution of regularity 7.3 satisfies the regularity assumptions 7.2 on the local time interval from Theorem 59 if $f \in C([0,T], H)$ and $f_t \in L^2(0,T,V')$. This can be seen as follows: $u_t \in L^2(0,T,V)$ and $u_{tt} \in L^2(0,T,V')$ imply $u_t \in C([0,T], H)$ [GR79, page 151, Theorem 1.1]. Hence $u_t^0 \in H$. As u^0 is a solution of the stationary Navier-Stokes equations with right hand side $-u_t^0 + f^0 \in H$, Theorem 3.6.1 on page 173 of [Soh01] yields $u^0 \in$

[5] In the linear case, there is in fact no difference between the almost explicit, the almost implicit and the implicit Euler schemes since there is no nonlinear term.

[6] See [AF03, page 250] for fractional order Sobolev spaces H^s with noninteger $s > 0$.

$V \cap H^2(G)^3$. Finally Theorem 59 applies and yields the asserted regularity. When applied to the Navier-Stokes SST space, the following theorem asserts the first order convergence of the almost implicit Euler scheme for the nonstationary Navier-Stokes equations.

Theorem 60 (Convergence of Almost Implicit Euler). *Let (V, L^∞) be an SST_∞ space, $T > 0$, and let $u : [0,T] \to V \cap L^\infty$ be a solution of the SST-generalized nonstationary Navier-Stokes equations 7.1 with initial value $u^0 \in V$ and force $f \in L^1(0,T,V')$. Let $N \in \mathbb{N}_{\geqslant 1}$, $h := \frac{T}{N}$, and let $v^0, v^1, \ldots, v^N$ be a sequence of solutions to the almost implicit Euler scheme 3.4 with initial value $v^0 = u^0$ and evaluation of the force defined by*[7]

$$f^k := \frac{1}{h}\int_{t_{k-1}}^{t_k} f\, d\tau \qquad (k \in \{1,2,\ldots,N\}).$$

Then, if u is of regularity 7.2, the approximation error satisfies

$$\begin{aligned}
&|e^n|^2 + \sum_{k=1}^{n} \left|e^k - e^{k-1}\right|^2 + h\left\|e^k\right\|^2 \\
&\leqslant h^2 \left(4\left(c_{HV\infty}^2 + c_{\infty VH}^2\right)|u|_{L^\infty(0,t_n,L^\infty)}^2 \int_0^{t_n} |u_t|^2\, d\tau + 4\int_0^{t_n} \|u_t\|^2\, d\tau\right) \\
&\qquad \exp\left(4c_{HV\infty}^2 |u|_{L^\infty(0,t_n,L^\infty)}^2\, t_n\right)
\end{aligned}$$

for every $n \in \{1,2,\ldots,N\}$, i. e. the almost implicit Euler scheme is first order convergent.

Proof. The values of all Bochner integrals in this proof don't depend on whether they are interpreted as to be defined with respect to the norm $|\cdot|$ or $\|\cdot\|$ because of Lemma 73 from sec. A.2. For every $\tau \in [0,T]$, due to $u_t \in V$ and the imbedding $V \hookrightarrow H' \subset V'$ from Definition 5, the equation $u_t(\varphi) = (u_t, \varphi)$ holds, where u_t is interpreted as an element of V' on the left and as an element of V on the right hand side of the equation. Let $1 \leqslant k \leqslant N$. Then for every $\varphi \in V$ the fact that v^{k-1}, v^k, and f^k solve the almost implicit Euler scheme 3.4, the fact that u and f solve the Navier-Stokes equations 7.1, and the fundamental theorem of calculus 75 yield

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} \left(\left(u - u^k, \varphi\right)\right) d\tau \\
&= \underbrace{\left(v^k - v^{k-1}, \varphi\right) + h\left(v^{k-1}, v^k, \varphi\right) + h\left(\left(v^k, \varphi\right)\right) - hf^k(\varphi)}_{=0} \\
&\qquad - h\left(\left(u^k, \varphi\right)\right) - \int_{t_{k-1}}^{t_k} \left((u_t, \varphi) + (u,u,\varphi) - f(\varphi)\right) d\tau \\
&= \left(e^k - e^{k-1}, \varphi\right) + \int_{t_{k-1}}^{t_k} \left(\left(v^{k-1}, v^k, \varphi\right) - (u,u,\varphi)\right) d\tau + h\left(\left(e^k, \varphi\right)\right).
\end{aligned}$$

[7] See sec. A.2 for the notion of Bochner integral.

Choosing $\varphi = 2e^k$ as a test function and taking $\left(e^k - e^{k-1}, 2e^k\right) = \left|e^k\right|^2 - \left|e^{k-1}\right|^2 + \left|e^k - e^{k-1}\right|^2$ into account yields

$$\begin{aligned}
&\left|e^k\right|^2 - \left|e^{k-1}\right|^2 + \left|e^k - e^{k-1}\right|^2 + 2h\left\|e^k\right\|^2 \\
&= \underbrace{2\int_{t_{k-1}}^{t_k} \left(u,u,e^k\right) - \left(v^{k-1},v^k,e^k\right) d\tau}_{=:\, I^k_{\text{convective}}} + \underbrace{2\int_{t_{k-1}}^{t_k} \left(\left(u-u^k,e^k\right)\right) d\tau}_{=:\, I^k_{\text{viscous}}} .
\end{aligned}$$

We split

$$\begin{aligned}
I^k_{\text{convective}} = {} & \underbrace{2\int_{t_{k-1}}^{t_k} \left(u-u^{k-1},u,e^k\right) d\tau}_{=:\, I^k_{c1}} + \underbrace{2\int_{t_{k-1}}^{t_k} \left(u^{k-1},u-u^k,e^k\right) d\tau}_{=:\, I^k_{c2}} \\
& + \underbrace{2h\left(u^{k-1}-v^{k-1},u^k,e^k\right)}_{=:\, I^k_{c3}} + 2h\underbrace{\left(v^{k-1},u^k-v^k,e^k\right)}_{= -\left(v^{k-1},e^k,e^k\right) = 0} .
\end{aligned}$$

The following estimations are valid for every $K \in \mathbb{N}_{>0}$. The fundamental theorem of calculus 75, Young's inequality A.4, and the case $r = 2$ of the exponential triangle inequality for integrals A.3 are used in the estimation

$$\begin{aligned}
\left|I^k_{c1}\right| &\leqslant 2c_{HV\infty}\int_{t_{k-1}}^{t_k} \left|u-u^{k-1}\right| \left\|e^k\right\| |u|_\infty \, d\tau \\
&\leqslant 2c_{HV\infty}\int_{t_{k-1}}^{t_k} \left|\int_{t_{k-1}}^{\tau} u_t d\vartheta\right| d\tau \left\|e^k\right\| |u|_{L^\infty(0,t_k,L^\infty)} \\
&\leqslant 2\left(c_{HV\infty}K^{\frac{1}{2}}h^{-\frac{1}{2}} |u|_{L^\infty(0,t_k,L^\infty)} \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{\tau} |u_t| \, d\vartheta d\tau\right)\left(K^{-\frac{1}{2}}h^{\frac{1}{2}}\left\|e^k\right\|\right) \\
&\leqslant c^2_{HV\infty}Kh^{-1} |u|^2_{L^\infty(0,t_k,L^\infty)} \left(h\int_{t_{k-1}}^{t_k} |u_t| \, d\vartheta\right)^2 + K^{-1}h\left\|e^k\right\|^2 \\
&\leqslant c^2_{HV\infty}Kh^2 |u|^2_{L^\infty(0,t_k,L^\infty)} \int_{t_{k-1}}^{t_k} |u_t|^2 \, d\tau + K^{-1}h\left\|e^k\right\|^2 ,
\end{aligned}$$

the estimation

$$\begin{aligned}
\left|I^k_{c2}\right| &\leqslant 2c_{\infty VH}\int_{t_{k-1}}^{t_k} \left|u^{k-1}\right|_\infty \left\|e^k\right\| \left|u-u^k\right| d\tau \\
&\leqslant 2c_{\infty VH} |u|_{L^\infty(0,t_{k-1},L^\infty)} \left\|e^k\right\| \int_{t_{k-1}}^{t_k} \left|\int_{\tau}^{t_k} u_t d\vartheta\right| d\tau \\
&\leqslant c^2_{\infty VH}Kh^2 |u|^2_{L^\infty(0,t_{k-1},L^\infty)} \int_{t_{k-1}}^{t_k} |u_t|^2 \, d\tau + K^{-1}h\left\|e^k\right\|^2 \qquad \left(\text{as for } I^k_{c1}\right) ,
\end{aligned}$$

the estimation

$$\begin{aligned}\left|I_{c3}^k\right| &\leqslant 2hc_{HV\infty}\left|u^{k-1}-v^{k-1}\right|\left\|e^k\right\|\left|u^k\right|_\infty\\ &\leqslant 2\left(c_{HV\infty}K^{\frac{1}{2}}h^{\frac{1}{2}}\left|e^{k-1}\right||u|_{L^\infty(0,t_k,L^\infty)}\right)\left(K^{-\frac{1}{2}}h^{\frac{1}{2}}\left\|e^k\right\|\right)\\ &\leqslant c_{HV\infty}^2Kh\,|u|^2_{L^\infty(0,t_k,L^\infty)}\left|e^{k-1}\right|^2+K^{-1}h\left\|e^k\right\|^2,\end{aligned}$$

and the estimation

$$\begin{aligned}\left|I_{\text{viscous}}^k\right| &\leqslant 2\int_{t_{k-1}}^{t_k}\left\|u-u^k\right\|d\tau\left\|e^k\right\|\\ &=2\int_{t_{k-1}}^{t_k}\left\|\int_\tau^{t_k}u_t d\vartheta\right\|d\tau\left\|e^k\right\|\\ &\leqslant 2\left(K^{\frac{1}{2}}h^{-\frac{1}{2}}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k}\|u_t\|\,d\vartheta d\tau\right)\left(K^{-\frac{1}{2}}h^{\frac{1}{2}}\left\|e^k\right\|\right)\\ &\leqslant Kh^{-1}\left(h\int_{t_{k-1}}^{t_k}\|u_t\|\,d\vartheta\right)^2+K^{-1}h\left\|e^k\right\|^2\\ &\leqslant Kh^2\int_{t_{k-1}}^{t_k}\|u_t\|^2\,d\tau+K^{-1}h\left\|e^k\right\|^2.\end{aligned}$$

Setting $K=4$ yields

$$\begin{aligned}&\underbrace{\left|e^k\right|^2}_{=:\,\alpha^k}-\left|e^{k-1}\right|^2+\underbrace{\left|e^k-e^{k-1}\right|^2+h\left\|e^k\right\|^2}_{=:\,b^k}\\ \leqslant\ &\underbrace{h^2\left(4\left(c_{HV\infty}^2+c_{\infty VH}^2\right)|u|^2_{L^\infty(0,t_k,L^\infty)}\int_{t_{k-1}}^{t_k}|u_t|^2\,d\tau+4\int_{t_{k-1}}^{t_k}\|u_t\|^2\,d\tau\right)}_{=:\,c^k}\\ &+\underbrace{4c_{HV\infty}^2h\,|u|^2_{L^\infty(0,t_k,L^\infty)}}_{=:\,d^k}\left|e^{k-1}\right|^2.\end{aligned}$$

Application of the following Lemma 61 proves the asserted error bound. □

Lemma 61 (Discrete Grönwall for Almost Implicit Euler)**.** *Let $N\in\mathbb{N}_{>0}$, $\alpha^0=0$, and $\alpha^k,b^k,c^k,d^k\geqslant 0$ for $k\in\{1,2,\dots,N\}$. If the inequality*

$$\alpha^k-\alpha^{k-1}+b^k\leqslant c^k+d^k\alpha^{k-1}$$

is satisfied for every $k\in\{1,2,\dots,N\}$, then for every $n\in\{1,2,\dots,N\}$ we have

$$\alpha^n+\sum_{k=1}^n b^k\leqslant\left(\sum_{k=1}^n c^k\right)\exp\left(\sum_{k=1}^n d^k\right).$$

Proof. Telescopic canceling and $\alpha^0 = 0$ yield

$$\alpha^n + \sum_{k=1}^{n} b^k \leqslant \sum_{k=1}^{n} \left(c^k + d^k \alpha^{k-1}\right). \tag{7.4}$$

The proof is complete if we prove that

$$\sum_{k=1}^{n} \left(c^k + d^k \alpha^{k-1}\right) \leqslant \left(\sum_{k=1}^{n} c^k\right) \exp\left(\sum_{k=1}^{n} d^k\right),$$

which can be accomplished by induction: The case $n = 1$ holds due to $\alpha^0 = 0$ and $d^1 \geqslant 0$. Inequality 7.4 is used for the first and the inductive hypothesis for the second estimation of the inductive step

$$\begin{aligned}
&\sum_{k=1}^{n+1} \left(c^k + d^k \alpha^{k-1}\right) \\
&\leqslant \left(\sum_{k=1}^{n} \left(c^k + d^k \alpha^{k-1}\right)\right) + c^{n+1} + d^{n+1} \sum_{k=1}^{n} \left(c^k + d^k \alpha^{k-1}\right) \\
&\leqslant \underbrace{\left(1 + d^{n+1}\right)}_{\leqslant \exp\left(d^{n+1}\right)} \left(\textstyle\sum_{k=1}^{n} c^k\right) \exp\left(\sum_{k=1}^{n} d^k\right) + \underbrace{c^{n+1}}_{\leqslant c^{n+1} \exp\left(\sum_{k=1}^{n+1} d^k\right)} \\
&\leqslant \left(\sum_{k=1}^{n+1} c^k\right) \exp\left(\sum_{k=1}^{n+1} d^k\right).
\end{aligned}$$

□

The following uniqueness result is a corollary of Theorem 60. If applied to the Navier-Stokes SST space, the corollary's assertion is a special case of an already known uniqueness result for the nonstationary Navier-Stokes equations [Tem84, page 297, Theorem 3.4].

Corollary 62 (Uniqueness)**.** *Let (V, L^∞) be an SST_∞ space, $T > 0$, $u^0 \in V$ and $f \in L^1(0, T, V')$. Then there is at most one solution $u : [0, T] \to V \cap L^\infty$ of regularity 7.2 to the SST-generalized nonstationary Navier-Stokes equations 7.1 with initial value u^0 and force f.*

Proof. Suppose $r : [0, T] \to V \cap L^\infty$ and $s : [0, T] \to V \cap L^\infty$ are two different solutions of regularity 7.2 with initial value u^0 and force f. Then there is $\vartheta \in (0, T]$ with $|r(\vartheta) - s(\vartheta)| =: \varepsilon > 0$ and thus a number $N \in \mathbb{N}_{\geqslant 1}$ such that $h := \frac{\vartheta}{N}$ is small enough as to satisfy the inequality

$$\begin{aligned}
\frac{\varepsilon^2}{4} > h^2 &\left(4\left(c_{HV\infty}^2 + c_{\infty VH}^2\right) |u|_{L^\infty(0,\vartheta,L^\infty)}^2 \int_0^\vartheta |u_t|^2 \, d\tau + 4 \int_0^\vartheta \|u_t\|^2 \, d\tau\right) \\
&\exp\left(4 c_{HV\infty}^2 |u|_{L^\infty(0,\vartheta,L^\infty)}^2 \vartheta\right)
\end{aligned}$$

for both $u \in \{r, s\}$. Theorem 19 implies that there is a sequence $v^0, v^1, \ldots, v^N$ of solutions to the almost implicit Euler scheme 3.4 with step size h, initial value $v^0 = u^0$, and evaluation of the force defined as in Theorem 60. The solution r with the sequence $v^0, v^1, \ldots, v^N$ as well as the solution s with the same sequence $v^0, v^1, \ldots, v^N$ satisfy the assumptions of Theorem 60 on the time interval $[0, \vartheta]$. Hence, due to the above choice of $h > 0$, the approximation errors satisfy

$$\left|v^N - r(\vartheta)\right|^2 < \frac{\varepsilon^2}{4} \qquad \text{and} \qquad \left|v^N - s(\vartheta)\right|^2 < \frac{\varepsilon^2}{4},$$

which contradicts $|r(\vartheta) - s(\vartheta)| = \varepsilon$. □

7.2. The product Crank-Nicolson scheme

In this section the approximations of the product Crank-Nicolson scheme are proven to converge of second order towards solutions of the SST-generalized Navier-Stokes equations under stronger regularity assumptions on the solutions than those used for the almost implicit Euler scheme in the last section. More precisely the regularity assumptions on the solutions are $u_t \in L^2(0, T, L^\infty)$ and $u_{tt} \in L^2(0, T, V)$.[8] The following Theorem 63 specifies conditions on the data u^0 and f that ensure the existence of such a solution in the case of the Navier-Stokes SST space.

Theorem 63. *Let V be the Navier-Stokes SST space from sec. 2.2 and suppose $G \subset \mathbb{R}^3$ has a C^6 boundary. Let $u^0 \in V \cap H^4(G)^3$, $T' > 0$, $f \in C\left([0, T'], H' \cap H^2(G)^3\right)$, $f_t \in C([0, T'], H')$, and $f_{tt} \in L^2(0, T', V')$. Moreover suppose that the initial value and the force at time $\tau = 0$ satisfy the compatibility condition*

$$(a, \varphi) + \left(u^0, u^0, \varphi\right) + \left(\left(u^0, \varphi\right)\right) = f^0(\varphi) \qquad (\varphi \in V) \tag{7.5}$$

with some initial acceleration $a \in V$. Then there is some $0 < T \leqslant T'$ such that the nonstationary Navier-Stokes equations with initial value u^0 and force f have one and only one solution $u \in C\left([0, T], V \cap H^4(G)^3\right)$ with $u_t \in C\left([0, T], H' \cap H^2(G)^3\right)$ and $u_{tt} \in L^2(0, T, V) \cap C([0, T], H')$.[9]

Proof. The result is obtained from [Tem82] as follows: Like in the proof of Theorem 59, statement (3.8) on page 86 of [Tem82] implies the unique existence of a solution $u \in C([0, T], V)$ for some $0 < T \leqslant T'$. The regularity assumptions of Theorem 3.1 for $m = 4$ on page 86 of [Tem82] are satisfied. The compatibility

[8] For the spaces $L^p(0, T, G)$ and $C^k([0, T], G)$ see sec. A.2.

[9] Note that $u_t \in C\left([0, T], H' \cap H^2(G)^3\right)$ implies $u_t \in C\left([0, T], L^\infty(G)^3\right)$ due to the Sobolev imbedding $H^2(G) \hookrightarrow C_B(G)$ (see [AF03, page 85, Theorem 4.12 for $n = k = 3$, $j = 0$, $m = 2$, $p = 2$]) and hence $u_t \in L^2\left(0, T, L^\infty(G)^3\right)$.

condition on the first time derivative $-\mathcal{A}u^0 - B(u^0, u^0) + f^0 \in V$ (first line of statement (3.11) on page 87 of [Tem82]) holds because it corresponds to compatibility condition 7.5, where $a = -\mathcal{A}u^0 - B(u^0, u^0) + f^0$. Statement (1.13) on page 75 of [Tem82] yields $\mathcal{A}u^0 \in H \cap H^2(G)^3$ and statement (3.5) on page 86 of [Tem82] yields $B(u^0, u^0) \in H \cap H^3(G)^3$. Since also $f^0 \in H \cap H^2(G)^3$, it follows that $a = -\mathcal{A}u^0 - B(u^0, u^0) + f^0 \in H \cap H^2(G)^3$. The compatibility condition on the second time derivative $-\mathcal{A}a - B(a, u^0) - B(u^0, a) + f_t^0 \in H$ (second line of statement (3.11) on page 87 of [Tem82]) holds because $\mathcal{A}a \in H$ due to statement (1.13) on page 75 of [Tem82], because $B(a, u^0) \in H \cap H^1(G)^3$ and $B(u^0, a) \in H \cap H^1(G)^3$ due to statement (3.5) on page 86 of [Tem82], and because $f_t^0 \in H$. Therefore the sufficiency part of the cited Theorem 3.1 yields $u \in C\left([0,T], V \cap H^4(G)^3\right)$ with $u_t \in C\left([0,T], H' \cap H^2(G)^3\right)$ and $u_{tt} \in C([0,T], H')$. The property $u_{tt} \in L^2(0, T, V)$ is not part of the assertions of the cited Theorem 3.1 but it is established in its proof [Tem82, page 88, statement (3.19)]. □

Remark 64. It is natural to ask whether nontrivial data u^0 and f that satisfy the compatibility condition 7.5 actually exist because otherwise Theorem 63 would be pointless. The answer is yes: First of all for every $f^0 \in V'$ and $a \in V$ there is at least one $u^0 \in V$ such that compatibility condition 7.5 is satisfied because of Theorem 18 with $g : V \to \mathbb{R}, \varphi \mapsto f^0(\varphi) - (a, \varphi)$, $w = 0$, $\lambda = \eta = 0$, and $\mu = 1$.[10] Under the additional assumptions that $f^0 \in H^2(G)^3$ and $a \in H^2(G)^3$ it can be shown that every such $u^0 \in V$ satisfies $u^0 \in H^4(G)^3$, see e.g. [Soh01, page 173, Theorem 3.6.1 for $k = 2$]. The idea to prescribe an initial acceleration a instead of an initial velocity u^0 also appears in [VZ13, pages 156-158].

The second order convergence of the Crank-Nicolson scheme for the nonstationary linear Stokes equations is proven with energy methods for solutions with regularity $u \in C\left([0,T], V \cap H^4(G)^3\right)$ in [Var92, page 51, Theorem 4.3].[11] A proof of the second order convergence of a family of linearized two-step methods for the nonstationary Navier-Stokes equations can be found in [GR79, page 191, Theorem 3.1]. The second order convergence of the product Crank-Nicolson scheme for a spatial discretization of the nonstationary Navier-Stokes equations is proven in [HR90, page 367, Theorems 4.1 and 4.2]. To our knowledge, the second order convergence of the product Crank-Nicolson scheme in the spatially continuous case where the Dirichlet norm can not be controlled by the L^2 norm has not been proven yet.

Theorem 65 (Convergence of Product Crank-Nicolson)**.** *Let (V, L^∞) be an SST_∞ space, $T > 0$, and let $u : [0,T] \to V \cap L^\infty$ be a solution of the SST-generalized nonstationary Navier-Stokes equations 7.1 with initial value $u^0 \in V$ and force $f \in L^1(0, T, V')$. Let $N \in \mathbb{N}_{\geqslant 1}$, $h := \frac{T}{N}$, and let $v^0, v^1, \ldots, v^N$ be a sequence of solutions*

[10] It is shown in Theorem 20 that the Navier-Stokes SST space satisfies the assumptions of Theorem 18.

[11] In the linear case, there is no difference between the sum, the linear and the product Crank-Nicolson schemes.

to the product Crank-Nicolson scheme 3.8 with initial value $v^0 = u^0$ and evaluation of the force defined by[12]

$$f^k := \frac{1}{h}\int_{t_{k-1}}^{t_k} f\, d\tau \qquad (k \in \{1, 2, \dots, N\})\,.$$

Moreover let

$$u_t \in L^2(0, T, L^\infty) \qquad \textit{and} \qquad u_{tt} \in L^2(0, T, V)\,.$$

Then, if

$$h \leqslant \frac{1}{4} c_{HV\infty}^{-2} |u|_{L^\infty(0,T,L^\infty)}^{-2}\,, \tag{7.6}$$

the approximation error satisfies

$$\begin{aligned}
&|e^n|^2 + \frac{1}{4}\sum_{k=1}^{n} h \left\|e^{k-1} + e^k\right\|^2 \\
&\leqslant h^4 \Bigg(2c_{HV\infty}^2 |u_t|_{L^\infty(0,t_n,|\cdot|)}^2 \int_0^{t_n} |u_t|_\infty^2\, d\tau \\
&\qquad\qquad + 8\left(c_{HV\infty} + c_{\infty VH}\right)^2 |u|_{L^\infty(0,t_n,L^\infty)}^2 \int_0^{t_n} |u_{tt}|^2\, d\tau \\
&\qquad\qquad\quad + 8\int_0^{t_n} \|u_{tt}\|^2\, d\tau \Bigg) \exp\left(8c_{HV\infty}^2 |u|_{L^\infty(0,t_n,L^\infty)}^2\, t_n\right)
\end{aligned}$$

for every $n \in \{1, 2, \dots, N\}$, i. e. the product Crank-Nicolson scheme is convergent of second order.

Proof. The values of all Bochner integrals in this proof don't depend on whether they are interpreted as to be defined with respect to the norm $|\cdot|$, $\|\cdot\|$, or $|\cdot|_\infty$ because of Lemma 74 from sec. A.2. The assumption that u be differentiable with respect to $|\cdot|_\infty$ implies that $u \in C(0, T, L^\infty)$ which in turn implies $u \in L^\infty(0, T, L^\infty)$ because $[0, T]$ is compact. The assumption that u_t be differentiable with respect to the Dirichlet norm $\|\cdot\|$ implies that $u \in C^1(0, T, V)$, which, again because of the

[12]See sec. A.2 for the notion of Bochner integral.

compactness of $[0,T]$, implies $u_t \in L^\infty(0,T,V)$. Let $1 \leqslant k \leqslant N$. The representation

$$
\begin{aligned}
& h^{-1}\int_{t_{k-1}}^{t_k}\left((t_k-\tau)\int_{t_{k-1}}^{\tau}(\vartheta-t_{k-1})\,u_{tt}d\vartheta+(\tau-t_{k-1})\int_{\tau}^{t_k}(t_k-\vartheta)\,u_{tt}d\vartheta\right)d\tau\\
&= h^{-1}\int_{t_{k-1}}^{t_k}\Bigg((t_k-\tau)\left[\vartheta u_t-u-t_{k-1}u_t\right]_{\vartheta=t_{k-1}}^{\tau}\\
&\qquad\qquad +(\tau-t_{k-1})\left[t_k u_t-\vartheta u_t+u\right]_{\vartheta=\tau}^{t_k}\Bigg)d\tau\\
&= t_k u^{k-1}-t_{k-1}u^k+h^{-1}\int_{t_{k-1}}^{t_k}\left(-hu-\tau u^{k-1}+\tau u^k\right)d\tau\\
&= \int_{t_{k-1}}^{t_k}\left(\frac{1}{2}\left(u^{k-1}+u^k\right)-u\right)d\tau
\end{aligned}
$$

based on the fundamental theorem of calculus 75 yields the estimate (valid if $|||\cdot|||$ stands either for the norm $|\cdot|$ or for the Dirichlet norm $\|\cdot\|$)

$$
\begin{aligned}
& \left|\left|\left|\int_{t_{k-1}}^{t_k}\left(\frac{1}{2}\left(u^{k-1}+u^k\right)-u\right)d\tau\right|\right|\right|\\
&\leqslant h^{-1}\int_{t_{k-1}}^{t_k}\Bigg(\left|\left|\left|(t_k-\tau)\int_{t_{k-1}}^{\tau}(\vartheta-t_{k-1})\,u_{tt}d\vartheta\right|\right|\right|\\
&\qquad\qquad +\left|\left|\left|(\tau-t_{k-1})\int_{\tau}^{t_k}(t_k-\vartheta)\,u_{tt}d\vartheta\right|\right|\right|\Bigg)d\tau\\
&\leqslant \int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{\tau}|||(\vartheta-t_{k-1})\,u_{tt}|||d\vartheta+\int_{\tau}^{t_k}|||(t_k-\vartheta)\,u_{tt}|||d\vartheta\right)d\tau\\
&\leqslant h\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{\tau}|||u_{tt}|||d\vartheta+\int_{\tau}^{t_k}|||u_{tt}|||d\vartheta\right)d\tau\\
&= h^2\int_{t_{k-1}}^{t_k}|||u_{tt}|||d\tau. \qquad (7.7)
\end{aligned}
$$

For every $\tau \in [0,T]$, due to $u_t \in V$ and the imbedding $V \hookrightarrow H' \subset V'$ from Definition 5, the equation $u_t(\varphi) = (u_t,\varphi)$ holds, where u_t is interpreted as an element of V' on the left and as an element of V on the right hand side of the equation. For every $\varphi \in V$ the fact that v^{k-1}, v^k, and f^k solve the product Crank-Nicolson scheme 3.8, the fact that u and f solve the Navier-Stokes equations 7.1, and the fundamental theorem of calculus 75 yield

$$
\begin{aligned}
&\int_{t_{k-1}}^{t_k} \left(\left(u - \frac{1}{2}\left(u^{k-1} + u^k\right), \varphi\right)\right) d\tau \\
&= \underbrace{\left(v^k - v^{k-1}, \varphi\right) + \frac{h}{4}\left(v^{k-1} + v^k, v^{k-1} + v^k, \varphi\right) + \frac{h}{2}\left(\left(v^{k-1} + v^k, \varphi\right)\right) - hf^k(\varphi)}_{=0} \\
&\quad - h\left(\left(\frac{1}{2}\left(u^{k-1} + u^k\right), \varphi\right)\right) - \int_{t_{k-1}}^{t_k} \left((u_t, \varphi) + (u, u, \varphi) - f(\varphi)\right) d\tau \\
&= \left(e^k - e^{k-1}, \varphi\right) + \int_{t_{k-1}}^{t_k} \left(\frac{1}{4}\left(v^{k-1} + v^k, v^{k-1} + v^k, \varphi\right) - (u, u, \varphi)\right) d\tau \\
&\quad + \frac{h}{2}\left(\left(e^{k-1} + e^k, \varphi\right)\right).
\end{aligned}
$$

Choosing $\varphi = e^{k-1} + e^k$ as a test function and taking $\left(e^k - e^{k-1}, e^{k-1} + e^k\right) = \left|e^k\right|^2 - \left|e^{k-1}\right|^2$ into account yields

$$
\begin{aligned}
&\left|e^k\right|^2 - \left|e^{k-1}\right|^2 + \frac{1}{2}h\left\|e^{k-1} + e^k\right\|^2 \\
&= \underbrace{\int_{t_{k-1}}^{t_k} \left(u, u, e^{k-1} + e^k\right) - \frac{1}{4}\left(v^{k-1} + v^k, v^{k-1} + v^k, e^{k-1} + e^k\right) d\tau}_{=:\, I^k_{\text{convective}}} \\
&\quad + \underbrace{\int_{t_{k-1}}^{t_k} \left(\left(u - \frac{1}{2}\left(u^{k-1} + u^k\right), e^{k-1} + e^k\right)\right) d\tau}_{=:\, I^k_{\text{viscous}}}.
\end{aligned}
$$

We split

$$
\begin{aligned}
I^k_{\text{convective}} = &\underbrace{\int_{t_{k-1}}^{t_k} \left(u - \frac{1}{2}\left(u^{k-1} + u^k\right), u, e^{k-1} + e^k\right) d\tau}_{=:\, I^k_{c1}} \\
&+ \underbrace{\int_{t_{k-1}}^{t_k} \left(\frac{1}{2}\left(u^{k-1} + u^k\right), u - \frac{1}{2}\left(u^{k-1} + u^k\right), e^{k-1} + e^k\right) d\tau}_{=:\, I^k_{c2}} \\
&+ \underbrace{h\left(\frac{1}{2}\left(u^{k-1} + u^k\right) - \frac{1}{2}\left(v^{k-1} + v^k\right), \frac{1}{2}\left(u^{k-1} + u^k\right), e^{k-1} + e^k\right)}_{= -h\left(\frac{1}{2}\left(e^{k-1} + e^k\right), \frac{1}{2}\left(u^{k-1} + u^k\right), e^{k-1} + e^k\right) =:\, I^k_{c3}} \\
&+ \underbrace{h\left(\frac{1}{2}\left(v^{k-1} + v^k\right), \frac{1}{2}\left(u^{k-1} + u^k\right) - \frac{1}{2}\left(v^{k-1} + v^k\right), e^{k-1} + e^k\right)}_{= -\frac{1}{2}\left(\frac{1}{2}\left(v^{k-1} + v^k\right), e^{k-1} + e^k, e^{k-1} + e^k\right) = 0}
\end{aligned}
$$

and sub-split

$$I_{c1}^k = \underbrace{\int_{t_{k-1}}^{t_k} \left(u - \frac{1}{2}\left(u^{k-1} + u^k\right), u - u^{k-1}, e^{k-1} + e^k\right) d\tau}_{=:\, I_{c1.1}^k} + \underbrace{\int_{t_{k-1}}^{t_k} \left(u - \frac{1}{2}\left(u^{k-1} + u^k\right), u^{k-1}, e^{k-1} + e^k\right) d\tau}_{=:\, I_{c1.2}^k} .$$

The following estimations are valid for every $K \in \mathbb{N}_{>0}$. The fundamental theorem of calculus 75, Young's inequality A.4, the case $r = 2$ of the exponential triangle inequality for integrals A.3, estimate 7.7, and inequality A.2 are used in the estimations

$$\begin{aligned}
\left|I_{c1.1}^k\right| &\leqslant c_{HV\infty} \int_{t_{k-1}}^{t_k} \left|u - \frac{1}{2}\left(u^{k-1} + u^k\right)\right| \left\|e^{k-1} + e^k\right\| \left|u - u^{k-1}\right|_\infty d\tau \\
&= 2\left(\frac{1}{2} c_{HV\infty} K^{\frac{1}{2}} h^{-\frac{1}{2}} \int_{t_{k-1}}^{t_k} \left|\frac{1}{2}\left(\int_{t_{k-1}}^{\tau} u_t d\vartheta - \int_{\tau}^{t_k} u_t d\vartheta\right)\right| \left|\int_{t_{k-1}}^{\tau} u_t d\vartheta\right|_\infty d\tau\right) \\
&\qquad \left(K^{-\frac{1}{2}} h^{\frac{1}{2}} \left\|e^{k-1} + e^k\right\|\right) \\
&\leqslant \frac{1}{4} c_{HV\infty}^2 K h^{-1} \left(\int_{t_{k-1}}^{t_k} \frac{1}{2}\left(\int_{t_{k-1}}^{\tau} |u_t|\, d\vartheta + \int_{\tau}^{t_k} |u_t|\, d\vartheta\right) \int_{t_{k-1}}^{\tau} |u_t|_\infty\, d\vartheta d\tau\right)^2 \\
&\qquad + K^{-1} h \left\|e^{k-1} + e^k\right\|^2 \\
&\leqslant \frac{1}{16} c_{HV\infty}^2 K h^{-1} \left(h^2 |u_t|_{L^\infty(0,t_k,|\cdot|)} \int_{t_{k-1}}^{t_k} |u_t|_\infty\, d\tau\right)^2 + K^{-1} h \left\|e^{k-1} + e^k\right\|^2 \\
&\leqslant \frac{1}{16} c_{HV\infty}^2 K h^4 |u_t|_{L^\infty(0,t_k,|\cdot|)}^2 \int_{t_{k-1}}^{t_k} |u_t|_\infty^2\, d\tau + K^{-1} h \left\|e^{k-1} + e^k\right\|^2,
\end{aligned}$$

$$
\begin{aligned}
&\left|I_{c1.2}^k\right| + \left|I_{c2}^k\right| \\
&\leqslant c_{HV\infty} \left| \int_{t_{k-1}}^{t_k} \left(u - \frac{1}{2}\left(u^{k-1} + u^k\right)\right) d\tau \right| \left\| e^{k-1} + e^k \right\| \left| u^{k-1} \right|_\infty \\
&\qquad + c_{\infty VH} \left| \frac{1}{2}\left(u^{k-1} + u^k\right)\right|_\infty \left\| e^{k-1} + e^k \right\| \left| \int_{t_{k-1}}^{t_k} \left(u - \frac{1}{2}\left(u^{k-1} + u^k\right)\right) d\tau \right| \\
&\leqslant 2 \left(\frac{1}{2} K^{\frac{1}{2}} h^{-\frac{1}{2}} h^2 \int_{t_{k-1}}^{t_k} |u_{tt}|\, d\tau \left(c_{HV\infty} \left| u^{k-1} \right|_\infty + c_{\infty VH} \left| \frac{1}{2}\left(u^{k-1} + u^k\right)\right|_\infty \right)\right) \\
&\qquad\qquad \left(K^{-\frac{1}{2}} h^{\frac{1}{2}} \left\| e^{k-1} + e^k \right\| \right) \\
&\leqslant \frac{1}{4} K h^{-1} h^4 \left(\int_{t_{k-1}}^{t_k} |u_{tt}|\, d\tau \right)^2 \left(\left(c_{HV\infty} + c_{\infty VH}\right) |u|_{L^\infty(0,t_k,L^\infty)} \right)^2 \\
&\qquad + K^{-1} h \left\| e^{k-1} + e^k \right\|^2 \\
&\leqslant \frac{1}{4} \left(c_{HV\infty} + c_{\infty VH}\right)^2 K h^4 |u|^2_{L^\infty(0,t_k,L^\infty)} \int_{t_{k-1}}^{t_k} |u_{tt}|^2\, d\tau + K^{-1} h \left\| e^{k-1} + e^k \right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
\left|I_{c3}^k\right| &\leqslant \frac{1}{2} c_{HV\infty} h \left| e^{k-1} + e^k \right| \left\| e^{k-1} + e^k \right\| \left| \frac{1}{2}\left(u^{k-1} + u^k\right)\right|_\infty \\
&= 2 \left(\frac{1}{4} c_{HV\infty} K^{\frac{1}{2}} h^{\frac{1}{2}} \left| e^{k-1} + e^k \right| \left| \frac{1}{2}\left(u^{k-1} + u^k\right)\right|_\infty \right) \left(K^{-\frac{1}{2}} h^{\frac{1}{2}} \left\| e^{k-1} + e^k \right\| \right) \\
&\leqslant \frac{1}{16} c_{HV\infty}^2 K h |u|^2_{L^\infty(0,t_k,L^\infty)} \left| e^{k-1} + e^k \right|^2 + K^{-1} h \left\| e^{k-1} + e^k \right\|^2 \\
&\leqslant \frac{1}{8} c_{HV\infty}^2 K h |u|^2_{L^\infty(0,t_k,L^\infty)} \left(\left| e^{k-1} \right|^2 + \left| e^k \right|^2 \right) + K^{-1} h \left\| e^{k-1} + e^k \right\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\left|I_{\text{viscous}}^k\right| &\leqslant \left\| \int_{t_{k-1}}^{t_k} \left(u - \frac{1}{2}\left(u^{k-1} + u^k\right)\right) d\tau \right\| \left\| e^{k-1} + e^k \right\| \\
&\leqslant h^2 \int_{t_{k-1}}^{t_k} \|u_{tt}\|\, d\tau \left\| e^{k-1} + e^k \right\| \\
&= 2 \left(\frac{1}{2} K^{\frac{1}{2}} h^{\frac{3}{2}} \int_{t_{k-1}}^{t_k} \|u_{tt}\|\, d\tau \right) \left(K^{-\frac{1}{2}} h^{\frac{1}{2}} \left\| e^{k-1} + e^k \right\| \right) \\
&\leqslant \frac{1}{4} K h^3 \left(\int_{t_{k-1}}^{t_k} \|u_{tt}\|\, d\tau \right)^2 + K^{-1} h \left\| e^{k-1} + e^k \right\|^2 \\
&\leqslant \frac{1}{4} K h^4 \int_{t_{k-1}}^{t_k} \|u_{tt}\|^2\, d\tau + K^{-1} h \left\| e^{k-1} + e^k \right\|^2 .
\end{aligned}
$$

Setting $K = 16$ yields

$$
\begin{aligned}
&\underbrace{\left|e^k\right|^2}_{=:\,\alpha^k} - \left|e^{k-1}\right|^2 + \underbrace{\frac{1}{4}h\left\|e^{k-1}+e^k\right\|^2}_{=:\,b^k} \\
&\leqslant \underbrace{\begin{aligned} h^4\Bigg(& c_{HV\infty}^2 \left|u_t\right|^2_{L^\infty(0,t_k,|\cdot|)} \int_{t_{k-1}}^{t_k} \left|u_t\right|^2_\infty d\tau \\
&+ 4\left(c_{HV\infty}+c_{\infty VH}\right)^2 \left|u\right|^2_{L^\infty(0,t_k,L^\infty)} \int_{t_{k-1}}^{t_k} \left|u_{tt}\right|^2 d\tau \\
&+4\int_{t_{k-1}}^{t_k} \left\|u_{tt}\right\|^2 d\tau\Bigg)\end{aligned}}_{=:\,c^k} \\
&\quad + \underbrace{2c_{HV\infty}^2 h \left|u\right|^2_{L^\infty(0,t_k,L^\infty)}}_{=:\,d^k} \left(\left|e^{k-1}\right|^2 + \left|e^k\right|^2\right).
\end{aligned}
$$

Since $d^k \leqslant \frac{1}{2}$ due to smallness assumption 7.6, the following Lemma 66 is applicable and yields the asserted error bound. ☐

Lemma 66 (Discrete Grönwall for Product Crank-Nicolson)**.** *Let $N \in \mathbb{N}_{>0}$, $\alpha^0 = 0$, and $\alpha^k, b^k, c^k, d^k \geqslant 0$ for $k \in \{1, 2, \ldots, N\}$. If the inequalities*

$$
\alpha^k - \alpha^{k-1} + b^k \leqslant c^k + d^k\left(\alpha^{k-1} + \alpha^k\right), \tag{7.8}
$$

$$
d^k \leqslant \frac{1}{2} \tag{7.9}
$$

are satisfied for every $k \in \{1, 2, \ldots, N\}$, then for every $n \in \{1, 2, \ldots, N\}$ we have

$$
\alpha^n + \sum_{k=1}^n b^k \leqslant \left(2\sum_{k=1}^n c^k\right) \exp\left(4\sum_{k=1}^n d^k\right).
$$

Proof. For every $k \in \{1, 2, \ldots, N\}$

$$
\begin{aligned}
\alpha^k - \alpha^{k-1} + b^k &= \frac{1}{1-d^k}\left(\alpha^k - \alpha^{k-1} + b^k - d^k\alpha^k + d^k\alpha^{k-1} - d^k b^k\right) \\
&\leqslant \frac{1}{1-d^k}\left(c^k + d^k\left(\alpha^{k-1}+\alpha^k\right) - d^k\alpha^k + d^k\alpha^{k-1} - d^k b^k\right) \\
&\leqslant 2c^k + 4d^k\alpha^{k-1},
\end{aligned}
$$

where the first estimation is based on inequality 7.8 and the second on smallness assumption 7.9. Telescopic canceling and $\alpha^0 = 0$ yield

$$
\alpha^n + \sum_{k=1}^n b^k \leqslant \sum_{k=1}^n \left(2c^k + 4d^k\alpha^{k-1}\right). \tag{7.10}
$$

The proof is complete if we prove that

$$\sum_{k=1}^{n}\left(2c^k + 4d^k\alpha^{k-1}\right) \leqslant \left(2\sum_{k=1}^{n} c^k\right)\exp\left(4\sum_{k=1}^{n} d^k\right),$$

which can be accomplished by induction: The case $n = 1$ holds due to $\alpha^0 = 0$ and $d^1 \geqslant 0$. Inequality 7.10 is used for the first and the inductive hypothesis for the second estimation of the inductive step

$$\begin{aligned}
&\sum_{k=1}^{n+1}\left(2c^k + 4d^k\alpha^{k-1}\right) \\
&\leqslant \left(\sum_{k=1}^{n}\left(2c^k + 4d^k\alpha^{k-1}\right)\right) + 2c^{n+1} + 4d^{n+1}\sum_{k=1}^{n}\left(2c^k + 4d^k\alpha^{k-1}\right) \\
&\leqslant \underbrace{\left(1 + 4d^{n+1}\right)}_{\leqslant \exp\left(4d^{n+1}\right)}\left(2\textstyle\sum_{k=1}^{n} c^k\right)\ \exp\left(4\displaystyle\sum_{k=1}^{n} d^k\right) + 2\underbrace{c^{n+1}}_{\leqslant c^{n+1}\exp\left(4\sum_{k=1}^{n+1} d^k\right)} \\
&\leqslant \left(2\sum_{k=1}^{n+1} c^k\right)\exp\left(4\sum_{k=1}^{n+1} d^k\right).
\end{aligned}$$

□

Remark 67. Of course, Theorem 65 admits a uniqueness corollary analogous to Corollary 62. But as the assumptions for the convergence of the product Crank-Nicolson scheme are stronger than those for the almost implicit Euler scheme, this corollary is a special case of Corollary 62 and therefore not stated.

A. Appendix

A.1. Inequalities

Lemma 68 (Exponential triangle inequality)**.** *Let* $n \in \mathbb{N}_{\geqslant 1}$*,* $1 \leqslant r < \infty$*, and* $x_1, \ldots, x_n \in \mathbb{R}$*. Then*

$$|x_1 + \cdots + x_n|^r \leqslant n^{r-1} \left(|x_1|^r + \cdots + |x_n|^r\right). \tag{A.1}$$

Proof. If $r > 1$, apply Hölder's inequality in $\mathbb{R}^n$ with exponents $\left(1 - r^{-1}\right)^{-1}$ and r to the product of the two vectors $(1)_{i=1}^n$ and $(x_i)_{i=1}^n$. □

Remark 69. If $(V, |\cdot|)$ is a normed space, the inequality

$$|a + b|^2 \leqslant 2\left(|a|^2 + |b|^2\right) \tag{A.2}$$

can be deduced from the above exponential triangle inequality A.1. But if $|\cdot|$ is induced by a scalar product $(\cdot, \cdot)$, it can also be deduced from the identity

$$|a + b|^2 + |a - b|^2 = 2\left(|a|^2 + |b|^2\right).$$

Lemma 70 (Exponential triangle inequality for integrals)**.** *Let* $a < b$*,* $1 \leqslant r < \infty$*, and* $f \in L^r(a, b)$*. Then* $f \in L^1(a, b)$ *and*

$$\left|\int_a^b f \, d\tau\right|^r \leqslant (b - a)^{r-1} \int_a^b |f|^r \, d\tau. \tag{A.3}$$

Proof. If $r > 1$, apply Hölder's inequality on $[a, b]$ with exponents $\left(1 - r^{-1}\right)^{-1}$ and r to the product of the two functions $\tau \mapsto 1$ and $\tau \mapsto f(\tau)$. □

Lemma 71 (Young's inequality)**.** *If* $a, b \in \mathbb{R}$*, then*

$$2|ab| \leqslant |a|^2 + |b|^2. \tag{A.4}$$

Proof. Young's inequality is a consequence of the identity

$$2|ab| + (|a| - |b|)^2 = |a|^2 + |b|^2.$$

□

A.2. Bochner integration

Definition 72 (Bochner integrability)**.** Let $a, b \in \mathbb{R}$ with $a < b$ and let $(G, |\cdot|)$ be a Banach space. Then $L^1(a, b, G, |\cdot|)$ denotes the space of all functions $u : [a, b] \to G$ that are Bochner integrable with respect to the norm $|\cdot|$ and the Lebesgue measure on $[a, b]$.[1] For $u \in L^1(a, b, G, |\cdot|)$,

$$\int_{[a,b]}^{|\cdot|} u \, dt$$

denotes the Bochner integral with respect to the norm $|\cdot|$. The shorter notations

$$L^1(a, b, G) \qquad \text{and} \qquad \int_a^b u \, dt$$

are used if there is no doubt about which norm is meant or if the value of the integral is the same for each norm in the particular context.

If a vector space is equipped with different norms there is a priori no reason why the notions of Bochner integrability and Bochner integral with respect to the different norms should coincide. The following two lemmas justify why it is possible to ignore the role of the norm in Bochner integration under certain conditions.

Lemma 73. *Let $(G, |\cdot|_G)$ be a Banach space with a subspace $F \subset G$ and $|\cdot|_F$ a norm on F such that $(F, |\cdot|_F)$ is a Banach space as well. Furthermore suppose that there is $C \in \mathbb{R}$ such that for all $x \in F$*

$$|x|_G \leqslant C \, |x|_F \, .$$

Let $a, b \in \mathbb{R}$ with $a < b$. Then $L^1(a, b, F) \subset L^1(a, b, G)$ and for every $u \in L^1(a, b, F)$

$$\int_{[a,b]}^{|\cdot|_G} u \, dt = \int_{[a,b]}^{|\cdot|_F} u \, dt. \tag{A.5}$$

Proof. Let $u \in L^1(a, b, F)$, i. e. let u be integrable with respect to $|\cdot|_F$. Then there is a sequence $(\varphi_j)_{j \geqslant 0}$ of simple functions $\varphi_j : [a, b] \to F$ that

1. is a Cauchy sequence with respect to the seminorm that maps every simple function $\varphi : [a, b] \to F$ to the number

$$\int_a^b |\varphi|_F \, dt$$

and

[1] See [AE01, pages 87 and 90] for the notions of Bochner integrability and Bochner integral.

2. such that $\left|\varphi_j(t) - u(t)\right|_F \xrightarrow{j\to\infty} 0$ for almost every $t \in [a,b]$.

The inequalities

$$\int_a^b \left|\varphi_i - \varphi_j\right|_G dt \leqslant C \int_a^b \left|\varphi_i - \varphi_j\right| dt$$

and $\left|\varphi_j(t) - u(t)\right|_G \leqslant C \left|\varphi_j(t) - u(t)\right|_F$ show that the same is true with respect to $|\cdot|_G$, i. e. $u \in L^1(a,b,G)$. Therefore not only

$$\left|\int_a^b \varphi_j dt - \int_{[a,b]}^{|\cdot|_F} u\,dt\right|_F \xrightarrow{j\to\infty} 0 \tag{A.6}$$

but also

$$\left|\int_a^b \varphi_j dt - \int_{[a,b]}^{|\cdot|_G} u\,dt\right|_G \xrightarrow{j\to\infty} 0$$

(note that the value of the integral of a simple function is a finite sum and hence does not depend on the chosen norm). As A.6 implies

$$\left|\int_a^b \varphi_j dt - \int_{[a,b]}^{|\cdot|_F} u\,dt\right|_G \leqslant C \left|\int_a^b \varphi_j dt - \int_{[a,b]}^{|\cdot|_F} u\,dt\right|_F \xrightarrow{j\to\infty} 0$$

and because limits in $(G, |\cdot|_G)$ are unique, A.5 holds. □

There is a similar result for the case of three norms:

Lemma 74. *Let $(H, |\cdot|_H)$ be a Banach space with two subspaces $F, G \subset H$, $|\cdot|_F$ a norm on F, and $|\cdot|_G$ a norm on G such that $(F, |\cdot|_F)$ and $(G, |\cdot|_G)$ are Banach spaces as well. Suppose that there is $C \in \mathbb{R}$ such that for all $x \in F$ and all $y \in G$*

$$|x|_H \leqslant C\,|x|_F \qquad \text{and} \qquad |y|_H \leqslant C\,|y|_G\,.$$

Let $a, b \in \mathbb{R}$ with $a < b$ and $u \in L^1(a,b,F) \cap L^1(a,b,G)$. Then

$$\int_{[a,b]}^{|\cdot|_F} u\,dt = \int_{[a,b]}^{|\cdot|_G} u\,dt.$$

Proof. Lemma 73 applied to each of the inclusions $F \subset H$ and $G \subset H$ yields

$$\int_{[a,b]}^{|\cdot|_F} u\,dt = \int_{[a,b]}^{|\cdot|_H} u\,dt = \int_{[a,b]}^{|\cdot|_G} u\,dt.$$

□

The following is the fundamental theorem of calculus for Banach space valued functions with Bochner integrable derivative.

Theorem 75 (Fundamental theorem of calculus)**.** *Let $(G, |\cdot|)$ be a Banach space, $a, b \in \mathbb{R}$ with $a < b$ and $u : [a, b] \to G$ a function that is differentiable at each point $\tau \in [a, b]$. Suppose that the derivative $u_t : [a, b] \to G$ is Bochner integrable. Then*

$$u(b) - u(a) = \int_a^b u_t \, d\tau.$$

Proof. Set $x = u(b) - u(a) - \int_a^b u_t \, d\tau$. The Hahn-Banach theorem (see e. g. [Rud73, page 58, Corollary of Theorem 3.3]) implies the existence of a continuous linear form $T : G \to \mathbb{R}$ with $T(x) = |x|$. Thus it is sufficient to show that $T(x) = 0$. Due to Theorem 2.11(iii) in [AE01, page 92], the form T commutes with the Bochner integral. Since T also commutes with derivation, it is sufficient to prove that

$$f(b) - f(a) = \int_a^b f_t \, d\tau$$

where $f := T \circ u$. The fundamental theorem of calculus for real-valued differentiable functions with Lebesgue integrable derivative, the proof of which can be found in [Rud87, Satz 7.21, page 179], completes the proof. □

Definition 76 (Bochner and C^k spaces)**.** Let $(G, |\cdot|)$ be a Banach space, $a, b \in \mathbb{R}$ with $a < b$, and $k \in \mathbb{N}$. Then $L^2(a, b, G, |\cdot|)$ denotes the space of all functions $u \in L^1(a, b, G)$ with $|u|^2 \in L^1(a, b, \mathbb{R})$ equipped with the seminorm

$$|u|_{L^2(a,b,G,|\cdot|)} := \left(\int_a^b |u|^2 \, dt \right)^{\frac{1}{2}},$$

$L^\infty(a, b, G, |\cdot|)$ the space of all functions $u \in L^1(a, b, G)$ such that $|u|$ is essentially bounded equipped with the seminorm

$$|u|_{L^\infty(a,b,G,|\cdot|)} := \operatorname*{ess\,sup}_{t \in [a,b]} |u(t)|,$$

and $C^k(a, b, G, |\cdot|)$ the space of all functions $u : [a, b] \to G$ that are k times continuously differentiable with respect to the norm $|\cdot|$. The shorter notations

$$L^2(a, b, G), \ L^\infty(a, b, G) \text{ or } L^\infty(a, b, |\cdot|), \text{ and } C^k(a, b, G)$$

are used in contexts where it is clear which norm or space is meant.[2]

[2] Usually, L^1-integrability is not required in the definition of L^2 or L^∞. But as $[a, b]$ is of finite measure, $L^2 \subset L^1$ and $L^\infty \subset L^1$ hold anyway. The benefit of requiring L^1-integrability is that Bochner measurability doesn't have to be made a subject of discussion.

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Index

Nomenclature

$((\cdot,\cdot))$	Dirichlet scalar product
$(\cdot,\cdot)$	L^2 scalar product
$(\cdot,\cdot,\cdot)$	trilinear form
$\|\cdot\|$	Dirichlet norm
$\lvert\cdot\rvert$	L^2 norm
κ	parameter κ, see Lemma 13
$u \cdot \nabla v$	convective term
V	SST space
H'	dual space w. r. t. $\lvert\cdot\rvert$, see Definition 4
V'	dual space w. r. t. $\|\cdot\|$, see Definition 4
$C^\infty_{0,\sigma}$	see Definition 6
u_t, $u_{tt}, \ldots$	time derivatives of u, see Definition 56
∂_t	differential operator with respect to time
v^k, u^k	superscript notation, see Definition 58

Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig, ohne unerlaubte Hilfe Dritter angefertigt und andere als die in der Dissertation angegebenen Hilfsmittel nicht benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen sind, habe ich als solche kenntlich gemacht. Dritte waren an der inhaltlich-materiellen Erstellung der Dissertation nicht beteiligt; insbesondere habe ich hierfür nicht die Hilfe eines Promotionsberaters in Anspruch genommen. Kein Teil dieser Arbeit ist in einem anderen Promotions- oder Habilitationsverfahren verwendet worden.